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NONEXPANSIVE BIJECTIONS TO THE UNIT BALL OF THE ℓ_1-SUM OF STRICTLY CONVEX BANACH SPACES

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Abstract

Extending recent results by Cascales *et al.* ['Plasticity of the unit ball of a strictly convex Banach space', *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **110**(2) (2016), 723–727], we demonstrate that for every Banach space X and every collection $Z_i, i \in I$, of strictly convex Banach spaces, every nonexpansive bijection from the unit ball of X to the unit ball of the sum of Z_i by ℓ_1 is an isometry.

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1. Introduction

This article is motivated by the challenging open problem, posed by Cascales *et al.* in 2016 [2], of whether the unit ball B_X of a Banach space X is *expand-contract plastic*, in other words, whether every nonexpansive bijective automorphism of B_X is an isometry. It looks surprising that such a general property, if true, remained unnoticed during the long history of the development of the theory of Banach spaces. On the other hand, if there is a counterexample, it is not an easy task to find it, because of known partial positive results. In the finite-dimensional case, the expand-contract plasticity of B_X follows from a compactness argument: every totally bounded metric space is expand-contract plastic [5]. For the infinite-dimensional case, the main result of [2] ensures expand-contract plasticity of the unit ball of every strictly convex Banach space, in particular, of Hilbert spaces and of all L_p with $1 < p < \infty$. An example of a not strictly convex infinite-dimensional space with the same property is presented in [3, Theorem 1]. This example is ℓ_1 and, more generally, $\ell_1(\Gamma)$, by a minor modification of the same proof.

In this paper we 'mix' results from [2, Theorem 2.6] and [3, Theorem 1] and demonstrate the expand-contract plasticity of the ball of the ℓ_1-sum of an arbitrary collection of strictly convex spaces. Moreover, we demonstrate a stronger result: for

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every Banach space X and every collection $Z_i, i \in I$, of strictly convex Banach spaces, every nonexpansive bijection from the unit ball of X to the unit ball of the ℓ_1-sum of the spaces Z_i is an isometry. Analogous results for nonexpansive bijections acting from the unit ball of an arbitrary Banach space to unit balls of finite-dimensional or strictly convex spaces, as well as to the unit ball of ℓ_1, were established recently in [6].

Our demonstration uses several ideas from the papers mentioned above, but elaborates them substantially to overcome the difficulties in this more general situation.

2. Notation and auxiliary statements

We first give the notation and results that we need in our exposition.

We deal with real Banach spaces. As usual, for a Banach space E we denote by S_E and B_E the unit sphere and the closed unit ball of E, respectively. A map $F\colon U \to V$ between metric spaces U and V is called *nonexpansive* if $\rho(F(u_1), F(u_2)) \le \rho(u_1, u_2)$ for all $u_1, u_2 \in U$, so in the case of a nonexpansive map $F\colon B_X \to B_Z$ we have $\|F(x_1) - F(x_2)\| \le \|x_1 - x_2\|$ for $x_1, x_2 \in B_X$.

For a convex set $M \subset E$, we denote by $\mathrm{ext}(M)$ the set of *extreme points* of M. Recall that $z \in \mathrm{ext}(M)$ if, for every nontrivial line segment $[u, v]$ containing z in its interior, at least one of the endpoints u, v does not belong to M. A space E is called *strictly convex* when $S_E = \mathrm{ext}(B_E)$. In strictly convex spaces, the triangle inequality is strict for all pairs of vectors with different directions, that is, $\|e_1 + e_2\| < \|e_1\| + \|e_2\|$, for all nonzero $e_1, e_2 \in E$ such that $e_1 \neq k e_2$ with $k \in (0, +\infty)$.

Let I be an index set and let $Z_i, i \in I$, be a fixed collection of strictly convex Banach spaces. We consider the sum of Z_i by ℓ_1 and denote it by Z. According to the definition, this means that Z is the set of all points $z = (z_i)_{i \in I}$, where $z_i \in Z_i$ for $i \in I$, such that the support $\mathrm{supp}(z) := \{i : z_i \neq 0\}$ is at most countable and $\sum_{i \in I} \|z_i\|_{Z_i} < \infty$. The space Z is equipped with the natural norm

$$\|z\| = \|(z_i)_{i \in I}\| = \sum_{i \in I} \|z_i\|_{Z_i}. \tag{2.1}$$

Even if I is uncountable, the sum in (2.1) reduces to a countable sum $\sum_{i \in \mathrm{supp}(z)} \|z_i\|_{Z_i}$ which does not depend on the order of its terms, so there is no need to introduce an ordering on I and to appeal to any definition for uncountable sums when we speak about the space Z.

We regard each Z_i as a subspace of Z by $Z_i = \{z \in Z\colon \mathrm{supp}(z) \subset \{i\}\}$. It is well known and easy to check that in this notation

$$\mathrm{ext}(B_Z) = \bigcup_{i \in I} S_{Z_i}.$$

Also, with this notation, each $z \in Z$ can be written uniquely as a sum $z = \sum_{i \in I} z_i$, $z_i \in Z_i$, with at most countably many nonzero terms and where the series converges absolutely.

Definition 2.1. Let E be a Banach space and let $H \subset E$ be a subspace. The linear projector $P\colon E \to H$ is *strict* if $\|P\| = 1$ and for any $x \in E \setminus H$ we have $\|P(x)\| < \|x\|$.

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Lemma 2.2. *Every strict projector $P\colon E \to H$ possesses the following property: for every $x \in E \setminus H$ and every $y \in H$, we have $\|P(x-y)\| < \|x-y\|$.*

Proof. If $x \notin H$, then $x - y \notin H$, and since the projector P is strict, it follows that $\|P(x-y)\| < \|x-y\|$. □

Consider a finite subset $J \subset I$ and an arbitrary collection $z = (z_i)_{i\in J}$ with $z_i \in S_{Z_i}$ for $i \in J$. For each of these z_i, pick a supporting functional $z_i^* \in S_{Z_i^*}$, that is, a norm-one functional with $z_i^*(z_i) = 1$. The strict convexity of Z_i implies that $z_i^*(x) < 1$ for all $x \in B_{Z_i} \setminus \{z_i\}$, $i \in J$. Set $z^* = (z_i^*)_{i\in J}$ and define the map

$$P_{z,z^*} : Z \to \operatorname{span}\{z_i, i \in J\}, \quad P_{z,z^*}((y_i)_{i\in I}) = \sum_{i\in J} z_i^*(y_i) z_i.$$

Lemma 2.3. *The map P_{z,z^*} is a strict projector onto* $\operatorname{span}\{z_i, i \in J\}$.

Proof. According to the definition, we have to check that:

(1) P_{z,z^*} is a projector on $\operatorname{span}\{z_i, i \in J\}$;
(2) $\|P_{z,z^*}\| = 1$; and
(3) if $(y_i)_{i\in I} \notin \operatorname{span}\{z_i, i \in J\}$, then $\|P_{z,z^*}((y_i)_{i\in I})\| < \|(y_i)_{i\in I}\|$.

Proof of (1). This is true since

$$\begin{aligned} P^2_{z,z^*}((y_i)_{i\in I}) &= P_{z,z^*}\Big(\sum_{i\in J} z_i^*(y_i) z_i\Big) = \sum_{i\in J} z_i^*(z_i^*(y_i) z_i) z_i \\ &= \sum_{i\in J} z_i^*(y_i) z_i^*(z_i) z_i = \sum_{i\in J} z_i^*(y_i) z_i = P_{z,z^*}((y_i)_{i\in I}). \end{aligned}$$

Proof of (2). Observe that

$$\|P_{z,z^*}((y_i)_{i\in I})\| = \Big\|\sum_{i\in J} z_i^*(y_i) z_i\Big\| = \sum_{i\in J} |z_i^*(y_i)| \le \sum_{i\in J} \|y_i\| \le \sum_{i\in I} \|y_i\| = \|(y_i)_{i\in I}\|. \tag{2.2}$$

Proof of (3). If there is $N \in I \setminus J$ such that $y_N \neq 0$, the statement is obvious by (2.2). If $y_N = 0$ for all $N \in I \setminus J$, then, since $y = \sum_{i\in J} y_i \notin \operatorname{span}\{z_i, i \in J\}$, there is a $j \in J$ such that $y_j \notin \operatorname{span}\{z_j\}$ and, consequently, $|z_j^*(y_j)| < \|y_j\|$ for this j. Thus, the first inequality in (2.2) is strict. □

Proposition 2.4 (Brower's invariance of domain principle [1]). *If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $f(U)$ is open in $\mathbb{R}^n$.*

Proposition 2.5 [3, Proposition 4]. *Let X be a finite-dimensional normed space and let V be a subset of B_X such that V is homeomorphic to B_X and $V \supset S_X$. Then $V = B_X$.*

Proposition 2.6 (Mankiewicz [4]). *If X, Y are real Banach spaces and $A \subset X$ and $B \subset Y$ are convex with nonempty interior, then every bijective isometry $F : A \to B$ can be extended to a bijective affine isometry $\tilde{F} : X \to Y$.*

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PROPOSITION 2.7 (Extracted from [2, Theorem 2.3] and [6, Theorem 2.1]). *Suppose $F : B_X \to B_Y$ is a nonexpansive bijection. Then:*

(1) $F(0) = 0$;
(2) $F^{-1}(S_Y) \subset S_X$; *and*
(3) *if $F(x)$ is an extreme point of B_Y, then $F(ax) = aF(x)$ for all $a \in (-1, 1)$.*

LEMMA 2.8 [6, Lemma 2.3]. *Let X, Y be Banach spaces and let $F: B_X \to B_Y$ be a bijective nonexpansive map such that $F(S_X) = S_Y$. Suppose that $V \subset S_X$ is such that $F(av) = aF(v)$ for all $a \in [-1, 1]$ and $v \in V$. If $A = \{tx : x \in V, t \in [-1, 1]\}$, then $F|_A$ is a bijective isometry between A and $F(A)$.*

LEMMA 2.9. *Let X, Y be real Banach spaces. Let $F : B_X \to B_Y$ be a bijective nonexpansive map such that $F(tv) = tF(v)$ for every $v \in F^{-1}(S_Y)$ and every $t \in [-1, 1]$. Then F is an isometry.*

PROOF. By Proposition 2.7, $F(0) = 0$ and $F^{-1}(S_Y) \subset S_X$. We show first that $F(S_X) \subset S_Y$, that is, $F(S_X) = S_Y$.

For arbitrary $x \in S_X$, consider the point $y = F(x)/\|F(x)\| \in S_Y$ and define $\hat{x} = F^{-1}(y)$. Then, with $t = \|F(x)\|$,

$$F(x) = ty = tF(\hat{x}) = F(t\hat{x}).$$

By injectivity, this implies $x = t\hat{x}$. Since $\|\hat{x}\| = 1 = \|x\|$, it follows that $\|F(x)\| = t = 1$, that is, $F(x) \in S_Y$.

Now we may apply Lemma 2.8 to $V = F^{-1}(S_Y) = S_X$ and

$$A = \{tx : x \in S_X, t \in [-1, 1]\} = B_X.$$

Since $F(A) = B_Y$, Lemma 2.8 shows that F is an isometry. □

3. Main result

THEOREM 3.1. *Let X be a Banach space, let $Z_i, i \in I$, be a fixed collection of strictly convex Banach spaces, let Z be the ℓ_1-sum of the collection $Z_i, i \in I$, and let $F: B_X \to B_Z$ be a nonexpansive bijection. Then F is an isometry.*

The crux of the proof is Lemma 3.2 below which analyses the behaviour of F on typical finite-dimensional parts of the unit ball.

Under the conditions of Theorem 3.1, consider a finite subset $J \subset I$ with $|J| = n$ and pick collections $z = (z_i)_{i \in J}$, $z_i \in S_{Z_i}, i \in J$, and $z^* = (z_i^*)_{i \in J}$, where each $z_i^* \in S_{Z_i^*}$ is a supporting functional for the corresponding z_i. Set $x_i = F^{-1}(z_i) \in S_X$. Denote by U_n and ∂U_n the unit ball and the unit sphere, respectively, of $\operatorname{span}\{x_i\}_{i \in J}$. Let V_n and ∂V_n be the unit ball and the unit sphere of $\operatorname{span}\{z_i\}_{i \in J}$.

LEMMA 3.2. *For every collection $(a_i)_{i \in J}$ of reals with $\sum_{i \in J} a_i x_i \in U_n$,*

$$\Big\|\sum_{i \in J} a_i x_i\Big\| = \sum_{i \in J} |a_i| \tag{3.1}$$

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(which means, in particular, that U_n is isometric to the n-dimensional unit ball of ℓ_1) and

$$F\Big(\sum_{i\in J} a_i x_i\Big) = \sum_{i\in J} a_i z_i. \tag{3.2}$$

Proof. We will use induction on n. Since $z_i \in \operatorname{ext} B_Z$, the case $n = 1$ of the Lemma follows from Proposition 2.7(3). We assume the validity of the Lemma for index sets of $n-1$ elements and prove it for $|J| = n$. Fix $m \in J$ and write $J_{n-1} = J \setminus \{m\}$. We claim that

$$F(U_n) \subset V_n. \tag{3.3}$$

To see this, consider $r \in U_n$. If r is of the form $a_m x_m$, the statement follows from Proposition 2.7(3). So we must consider $r = \sum_{i\in J} a_i x_i$ with $\sum_{i\in J} |a_i| \le 1$ and $\sum_{i\in J_{n-1}} |a_i| \ne 0$. Denote the expansion of $F(r)$ by $F(r) = (v_i)_{i\in I}$. For the element

$$r_1 = \sum_{i\in J_{n-1}} \frac{a_i}{\sum_{j\in J_{n-1}} |a_j|} x_i,$$

by the induction hypothesis,

$$F(r_1) = \sum_{i\in J_{n-1}} \frac{a_i}{\sum_{j\in J_{n-1}} |a_j|} z_i.$$

On the one hand,

$$\Big\|\sum_{i\in J} a_i x_i\Big\| \le \sum_{i\in J} |a_i|,$$

and on the other hand,

$$\begin{aligned}\Big\|\sum_{i\in J} a_i x_i\Big\| &= \Big\|\sum_{i\in J_{n-1}} a_i x_i - (-a_m x_m)\Big\| \ge \Big\|F\Big(\sum_{i\in J_{n-1}} a_i x_i\Big) - F(-a_m x_m)\Big\| \\ &= \Big\|\sum_{i\in J_{n-1}} a_i z_i - a_m z_m\Big\| = \sum_{i\in J} |a_i|.\end{aligned}$$

Thus, (3.1) is demonstrated and we may write the following inequalities.

$$\begin{aligned}2 &= \Big\|F(r_1) - \frac{a_m}{|a_m|} z_m\Big\| \le \Big\|F(r_1) - \sum_{i\in J} v_i\Big\| + \Big\|\sum_{i\in J} v_i - F\Big(\frac{a_m}{|a_m|} x_m\Big)\Big\| \\ &= \|F(r_1) - F(r)\| + \Big\|F(r) - F\Big(\frac{a_m}{|a_m|} x_m\Big)\Big\| - 2\Big\|\sum_{i\in I\setminus J} v_i\Big\| \\ &\le \|F(r_1) - F(r)\| + \Big\|F(r) - F\Big(\frac{a_m}{|a_m|} x_m\Big)\Big\| \\ &\le \Big\|\sum_{i\in J_{n-1}} \frac{a_i}{\sum_{j\in J_{n-1}} |a_j|} x_i - \sum_{i\in J} a_i x_i\Big\| + \Big\|\sum_{i\in J} a_i x_i - \frac{a_m}{|a_m|} x_m\Big\| \\ &\le \sum_{i\in J_{n-1}} \Big|a_i - \frac{a_i}{\sum_{j\in J_{n-1}} |a_j|}\Big| + |a_m| + \sum_{i\in J_{n-1}} |a_i| + \Big|a_m - \frac{a_m}{|a_m|}\Big| \\ &= \sum_{i\in J_{n-1}} |a_i|\Big(1 + \Big|1 - \frac{1}{\sum_{j\in J_{n-1}} |a_j|}\Big|\Big) + |a_m|\Big(1 + \Big|1 - \frac{1}{|a_m|}\Big|\Big) = 2.\end{aligned}$$

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So all the inequalities in this chain are in fact equalities, which implies that

$$F(r) = \sum_{i \in J} v_i \quad \text{and} \quad \|F(r_1) - F(r)\| + \left\| F(r) - F\left(\frac{a_m}{|a_m|} x_m\right)\right\| = 2.$$

Our goal is to show $F(r) \in V_n$. Suppose, by contradiction, that $F(r) = \sum_{i \in J} v_i \notin V_n$ and, for convenience, set $s = \sum_{j \in J_{n-1}} |z_j^*(v_j)|$. Using the notation of Lemma 2.3,

$$\begin{aligned}
2 &= \left\| F\left(\sum_{i \in J_{n-1}} \frac{z_i^*(v_i)}{s} x_i\right) - F(r)\right\| + \left\| F(r) - F\left(\frac{z_m^*(v_m)}{|z_m^*(v_m)|} x_m\right)\right\| \\
&= \left\| \sum_{i \in J_{n-1}} \left(\frac{z_i^*(v_i)}{s} z_i - v_i\right) - v_m \right\| + \left\| \sum_{i \in J_{n-1}} v_i + v_m - \frac{z_m^*(v_m)}{|z_m^*(v_m)|} z_m \right\| \\
&> \left\| P_{z,z^*}\left(\sum_{i \in J_{n-1}} \left(\frac{z_i^*(v_i)}{s} z_i - v_i\right) - v_m\right)\right\| + \left\| P_{z,z^*}\left(\sum_{i \in J_{n-1}} v_i + v_m - \frac{z_m^*(v_m)}{|z_m^*(v_m)|} z_m\right)\right\| \\
&= \left\| \sum_{i \in J_{n-1}} \left(\frac{z_i^*(v_i)}{s} - z_i^*(v_i) z_i\right) - z_m^*(v_m) z_m \right\| + \left\| \sum_{i \in J_{n-1}} z_i^*(v_i) z_i + x_m^*(v_m) - \frac{z_m^*(v_m)}{|z_m^*(v_m)|} z_m \right\| \\
&= \sum_{i \in J_{n-1}} \left| z_i^*(v_i) - \frac{z_i^*(v_i)}{s}\right| + |z_m^*(v_m)| + \sum_{i \in J_{n-1}} |z_i^*(v_i)| + \left| z_m^*(v_m) - \frac{z_m^*(v_m)}{|z_m^*(v_m)|}\right| \\
&= \sum_{i \in J_{n-1}} |z_i^*(v_i)| \left(1 + \left|1 - \frac{1}{s}\right|\right) + |z_m^*(v_m)| \left(1 + \left|1 - \frac{1}{|z_m^*(v_m)|}\right|\right) = 2.
\end{aligned}$$

Observe that we have written the strict inequality in this chain because of Lemmas 2.3 and 2.2. The above contradiction means that our assumption was wrong and establishes the claim (3.3).

Further, we are going to prove the inclusion

$$\partial V_n \subset F(U_n). \tag{3.4}$$

We will argue by contradiction. Suppose there is a point $\sum_{i \in J} t_i \in \partial V_n \setminus F(U_n)$ and write $\tau = F^{-1}(\sum_{i \in J} t_i)$. Then $\|\sum_{i \in J} t_i\| = 1$ and $\tau \notin U_N$. Rewrite

$$\sum_{i \in J} t_i = \sum_{i \in J} \|t_i\| \, \hat{t}_i, \qquad \hat{t}_i \in S_{Z_i}.$$

Pick supporting functionals $t_i{}^*$ for the points $\hat{t}_i$, $i \in J$, and write $t = (\hat{t}_i)_{i \in J}$ and $t^* = (t_i{}^*)_{i \in J}$. We claim that $F(\alpha\tau) \in V_n$ for all $\alpha \in [0, 1]$. Indeed, if $F(\alpha\tau) \notin V_n$ for some α and $F(\alpha\tau) = \sum_{i \in I} w_i$, we deduce from Lemmas 2.3 and 2.2 the following contradiction:

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$$
\begin{aligned}
1 = \|0-\alpha\tau\| + \|\alpha\tau-\tau\| &\geq \Big\|0-\sum_{i\in I} w_i\Big\| + \Big\|\sum_{i\in I} w_i - \sum_{i\in J} t_i\Big\| \\
&= 2\Big\|\sum_{i\in I\setminus J} w_i\Big\| + \Big\|\sum_{i\in J} w_i\Big\| + \Big\|\sum_{i\in J} w_i - \sum_{i\in J} t_i\Big\| \\
&> \Big\|P_{t,t^*}\Big(\sum_{i\in J} w_i\Big)\Big\| + \Big\|P_{t,t^*}\Big(\sum_{i\in J} w_i\Big) - \sum_{i\in J} t_i\Big\| \\
&= \Big\|\sum_{i\in J} t_i^*(w_i)\hat{t}_i\Big\| + \Big\|\sum_{i\in J} t_i^*(w_i)\hat{t}_i - \sum_{i\in J} t_i\Big\| \\
&= \sum_{i\in J} |t_i^*(w_i)| + \sum_{i\in J} \big|\|t_i\| - t_i^*(w_i)\big| \geq \sum_{i\in J} \|t_i\| = 1.
\end{aligned}
$$

Note that $F(U_n)$ contains a relative neighbourhood of zero in V_n (by Propositions 2.7(1) and 2.4), so the continuous curve $\{F(\alpha\tau)\colon \alpha \in [0,1]\}$ connecting zero with $\sum_{i\in J} t_i$ in V_n has a nontrivial intersection with $F(U_n)$. This implies that there is an $a \in [0,1]$ such that $F(a\tau) \in F(U_n)$. Since $a\tau \notin U_n$, this contradicts the injectivity of F and establishes (3.4).

Now, (3.3) and (3.4) together with Lemma 2.5 imply $F(U_n) = V_n$. Observe, that U_n and V_n are isometric to the unit ball of the n-dimensional ℓ_1, so they can be considered as two copies of the same compact metric space. The expand-contract plasticity of totally bounded metric spaces [5] implies that every bijective nonexpansive map from U_n onto V_n is an isometry. In particular, F maps U_n onto V_n isometrically. Finally, by Lemma 2.6, the restriction of F to U_n extends to a linear map from $\operatorname{span}\{x_i, i \in J\}$ to $\operatorname{span}\{z_i, i \in J\}$, which evidently implies (3.2). □

Proof of Theorem 3.1. Our aim is to apply Lemma 2.9. To satisfy the conditions of the lemma, for every $z \in S_Z$ we must consider $y = F^{-1}(z)$ and show that

$$F(ty) = tz \tag{3.5}$$

for every $t \in [-1,1]$. To this end, let $J_z = \operatorname{supp}(z)$ and write

$$z = \sum_{i\in J_z} z_i = \sum_{i\in J_z} \|z_i\|\tilde{z}_i,$$

where $\tilde{z}_i \in S_{Z_i}$. Also, for $i \in J_z$, set

$$x_i := F^{-1}(\tilde{z}_i) \in S_X.$$

If J_z is finite, formula (3.2) of Lemma 3.2 implies that

$$y = F^{-1}(z) = F^{-1}\Big(\sum_{i\in J_z} \|z_i\|\tilde{z}_i\Big) = \sum_{i\in J_z} \|z_i\| x_i,$$

and

$$F(ty) = F\Big(\sum_{i\in J_z} t\|z_i\| x_i\Big) = \sum_{i\in J_z} t\|z_i\|\tilde{z}_i = tz,$$

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which gives (3.5) in this case. It remains to prove (3.5) if J_z is countable. In this case, $J_z = \{i_1, i_2, \ldots\}$ and we consider the finite subsets $J_n = \{i_1, i_2, \ldots, i_n\}$. For these finite subsets, $\sum_{i \in J_n} \|z_i\| \le 1$, so $\sum_{i \in J_n} \|z_i\| x_i \in U_n := B_{\operatorname{span}\{x_i\}_{i \in J_n}}$ and, by Lemma 3.2,

$$F\Big(\sum_{i \in J_n} \|z_i\| x_i\Big) = \sum_{i \in J_n} \|z_i\| \tilde{z}_i.$$

Passing to the limit as $n \to \infty$,

$$F\Big(\sum_{i \in J_z} \|z_i\| x_i\Big) = \sum_{i \in J_z} \|z_i\| \tilde{z}_i = z, \quad \text{that is, } y = F^{-1}(z) = \sum_{i \in J_z} \|z_i\| x_i.$$

One more application of formula (3.2) of Lemma 3.2 gives

$$F\Big(\sum_{i \in J_n} t\|z_i\| x_i\Big) = \sum_{i \in J_n} t\|z_i\| \tilde{z}_i,$$

which after passing to the limit ensures (3.5) since

$$F(ty) = F\Big(\lim_{n \to \infty} \sum_{i \in J_n} t\|z_i\| x_i\Big) = \lim_{n \to \infty} \sum_{i \in J_n} t\|z_i\| \tilde{z}_i = \sum_{i \in J_z} t\|z_i\| \tilde{z}_i = tz.$$

Thus we can apply Lemma 2.9 to F which completes the proof of the theorem. □

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