ON BIRKHOFF'S PROBLEM 73 FOR MONOIDS

Barron Brainerd

(received April 29, 1960)

Introduction. Birkhoff in [2] poses the following problem:

"Problem 73. Find necessary and sufficient conditions in order that the correspondence between the congruence relations and the (neutral) ideals of a lattice be one-one".

This problem has been solved by Areškin [1] and Hashimoto [3]. Essentially the conditions reduce to the requirement that the lattice be a generalized Boolean algebra.

Analogous problems may be stated for other algebraic systems. In particular we wish to discuss the problem for monoids. A set M together with a binary operation called multiplication is a monoid if the multiplication is associative and there is a multiplicative identity usually called 1 in M. Essentially a monoid satisfies all the axioms for a group except the inverse axiom.

A congruence relation ϕ on a monoid M is an equivalence relation on M which preserves multiplication, that is, if $a \phi b$ and $c \phi d$, then $ac \phi bd$. Here $a \phi b$ should be read "a congruent to b mod ϕ ". The set of elements congruent to 1 mod ϕ is called the kernel of ϕ and is denoted $\ker \phi$. A submonoid B of M is called normal if it is a kernel. If M is a group, then B is a normal subgroup.

The analogue to problem 73 may be stated as follows:

Find necessary and sufficient conditions for the correspondence $\phi \rightarrow \ker \phi$ between the congruence relations and the normal submonoids of a monoid to be one-one. For groups the correspondence is automatically one-one. We propose to solve this problem by elementary methods for a wide class of monoids

https://doi.org/10.4153/CMB-1960-026-3 Published online by Cambridge University Press

(namely those in which the non-invertible elements constitute an ideal). This class includes finite monoids, commutative monoids, and monoids which obey either a right or left cancellation law.

A partial solution of this problem exists in the literature for another class of generalized groups. Preston [4] has shown that for inverse semigroups the correspondence $\phi \to \ker \phi$ between congruence relations and normal subsemigroups is one-one. He however uses a different generalization of normal subgroup than is used here.

2. Definitions and preliminary remarks. In this section the condition (α) is stated and certain classes of monoids are shown to satisfy it.

Each monoid M contains a set

$$M^* = \{ a \in M \mid \exists b \in M \ [ab = ba = 1] \}$$

which is easily seen to be a subgroup of M, that is, a submonoid satisfying the inverse axiom. The set theoretic complement $N = M - M^*$ of M^* in M is called the hull of M. If M_R and M_L stand for the sets containing those elements of M which have inverses on the right and left respectively, then $M^* = M_L \cap M_R$.

A subset B of a monoid M is an ideal if $BM \subseteq B$ and $MB \subseteq B$; the null set is not excluded from the class of ideals.

A monoid is said to satisfy condition (α) if its hull is an ideal.

It is obvious that a commutative monoid satisfies (α). To show that finite monoids and monoids which obey a cancellation law also satisfy condition (α) we need the following lemma.

LEMMA 2.1. The following statements are equivalent: (i) $M_R \subseteq M_L$, (ii) $M_L \subseteq M_R$, (iii) N is an ideal.

Proof.

(i) $\Rightarrow$ (ii). If $a \in M_L$, then $a'a = 1$. Since $a' \in M_R$ and $M_R \subseteq M_L$, it follows that $a' \in M^*$ and $M_L = M^*$. Thus $M_L \subseteq M_R$.

(ii) => (iii). By the symmetry of the situation, $M_L \subseteq M_R$ implies $M - M_L = N = M - M_R$ and so N is an ideal.

(iii) => (i). If $a \in M_R$, then $aa' = 1$ and because N is an ideal $a \notin N$. Therefore $M_R = M$ and $M_R \subseteq M_L$.

PROPOSITION 2.2. Any finite monoid M satisfies condition (α).

Proof. For each $a \in M$ consider the function $\bar{a} : x \to ax$ of M into M. If $a \in M_L$, then clearly $\bar{a}$ is one-one and hence is (by the finiteness of M) a permutation on M. Therefore there is an element a'' such that $\bar{a}(a'') = aa'' = 1$. Thus $M_L \subseteq M_R$. From lemma 2.1 it then follows that N is an ideal.

PROPOSITION 2.3. If a monoid M obeys either the right or left cancellation law, then M satisfies condition (α).

Proof. Suppose M obeys the right cancellation law. Then if $a \in M_R$, $aa' = 1$ for some $a' \in M$, and $a'aa' = a'1 = 1a'$. By the cancellation law $a'a = 1$ so $a \in M_L$. Therefore $M_R \subseteq M_L$ and N is an ideal by lemma 2.1.

If M obeys the left cancellation law then by a similar argument it can be shown that N is an ideal in this case as well.

3. Main result. The main result of this note rests on the following easily verifiable observation: If B is an ideal of a monoid M, then the relation $\tau_B = \{(a, b) \in M \times M \mid a = b$ or both $a \in B$ and $b \in B\}$ is a congruence relation on M.

THEOREM. Let M be a monoid which satisfies condition (α). The correspondence $\phi \to \ker \phi$ between congruence relations and normal submonoids of M is one-one if and only if the hull N of M contains at most one element.

Proof. If the correspondence is one-one, then the congruence relation τ_N must be the identity relation on M. This follows because $\ker \tau_N = \{1\}$ which is the kernel of the identity relation. Thus N can contain at most one element.

Conversely if N contains at most one element, then either M is a group, in which case the correspondence is one-one, or $N = \{e\}$ where $ea = ae = e$ for all $a \in M$. In the latter case either $e \in \ker \phi$ or $e \phi a$ implies $e = a$. The set $\ker \phi$ contains e if and only if ϕ is the trivial congruence relation, that is the congruence relation with only one congruence class. Thus every non-trivial congruence relation is determined entirely by its behaviour on M^*. Since every non-trivial normal submonoid of M is a normal subgroup of M^*, it follows that the mapping $\phi \rightarrow \ker \phi$ is one-one when restricted to the non-trivial congruence relations. Since the only congruence relation with kernel M is the trivial one, the mapping $\phi \rightarrow \ker \phi$ is one-one in general.

REFERENCES

1. G. Ya. Areškin, On congruence relations in distributive lattices with zero elements, Dokl. Akad. Nauk S.S.S.R. (NS), 90 (1953), 485-486.

2. G. Birkhoff, Lattice Theory, second edition, (New York, 1948.)

3. J. Hashimoto, Ideal theory for lattices, Math. Japon. 2 (1952), 149-186.

4. G.B. Preston, Inverse semi-groups, J. London Math. Soc. 29 (1954), 396-403.

University of Toronto

https://doi.org/10.4153/CMB-1960-026-3 Published online by Cambridge University Press