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SEMI-SIMPLE ARTINIAN RINGS OF FIXED POINTS

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Let G be a finite group of automorphisms of the ring R, and let R^G denote the ring of fixed points of G in R; that is, $R^G=\{x \in R \mid x^g = x, \forall g \in G\}$. Let $|G|$ denote the order of G. In this note, we prove the following:

THEOREM. *Assume that R has no nilpotent ideals and no $|G|$-torsion. Then if R^G is semi-simple Artinian, R is semi-simple Artinian.*

The proof uses a recent theorem of G. Bergman and I. M. Isaacs, which we state here for convenience:

PROPOSITION 1 ([1], p. 76). *Let G be a finite group of automorphisms acting on R, and assume that R has no $|G|$-torsion. Then if $R^G=(0)$, R is nilpotent.*

In all that follows, we will assume that R is semi-prime (i.e., has no nilpotent ideals) and has no $|G|$-torsion.

LEMMA 1. *If $I\neq(0)$ is a right (left) ideal of R invariant under G, then $I \cap R^G\neq(0)$.*

Proof. Since I is G-invariant, G acts as a group of automorphisms of I. Thus, if $I \cap R^G=(0)$, I is nilpotent by Proposition 1. This is impossible since R is semi-prime.

LEMMA 2. *If R^G has a unit element e, then e is a unit for R.*

Proof. Consider $I=\{y-ey \mid y \in R\}$. Since $e \in R^G$, I is a right ideal of R invariant under G. Hence by lemma 1, $I \cap R^G\neq 0$ if $I\neq 0$. But if $0\neq y-ey \in R^G$, then $0=e(y-ey)=y-ey$, a contradiction. Thus $I=0$, and so $y=ey$, $\forall y \in R$. Similarly $y=ye$, $\forall y \in R$.

LEMMA 3. *If R^G is semi-simple, then R is semi-simple.*

Proof. Let $J(R)$ denote the Jacobson radical of R. Now $J(R)$ is invariant under any automorphism of R, so in particular it is G-invariant. Thus if $J(R)\neq 0$, $J(R) \cap R^G\neq 0$ by lemma 1. But if $x \in J(R)$ is fixed by all $g \in G$, its quasi-inverse must also be fixed, so is in R^G. Hence $J(R) \cap R^G$ is a quasi-regular ideal in R^G, which is semisimple. This is a contradiction unless $J(R)=0$.

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Proof of the Theorem. Since R^G is (right) Artinian, we may choose a finite family of maximal right ideals of R, $\rho_1, \ldots, \rho_m$, so as to minimize $I=R^G \cap (\bigcap_{i=1}^m \rho_i)$. We claim that $I=(0)$. For if not, since R^G is semi-simple, we may write $I=eR^G$, where e is a non-zero idempotent of R^G. Then $(1-e)R$ will be a proper right ideal of R, and we can find a maximal right ideal $\rho_{m+1} \subset R$ containing $1-e$. Clearly $e \notin \rho_{m+1}$, hence $R^G \cap (\bigcap_{i=1}^{m+1} \rho_i)$ is properly smaller than I, contradicting the assumption that I is minimal. Thus $I=(0)$.

Now consider the right ideal of R given by $\rho=\bigcap_{\substack{i\leq m\\ g\in G}} \rho_i^g$. ρ is G-invariant, and $\rho \cap R^G \subseteq I=(0)$, hence by Lemma 1, $\rho=(0)$. But since (0) is the intersection of finitely many maximal right ideals of R, R has finite composition length as a right module over itself (since there is a natural R-module embedding of R in the finite sum of simple modules $S=\sum_{i,g} \oplus R/\rho_i^g$). Hence R is Artinian as a right R-module, hence as a ring.

By Lemma 3, R is semi-simple since R^G is, and thus the theorem is proved.

The proof of the theorem shows a little more. Namely, if length $_R(R)$ denotes the length of a composition series for R as a right R-module, then m can be chosen $\leq$length $_{R^G}(R^G)$. Thus, we have:

COROLLARY. *Let R and R^G be as in the theorem. Then length* $_R(R) \leq |G|$ *length* $_{R^G}(R^G)$.

In the special case when G is a solvable group, Cohen has used the Theorem to prove that if R is semi-prime with no $|G|$-torsion and R^G is Goldie, then R must also be a Goldie ring [2]. It has recently been announced by V. K. Harchenko [3] that this result is true for any finite group G.

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