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POLYNOMIAL RINGS WITH THE OUTER PRODUCT PROPERTY

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Note. We shall assume throughout that all rings R are noetherian. This property is not used in some of the lemmas but it intercedes before the main theorem. We retain this assumption to ease the exposition.

Introduction. In [**3**] Lissner defined a class of rings called *outer product rings*, (OP-rings). These are commutative rings R with identity for which *every* exterior vector $v \in \bigwedge^{n-1}R^n$ is decomposable, i.e., $v = v_1 \wedge \ldots \wedge v_{n-1}$ with $v_i \in R^n$, $i = 1, \ldots, n-1$.

If we look only at those vectors $v \in \bigwedge^{n-1}R^n$ whose co-ordinates with respect to any basis of $\bigwedge^{n-1}R^n$ generate the unit ideal in R and consider those rings R for which all vectors of this type are decomposable, we obtain the class of rings which have been referred to as *Hermite-rings* (H-rings, see also Lissner [**3**]). This class of H-rings evidently contains the class of OP-rings.

Proposition A. *R is an H-ring if and only if for any n elements of R, $a_1, \ldots, a_n$, such that the ideal they generate in R is R, there exists an invertible $n \times n$ matrix with first row* $(a_1, \ldots, a_n)$.

Proof. The reader should refer to [**3**, § 2, Proposition 2.1 and Corollary] for a proof of this statement.

One of the major reasons for considering H-rings is contained in the following proposition.

Proposition B. *The following two statements are equivalent.*

(i) *R is an H-ring.*

(ii) *If P is a finitely generated projective R-module such that $P \oplus R^m \cong R^s$ for two integers m, s, then $P \cong R^{s-m}$.*

Proof. See [**11**, Proposition 12.2, p. 185].

In [**7**] Serre asked if every f.g. projective R-module, $R = k[X_1, \ldots, X_n]$ where k is a field, is necessarily free. Serre showed [**7**, Proposition 10] that if P is a f.g. projective $k[X_1, \ldots, X_n]$-module then $P \oplus R^m \cong R^s$ for some integers m, s, (depending on P). In view of Proposition B above, Serre's question amounts to asking whether $k[X_1, \ldots, X_n]$ is an H-ring.

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In [**10**], Seshadri showed that for $n = 2$, finitely generated projective $k[X_1, \ldots, X_n]$-modules are indeed free, while in [**4**] Lissner showed that $R[X]$, when R is a principal ideal domain (P.I.D.), is an OP-ring, thereby obtaining Seshadri's theorem as a consequence.

Our aim in this paper is to classify all those noetherian rings R such that $R[X]$ is an OP-ring. This is accomplished via the following theorem.

THEOREM. *Let R be a noetherian ring. The following are equivalent:*

(i) *$R[X]$ is an* OP-*ring.*

(ii) *R is a direct sum of rings of global dim $\leqq 1$ and special principal ideal rings.*

In the second section we completely classify finitely generated projective $R[X]$-modules, when $R[X]$ is an OP-ring.

1. The outer product property. We recall a few lemmas.

LEMMA 1. *Let R be an* OP-*ring and $\phi : R \rightarrow S$ be a surjection. Then S is an* OP-*ring.*

Proof. See [**13**, Proposition 1.3].

LEMMA 2. *Let R be a ring and let S be a multiplicatively closed subset of R ($0 \notin S$, $1 \in S$). If R is an* OP-*ring then so is R_S.*

Proof. See [**5**, Proposition 4.5].

COROLLARY *Let $R[X]$ be an* OP-*ring and $p \subset R$ be a prime ideal. Then $R_p[X]$ is an* OP-*ring.*

Proof. The proof is clear from Lemma 2.

In view of the corollary to Lemma 2 it is appropriate to begin considering OP-rings of the form $R[X]$ in the case that R is local.

The following lemma is probably well-known, but inasmuch as there appears no proof in the literature we shall include a proof here for completeness.

We recall the following definition.

Definition. Let (R, m) be a local ring. The *v-dimension* (*v-dim*) of R is $= \dim_{R/m} m/m^2$.

It is well-known (see, e.g., [**6**, p. 189]) that the v-dimension of R is equal to the minimal number of elements in a generating set for m.

LEMMA 3. *Let (R, m) be a local ring of v-dimension $= s$. Then there exists a maximal ideal in $A = R[X]$, say p, such that A_p has* v-dim $= s + 1$.

Proof. We suppose $m = (u_1, \ldots, u_s)$ and let $p \subset A$, $p = (u_1, \ldots, u_s, X)$. It is clear that p is a maximal ideal of A. Let $\phi : A \rightarrow A_p$ be the canonical map, $\phi(f(x)) = f(x)/1$. Evidently $\phi(p)A_p$ is the maximal ideal of A_p and is generated by $\{u_1/1, u_2/1, \ldots, u_s/1, x/1\}$. If we can show that *this* set of generators is minimal we will know that v-dimension $A_p = s + 1$, since in a local ring all minimal generating sets have the same number of elements.

Claim 1. $x/1 \notin (u_1/1, u_2/1, \ldots, u_s/1)$ in A_p: Suppose

$$\frac{x}{1} = \frac{f_1}{g_1}\frac{u_1}{1} + \ldots + \frac{f_s}{g_s}\frac{u_s}{1}$$

where $f_i, g_i \in R[X]$, and $g_i \notin p$. Then

$$\frac{x}{1} = \frac{f_1g_2 \ldots g_s u_1 + g_1f_2g_3 \ldots g_s u_2 + \ldots + g_1g_2 \ldots g_{s-1}f_s u_s}{g_1g_2 \ldots g_s}.$$

From the definition of equality in A_p, there exists $k \in R[X]$, $k \notin p$, such that

$$(*) \quad kg_1g_2 \ldots g_s x = kf_1g_2 \ldots g_s u_1 + kg_1f_2g_3 \ldots g_s u_2 + \ldots + kg_1g_2 \ldots g_{s-1}f_s u_s.$$

Since $g_i \notin p$, the constant term of g_i is *not* in $m \subset R$; similarly for k. Thus, on the left side of equation $(*)$, the coefficient of x is (const. term of k) $\cdot \prod_{i=1}^{s}$ (const. term g_i). Since all these factors are in R but not in m, and m is prime, the coefficient of x on the left side of $(*)$ is an element of R not in m. Evidently all the coefficients on the right side of $(*)$ are in m. This contradiction establishes the claim.

Claim 2. $u_1/1 \notin (u_2/1, \ldots, u_s/1, x/1)$: (We shall observe that the argument given here does not depend on u_1, but would work for any u_i.) Suppose $u_1/1 \in (u_2/1, \ldots, u_s/1, x/1)$; then

$$\frac{u_1}{1} = \frac{f_1}{g_1}\cdot\frac{x}{1} + \frac{f_2}{g_2}\cdot\frac{u_2}{1} + \ldots + \frac{f_s}{g_s}\cdot\frac{u_s}{1}, \; f_i, g_i \in R[X] \quad \text{and} \quad g_i \notin p.$$

Thus,

$$\frac{u_1}{1} = \frac{f_1g_2 \ldots g_s x + g_1f_2g_3 \ldots g_s u_2 + \ldots + g_1g_2 \ldots g_{s-1}f_s u_s}{g_1g_2 \ldots g_s}.$$

Again by the definition of equality in A_p, there exists $k \in R[X]$, $k \notin p$ such that

$$(\dagger) \quad u_1kg_1g_2 \ldots g_s = kf_1g_2 \ldots g_s x + kg_1f_2g_3 \ldots g_s u_2 + \ldots + kg_1g_2 \ldots g_{s-1}f_s u_s.$$

As before, the constant term of $kg_1 \ldots g_s$ is in R not in m, call it t_0. Now $t_0u_1 \neq 0$, since u_1 is part of a minimal generating set for m and $t_0 \notin m$.

The constant term on the right side of $(\dagger)$ is $\neq 0$ and $t_0u_1 =$ (const. term of $kg_1f_2g_3 \ldots g_s) \cdot u_2 + \ldots +$ (const. term of $kg_1g_2 \ldots g_{s-1}f_s) \cdot u_s$.

This equation contradicts the minimality of the generating set $\{u_1, \ldots, u_s\}$ of m, and establishes Claim 2.

Thus $\phi(p)A_p$ needs $s + 1$ generators and so v-dimension $A_p = s + 1$.

PROPOSITION 1. *If (R, m) is a local ring of v-dimension $\geqq 2$, then $R[X]$ is not an* OP-*ring.*

Proof. If $R[X]$ is an OP-ring, then for any prime ideal $p \subset R[X]$, $R[X]_p$ is an OP-ring [**5**, Proposition 4.5]. By Lemma 3, there exists maximal ideal $p \subset R[X]$ such that v-dimension $R[X]_p = (v\text{-dimension } R) + 1 > 2$. But local OP-rings must have v-dim $\leqq 2$ [**13**].

Thus, $R[X]$ is not an OP-ring.

PROPOSITION 2. *If R is a ring of Krull dimension $\geqq 2$, then $R[X]$ is not an* OP-*ring.*

Proof. If Krull dimension $R \geqq 2$ there exists a maximal ideal m in R such that Krull dimension $R_m \geqq 2$. Since Krull dimension $R_m \leqq v$-dimension R_m, we see by Proposition 1 that $R_m[X]$ is not an OP-ring. Thus, $R[X]$ is not an OP-ring by Lemma 2.

In view of Proposition 2 we need only consider rings R of Krull Dimension $\leqq 1$.

If Krull dimension $R = 0$ then all prime ideals in R are maximal and R is a finite direct sum of *primary* rings, i.e., rings with exactly one prime ideal, (finite because R is noetherian). We recall the following proposition.

PROPOSITION 3. *If $R = R_1 \oplus \ldots \oplus R_s$ then, R is an* OP-*ring* $\Leftrightarrow$ *R_i is an* OP-*ring for each $i = 1, \ldots, s$.*

Proof. See [**5**, Theorem 5.4].

So, if R has Krull dimension $= 0$, $R = R_1 \oplus \ldots \oplus R_s$, where the R_i are all primary rings and $R[X] = R_1[X] \oplus \ldots \oplus R_s[X]$. By Proposition 3, $R[X]$ is an OP-ring if and only if $R_i[X]$ is an OP-ring for each $i = 1, 2, \ldots, s$.

PROPOSITION 4. *Let R be a primary ring with prime ideal p. $R[X]$ is an* OP-*ring* $\Leftrightarrow$ *R is a special principal ideal ring or R is a field.* (Recall that R is a special principal ideal ring if R has a unique prime ideal, $p = (u)$, which is nilpotent.)

Proof. $\Rightarrow$: Since $R[X]$ is an OP-ring so is R, by Lemma 1. A primary ring is a local ring and so by Proposition 1, v-dim $R < 2$. If v-dimension $R = 0$ then R is a field. So we may assume that v-dimension $R = 1$. In that case p $= (u)$ and p is nilpotent, i.e., R is a special principal ideal ring.

$\Leftarrow$: If R is a field then $R[X]$ is a principal ideal domain and so is an OP-ring [**3**, Theorem 2.2]. If R is a special principal ideal ring then R is a complete local ring and hence by a theorem of I. S. Cohen [**2**], R is the homomorphic image of a regular local ring of Krull dimension equal to the v-dimension of R. Let A be such a regular local ring of Krull dimension $= 1$, and $\phi : A \rightarrow R$ the surjection provided by the theorem. Now A is a principal ideal domain and so $A[X]$ is an OP-ring [**4**]. We extend ϕ to a homomorphism, which we will also call ϕ, $\phi : A[X] \rightarrow R[X]$. This new ϕ is also a surjection. Since $A[X]$ is an OP-ring, so is $R[X]$.

Thus, if R has Krull dimension 0, $R[X]$ is an OP-ring if and only if R is a finite direct sum of fields and principal ideal rings.

It remains only to consider rings R of Krull dimension = 1. We may as well assume that the prime spectra of the rings considered are connected, for if not then R is a finite sum of rings of Krull dimension $\leqq 1$, whose prime spectra are connected. We have already considered the case of Krull dimension zero. Since the OP property is invariant under direct sums we are reduced to considering the problem in one summand.

PROPOSITION 5. *Let R be a ring with connected prime spectrum and Krull dimension one.*

$R[X]$ is an OP-*ring* $\Leftrightarrow$ *R has global dimension one.*

Proof. $\Leftarrow$: If R has global dimension one then R is a direct sum of Dedekind domains [**1**, Proposition 4.13]. Since the prime spectrum of R is connected, R is a Dedekind domain. That $R[X]$ is an OP-ring follows from [**12**, Theorem 1.2].

$\Rightarrow$: Let m be any maximal ideal of R. Suppose that m is also a minimal prime ideal of R. Since R is noetherian there are only finitely many minimal prime ideals of R, say $m, p_1, \ldots, p_s$. In Spec(R) we let

$$V(\mathfrak{A}) = \{\text{prime } q \subset R | \mathfrak{A} \subseteq q\}$$

for any ideal $\mathfrak{A}$. The $V(\mathfrak{A})$ are all closed sets in Spec(R) and Spec(R) = $(\{m\}) \cup (\bigcup_{i=1}^{s} V(p_i))$. Clearly $\{m\} \cap V(p_i) = \phi, i = 1, \ldots, s$. Since $\bigcup_i V(p_i)$ is closed in Spec(R) we have that $\{m\}$ is open. Since $\{m\} = V(m)$ we also have that $\{m\}$ is closed. This is a contradiction since Spec(R) is connected. Thus m is not a minimal prime of R and so the Krull dimension of R_m is 1.

Since $R[X]$ is an OP-ring so is $R_m[X]$ by Lemma 2. Hence, by Proposition 1, v-dimension $R_m < 2$. Since in any event Krull dimension $R_m \leqq v$-dimension R_m, we have both dimensions = 1. Hence global dimension $R_m = 1$. Since this maximal ideal was arbitrarily chosen and since global dimension R = sup global dimension R_m, (m varying over the maximal ideals of R) we have global dimension $R = 1$.

Thus, if R has Krull dimension 1 and $R[X]$ is an OP-ring then R is a finite sum of rings of global dimension 1, special PIR's and fields. This, together with our previous results proves:

THEOREM 1. *Let R be a commutative noetherian ring with* 1. *$R[X]$ is an* OP-*ring* $\Leftrightarrow$ *R is a direct sum of rings of global dimension* $\leqq 1$ *and special principal ideal rings.*

The following corollary is an immediate consequence.

COROLLARY 1.1. *If R is a domain the following are equivalent.*

(1) *R is a Dedekind domain* (*possibly a field*).

(2) *$R[X]$ is an* OP-*ring.*

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(Note: The referee has brought to my attention a recent article by Kleiner in Mat. Sb. *84*, No. 4 (1971), 526–536, in which he also obtains Corollary 1.1 above. Our Theorem 1 then, generalizes his result.)

2. Projective $R[X]$-modules when $R[X]$ is an OP-ring. In view of Theorem 1 of the previous section if one wishes to consider the structure of finitely generated projective $R[X]$-modules when $R[X]$ is an OP-ring it suffices to consider the structure of finitely generated projective $R[X]$ modules when R is a connected ring of global dimension $\leqq 1$ and when R is a special principal ideal ring.

In view of the theorem of Serre [**8**], we know precisely what the situation is when global dimension of R is $\leqq 1$. If R is a field or a principal ideal domain then all finitely generated projectives are free; while if R is a Dedekind domain, not a field or a principal ideal domain, then all finitely generated projective $R[X]$-modules have the form $(R[X])^s \oplus I$ where I is a projective ideal of $R[X]$.

The only case left to consider is the case when R is a special principal ideal ring.

THEOREM 2. *Let R be a special principal ideal ring. Then finitely generated projective $R[X]$-modules are free.*

Proof. Let $p = (u) \subset R$ be the unique prime ideal of R, and let $p[X] = \{a_0 + a_1x + \ldots + a_nx^n | a_i \in p\}$.

Then p is nilpotent, say $p^m = 0$, so $p[X]^m = 0$ also, and $R[X]$ is therefore trivially complete with respect to the $p[X]$-topology. Proposition 2.29 of [**11**] then applies and

$$K_0(R[X]) \cong K_0\left(\frac{R[X]}{p[X]}\right).$$

Since

$$\frac{R[X]}{p[X]} \cong \frac{R}{p}[X] \quad \text{and} \quad K_0\left(\frac{R}{p}[X]\right) \cong Z,$$

it follows that $K_0(R[X]) \cong Z$, i.e., every finitely generated projective $R[X]$-module has a free complement. Since $R[X]$ is an H-ring we have, by Proposition B, that every finitely generated projective $R[X]$-module is then free.

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