ON NON-CROSS VARIETIES OF GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Our title has become something of a misnomer, however we retain it since drafts of this note have been quoted with it.

Unless otherwise stated our terminology and notation follow that in Hanna Neumann's book [12].

The Oates-Powell Theorem ([12] p. 151) allows us to say that a variety is *Cross* if and only if it can be generated by a finite group, and to assert that the laws of a Cross variety are finitely based. A variety is *just-non-Cross* if it is not Cross but every proper subvariety of it is Cross.

We asked in [9]: what non-Cross varieties have just-non-Cross subvarieties? The answer is: all of them.

THEOREM 1. *Every non-Cross variety has a just-non-Cross subvariety.*

The proof is an easy application of Zorn's Lemma. If $\{\mathfrak{B}_\lambda : \lambda \in \Lambda\}$ is a descending chain of non-Cross subvarieties of a non-Cross variety such that the intersection $\mathfrak{D} = \bigwedge\{\mathfrak{B}_\lambda : \lambda \in \Lambda\}$ is properly contained in each $\mathfrak{B}_\lambda$, then the union of the corresponding chain $\{B_\lambda : \lambda \in \Lambda\}$ of fully invariant subgroups of the word group X_∞ ([12] p. 4) is not finitely generated, hence $\mathfrak{D}$ is not finitely based, and a fortiori $\mathfrak{D}$ is still non-Cross.

In [9] we claimed that for every prime p the product variety $\mathfrak{A}_p\mathfrak{A}_p$ is just-non-Cross ($\mathfrak{A}_p$ is the variety of abelian groups of exponent dividing p). Here we substantiate this as a consequence of a detailed description, in section 2, of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$.

The variety $\mathfrak{A}$ of all abelian groups and the varieties $\mathfrak{A}_p\mathfrak{A}_p$ are just-non-Cross and nilpotent-by-abelian. The converse is also true.

THEOREM 2. *The only nilpotent-by-abelian just-non-Cross varieties are* $\mathfrak{A}$ *and the* $\mathfrak{A}_p\mathfrak{A}_p$.

This theorem is related to the so-called external result we state in section 3, and is proved with it in section 5.

2. The subvariety lattice of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$

In this section we give a description of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. Proofs are deferred to section 4.

Lattice terminology follows Birkhoff [1].

We begin with some notation. The set of positive integers is denoted by P. As usual $\mathfrak{A}_n$, $\mathfrak{B}_n$, $\mathfrak{N}_n$ denote, respectively, the variety of abelian groups of exponent dividing n, the variety of groups of exponent dividing n, and the variety of groups of nilpotency class at most n. The variety of all groups will, for convenience, be denoted $\mathfrak{N}_\omega$. Our description of the subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ will be in terms of these varieties and one more family whose members will be denoted $\mathfrak{N}_{n*}$. The variety $\mathfrak{N}_{n*}$ is the subvariety of $\mathfrak{N}_n$ defined by the additional law $\prod_{s=2}^n [x_s, x_1, \cdots, x_{s-1}, x_{s+1}, \cdots, x_n]$. Note that $\mathfrak{N}_{n*} \supseteq \mathfrak{N}_{n-1}$. For any particular prime p only certain of these additional varieties are needed, namely those for which n is at least 3 and is divisible by p. We therefore introduce for each prime p an ordered extension $P(p)$ of P defined by:

$$P(p) = \{1, \cdots, p-1, p*, p, \cdots, pr-1, pr*, pr, \cdots, \omega\} \quad \text{for } p \text{ odd},$$
$$P(2) = \{1, 2, 3, 4*, 4, \cdots, 2r-1, 2r*, 2r, \cdots, \omega\}$$

with the order as indicated. The $P(p)$ and $\{0, 1, \cdots, \alpha+1\}$ taken in this order may be considered as lattices – we do this. For each p the varieties $\mathfrak{B}_{p^\beta}$ and $\mathfrak{A}_{p^\tau}\mathfrak{N}_\nu$ for ν in $P(p)$ play a distinguished role. We denote them $\mathfrak{B}(\beta)$ and $\mathfrak{N}(\tau, \nu)$ respectively.

With each subvariety $\mathfrak{V}$ of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ we associate an element $\boldsymbol{\beta}(\mathfrak{V})$ of $\{0, \cdots, \alpha+1\}$ and elements $\boldsymbol{\nu}(0, \mathfrak{V}), \cdots, \boldsymbol{\nu}(\alpha-1, \mathfrak{V})$ of $P(p)$ as follows:

$$\boldsymbol{\beta}(\mathfrak{V}) = \min\{\beta : \mathfrak{V} \subseteq \mathfrak{B}(\beta)\};$$

for $\tau \in \{0, \cdots, \alpha-1\}$,

$$\boldsymbol{\nu}(\tau, \mathfrak{V}) = \min\{\nu : \mathfrak{V} \subseteq \mathfrak{N}(\tau, \nu)\}.$$

The subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ are characterized by the above invariants:

2.1 *If $\mathfrak{V}$ is a subvariety of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$, then*

$$\mathfrak{V} = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}(\boldsymbol{\beta}(\mathfrak{V})) \wedge \bigwedge_{\tau=0}^{\alpha-1} \mathfrak{N}(\tau, \boldsymbol{\nu}(\tau, \mathfrak{V})).$$

If $\alpha = 1$, it follows that every proper subvariety of $\mathfrak{A}_p\mathfrak{A}_p$ is nilpotent, and hence Cross because it has finite exponent. As $\mathfrak{A}_p\mathfrak{A}_p$ is obviously not Cross, this yields the following.

THEOREM 3. *For every prime p the variety $\mathfrak{A}_p\mathfrak{A}_p$ is just-non-Cross.*

This discharges a debt incurred in [9]. The proof here – due primarily to one of us (MFN) – supersedes an earlier one which motivated the papers [4], [5] (and in which the result was also announced).

It is clear that for all subvarieties $\mathfrak{U}$, $\mathfrak{V}$ of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$

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$$\boldsymbol{\beta}(\mathfrak{U} \vee \mathfrak{V}) = \max \{\boldsymbol{\beta}(\mathfrak{U}), \boldsymbol{\beta}(\mathfrak{V})\}$$

and

$$\boldsymbol{\nu}(\tau, \mathfrak{U} \vee \mathfrak{V}) = \max \{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}$$

for all τ in $\{0, \cdots, \alpha-1\}$. The next point to prove is that the corresponding result for meets also holds.

2.2 *For all subvarieties* $\mathfrak{U}$, $\mathfrak{V}$ *of* $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$

$$\boldsymbol{\beta}(\mathfrak{U} \wedge \mathfrak{V}) = \min \{\boldsymbol{\beta}(\mathfrak{U}), \boldsymbol{\beta}(\mathfrak{V})\}$$

and

$$\boldsymbol{\nu}(\tau, \mathfrak{U} \wedge \mathfrak{V}) = \min \{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}$$

for all τ *in* $\{0, \cdots, \alpha-1\}$.

Now it follows from 2.1 that the mapping $\chi : \mathfrak{V} \mapsto (\boldsymbol{\beta}(\mathfrak{V}), \boldsymbol{\nu}(0, \mathfrak{V}), \cdots, \boldsymbol{\nu}(\alpha-1, \mathfrak{V}))$ is an embedding of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ into the direct product Λ of the lattice $\{0, \cdots, \alpha+1\}$ with α copies of $P(p)$. A sublattice of a direct product of distributive lattices with descending chain condition is a distributive lattice with descending chain condition.

THEOREM 4. *The lattice of subvarieties of* $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ *is distributive with descending chain condition.*

The description of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ is now completed by giving its image under χ. Let Σ be the subset of the direct product lattice Λ defined by:

$(\beta, \nu_0, \cdots, \nu_{\alpha-1}) \in \Sigma$ *if and only if*

$$\begin{array}{ll}
\nu_\beta = \cdots = \nu_{\alpha-1} = 1 & \textit{for } \beta < \alpha, \\
\nu_{\beta-1} < p & \textit{for } 1 \leqq \beta \leqq \alpha; \\
\nu_{\tau+1} \leqq \begin{cases} \nu_\tau \\ \nu_\tau - p + 1 \\ pr \\ 2 \end{cases} & \begin{array}{l} \textit{for } \nu_\tau \in \{1, \omega\}, \\ \textit{for } \nu_\tau \in P \textit{ and } \nu_\tau > p, \\ \textit{for } \nu_\tau = p(r+1)* \textit{ with } r \in P, \\ \textit{for } 2 \leqq \nu_\tau \leqq p; \end{array} \\
\nu_{\tau+2} = 1 & \textit{for } \nu_\tau \leqq 2p-1.
\end{array}$$

2.3 *The image of* χ *is* Σ.

While the description of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ afforded by all this is adequate, it is somewhat *ad hoc*. Because the lattice is distributive with descending chain condition, it follows (cf. section 2 of Chapter VIII of [1] – suitably corrected) that every element of the lattice can be uniquely written as an irredundant finite join of (finitely) join-irreducible elements. Moreover, a finite set of join-irreducibles gives its join irredundantly if and only if no two distinct elements of the set are comparable. Hence such a lattice can easily be reconstructed from the partially ordered set of its join-irreducible elements. The reconstruction can be carried out so as to yield a faithful representation of the lattice in the lattice of all subsets of the set of its join-irreducible elements. These facts suggest that a canonical way of describing such lattices is to give the partially ordered sets of their join-irreducible elements. We do this for the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. An advantage of this approach is that our results are then more readily comparable with related results of Brooks [2] and Bryce [3], and better suited for the extension of the present results to a description of the subvarieties of $\mathfrak{A}\mathfrak{A}_n$ with square-free n (to be given in [10]).

Given 2.3 and the explicit description of the sublattice Σ of Λ, it is an elementary exercise to derive the desired information. We simply give the result after a hint to the derivation we used.

If an element $(\beta, \nu_0, \cdots, \nu_{\alpha-1})$ of Σ is join-irreducible, then $(\beta', \nu_0, \cdots, \nu_{\alpha-1}) \notin \Sigma$ for $\beta' < \beta$ because

$$(\beta, \nu_0, \cdots, \nu_{\alpha-1}) = (\beta, 1, \cdots, 1) \vee (\beta', \nu_0, \cdots, \nu_{\alpha-1}).$$

Similarly $(\beta, \nu_0, \cdots, \nu_{\tau-1}, \mu, \nu_{\tau+1}, \cdots, \nu_{\alpha-1}) \notin \Sigma$ for $\mu < \nu_\tau$ and $\tau \in \{0, \cdots, \alpha-2\}$. Hence if $\nu_{\alpha-1} = \nu \neq 1$, the conditions defining Σ determine $\beta, \nu_0, \ldots, \nu_{\alpha-2}$:

$$\beta = \begin{cases} \alpha & \text{for } \nu < p, \\ \alpha+1 & \text{for } \nu \geqq p; \end{cases}$$

$$\nu_\tau = \begin{cases} \omega & \text{for } \nu = \omega, \\ \langle \nu \rangle + (p-1)(\alpha-1-\tau) & \text{for } \nu \neq \omega \quad \text{except } \nu = 2, \tau \in \{\alpha-3, \alpha-2\}; \end{cases}$$

$$\nu_{\alpha-2} = 2,\ \nu_{\alpha-3} = 2p* \qquad \text{for } \nu = 2:$$

here, and in the sequel, $\nu \mapsto \langle \nu \rangle$ denotes the mapping of $P(p) \setminus \{\omega\}$ to P which is the identity on P and for which $\langle pr* \rangle = pr$ whenever $pr* \in P(p) \setminus P$. Finally, if $\nu_{\alpha-1} = 1$, then $\beta = \alpha+1$ or the corresponding variety lies in $\mathfrak{A}_{p^{\alpha-1}}\mathfrak{A}_p$; if $\beta = \alpha+1$, then $\nu_0 = \cdots = \nu_{\alpha-2} = 1$. It is straightforward to check that the resulting elements of Σ are join-irreducible.

We can now describe the partially ordered set $J(p^\alpha)$ of the join-irreducible subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. Clearly $J(p^0)$ consists of $\mathfrak{E}$ and $\mathfrak{A}_p$ with $\mathfrak{E} \subset \mathfrak{A}_p$. For α in P the set $J(p^\alpha)$ consists of $J(p^{\alpha-1})$ and for each ν in $P(p)$ a variety $\mathfrak{J}(p^\alpha, \nu)$ defined as follows:

$$\mathfrak{J}(p^\alpha, 1) = \mathfrak{A}_{p^{\alpha+1}};$$

$$\mathfrak{J}(p^\alpha, 2) = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}_{p^\alpha} \wedge \mathfrak{R}_{2+(p-1)(\alpha-1)} \wedge \mathfrak{A}_{p^{\alpha-3}}\mathfrak{R}_{2p*} \wedge \mathfrak{A}_{p^{\alpha-2}}\mathfrak{R}_2$$

here the second term must be omitted when $p = 2$, and the fourth and fifth when they are not meaningful (also, the third term is redundant when α is 2 or 3); for $\nu \in P(p) \setminus \{1, 2, \omega\}$,

$$\mathfrak{J}(p^\alpha, \nu) = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}_{p^\alpha} \wedge \mathfrak{R}_{\langle \nu \rangle+(p-1)(\alpha-1)} \wedge \mathfrak{A}_{p^{\alpha-1}}\mathfrak{R}_\nu$$

here the second term must be omitted when $\nu \geqq p$ and the last term is redundant when $\nu \in P$; and

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$$\mathfrak{J}(p^\alpha, \omega) = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p.$$

Note that the only non-nilpotent join-irreducible varieties in $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ are the $\mathfrak{A}_{p^\tau}\mathfrak{A}_p$ with $\tau \in \{1, \cdots, \alpha\}$. In contrast to this Brooks [2] has shown that there is an infinite number of non-nilpotent join-irreducible subvarieties in $\mathfrak{A}_p\mathfrak{A}_{p^2}$.

It is a routine matter to check that the partial order on $J(p^\alpha)$ is generated by that on $J(p^{\alpha-1})$ and the inclusions:

$$\begin{aligned}
&\mathfrak{A}_p \subset \mathfrak{J}(p^\alpha, 1) \subset \mathfrak{J}(p^\alpha, p),\\
&\mathfrak{A}_p \subset \mathfrak{J}(p^\alpha, 2),\\
&\mathfrak{J}(p^\alpha, \mu) \subset \mathfrak{J}(p^\alpha, \nu) \quad \text{whenever } \mu, \nu \in P(p) \text{ and } 2 \leqq \mu < \nu;
\end{aligned}$$

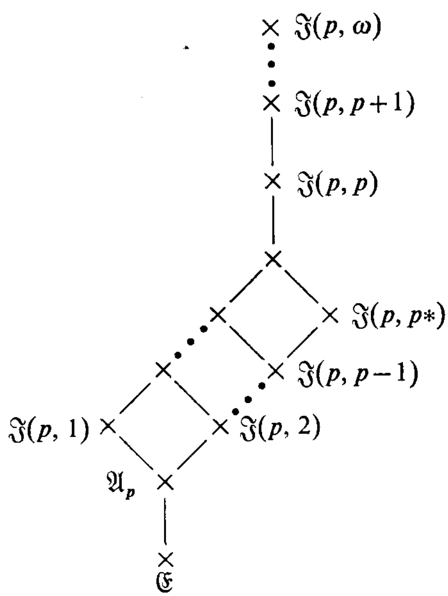
if $\alpha > 1$ then also

$$\begin{aligned}
&\mathfrak{J}(p^{\alpha-1}, 1) \subset \mathfrak{J}(p^\alpha, 1),\\
&\mathfrak{J}(p^{\alpha-1}, 1) \subset \mathfrak{J}(p^\alpha, 2),\\
&\mathfrak{J}(p^{\alpha-1}, 2) \subset \mathfrak{J}(p^\alpha, 2),\\
&\mathfrak{J}(p^{\alpha-1}, \langle\nu\rangle + p - 1) \subset \mathfrak{J}(p^\alpha, \nu) \quad \text{for all } \nu \text{ in } P(p)\setminus\{1, 2, \omega\},\\
&\mathfrak{J}(p^{\alpha-1}, \omega) \subset \mathfrak{J}(p^\alpha, \omega);
\end{aligned}$$

and if $\alpha > 2$ then

$$\mathfrak{J}(p^{\alpha-2}, 2p*) \subset \mathfrak{J}(p^\alpha, 2).$$

It is easy to indicate diagrammatically the lattice in the case $\alpha = 1$ and, say, $p \neq 2$:



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3. External result on $\mathfrak{A}_p\mathfrak{A}_p$

By an external result on a variety $\mathfrak{B}$ we mean a result of the form: A variety which does not contain $\mathfrak{B}\cdots$. For example, a variety which does not contain $\mathfrak{A}$ has finite exponent. For $\mathfrak{A}_p\mathfrak{A}_p$ we can prove the following.

THEOREM 5. *A soluble variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ cannot contain any non-nilpotent p-group and therefore has a bound on the nilpotency class of its p-groups.*

The proof is given in section 5.

One might hope for a stronger result which we state as a problem.

In a variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ is every locally finite p-group nilpotent?

The local finiteness is needed in view of the result of Novikov-Adyan [13] which implies that for all large enough primes p there are infinite finitely generated groups of exponent p. Note also that this result implies the existence of just-non-Cross varieties of exponent p.

A special case of the above problem is the well-known question: is there a bound on the nilpotency class of finite groups of exponent p?

It is perhaps worth recording some consequences of Theorem 5.

COROLLARY 1. *A soluble variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ has a bound on the nilpotency class of nilpotent torsion free groups in it.*

COROLLARY 2. *A soluble variety in which the nilpotent groups do not form a subvariety contains $\mathfrak{A}_p\mathfrak{A}_p$ for some prime p.*

This discharges another debt incurred in [9].

COROLLARY 3. *The variety generated by the two-generator free metabelian-of-exponent-q groups for an infinite set of primes q contains $\mathfrak{A}_p\mathfrak{A}_p$ for some prime p.*

The last statement is in fact valid for all p but this requires additional argument which is not given in this note.

4. Proofs for section 2

Most of the discussion is set in a free group H of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ of countably infinite rank freely generated by $\{a_i : i \in P\}$. Much of the argument will involve the verbal subgroups $\mathfrak{A}_p(H)$, $\mathfrak{B}(\beta)(H)$ and $\mathfrak{N}(\tau, \nu)(H)$; we denote them A_p, $B(\beta)$, $N(\tau, \nu)$ respectively.

We first write down relationships between the subgroups $N(\tau, \nu)$. The first two are obvious:

4.01 *For all τ in $\{0\} \cup P$ and all $\mu \leqq \nu$ in $P(p)$,*

$$N(\tau, \mu) \geqq N(\tau, \nu) \text{ and } \mathfrak{A}_p(N(\tau, \nu)) = N(\tau+1, \nu).$$

Further relations are easy consequences of some well-known results about

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commutators. We record here all such results which are used frequently in what follows. Notation: $[u, v] = u^{-1}v^{-1}uv$, $[u, v, w] = [[u, v], w]$, $[u, 0v] = u$ and $[u, nv] = [u, (n-1)v, v]$ for all n in P. The identity is denoted e.

4.02 *For $h, h_1, h_2, \cdots$ in H and d in A_p,*

$$[h_1, h_2h_3] = [h_1, h_2][h_1, h_3][h_1, h_2, h_3];$$
$$[h_1h_2, h_3] = [h_1, h_3][h_2, h_3][h_1, h_3, h_2];$$
$$[h_1, h_2, h_3] = [h_1, h_3, h_2][h_3, h_2, h_1];$$
$$[d, h_1, \cdots, h_m] = [d, h_{1\pi}, \cdots, h_{m\pi}]$$

for all m in P and all permutations π of $\{1, \cdots, m\}$;

$$[h_1, h_2^m] = \prod_{i=1}^{m} [h_1, ih_2]^{m!/i!(m-i)!} \qquad \text{for all } m \text{ in } P;$$

in particular, since $[d, h^p] = e$ and $[h_1^p, h_2^p] = e$,

$$\prod_{i=1}^{p} [d, ih]^{p!/i!(p-i)!} = e$$

and

$$\prod_{i=1}^{p} \prod_{j=1}^{p} [h_2, ih_1, (j-1)h_2]^{(p!)^2/i!j!(p-i)!(p-j)!} = e.$$

The last two equations have the following immediate consequences.

4.03 *For all τ and all n in P,*

$$N(\tau+1, n+1) \leqq N(\tau, n+p) \text{ and } N(\tau+2, 1) \leqq N(\tau, 2p-1).$$

In fact a little more is true.

4.04 *For all τ and all r in P,*

$$N(\tau+1, pr) \leqq N(\tau, p(r+1)*).$$

Before proving this, we introduce some further notation. For s, n in P with $2 \leqq s < n$, let

$$b(s, s) = [a_s, a_1, a_2, \cdots, a_{s-1}]$$

and

$$b(s, n) = [b(s, n-1), a_n].$$

We denote by ι the identity endomorphism of H, and by $\pi_{i,j}$ with i, j in P the endomorphism which fixes all the generators except a_i, a_j which it interchanges.

PROOF OF 4.04. Let ψ be the endomorphism of H which maps a_j to a_{pr+1} if $pr+1 \leqq j \leqq p(r+1)$ and to a_j otherwise. From

$$\prod_{s=2}^{p(r+1)} (b(s, p(r+1))\psi)^{p^\tau} \in N(\tau, p(r+1)*)$$

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it is easy to derive, using 4.02 and the inclusions

$$N(\tau+1, pr+1) \leqq N(\tau, p(r+1)) \leqq N(\tau, p(r+1)*)$$

(which hold on account of 4.03 and 4.01), that

$$h = \prod_{s=2}^{pr} b(s, pr+1)^{p^{\tau+1}} \in N(\tau, p(r+1)*).$$

Then, applying $\iota - \pi_{2, pr+1}$ to h and using 4.02, one gets

$$[a_2, a_{pr+1}, a_1, a_3, a_4, \cdots, a_{pr}]^{p^{\tau+1}} \in N(\tau, p(r+1)*)$$

and the result follows.

The story is completed by obtaining suitable generating sets for the $N(\tau, \nu)$.

Let $\mathscr{B}$ be the subset of H defined by: $b \in \mathscr{B}$ if and only if $b = [a_i, m_j a_j, m_{j+1} a_{j+1}, \cdots, m_s a_s]$ where $i > j \leqq s$; $m_j - 1, m_{j+1}, \cdots, m_s \in \{0, \cdots, p-1\}$, $m_s \neq 0$; and if $m_j = p$ then firstly $i \leqq s$ implies $m_i < p-1$ and secondly $m_k = 0$ whenever $j < k < i$ and $k \leqq s$.

4.05 *The set* $\mathscr{B} \cup \{a_i^p : i \in P\}$ *is a free generating set for* A_p *as free* $\mathfrak{A}_{p^\alpha}$*-group.*

Proof. It follows easily from 4.02 that A_p is generated by $\mathscr{B}^* = \mathscr{B} \cup \{a_i^p : i \in P\}$. If there were a non-trivial relation between the elements of $\mathscr{B}^*$ this would involve only finitely many of $\{a_i : i \in P\}$. It therefore suffices to consider for each k in P the subgroup H_k of H generated by $\{a_1, \cdots, a_k\}$ and to show that $\mathfrak{A}_p(H_k) \cap \mathscr{B}^*$ is independent in $\mathfrak{A}_p(H_k)$. By the Schreier formula for the rank of subgroups of absolutely free groups, $\mathfrak{A}_p(H_k)$ has rank $(k-1)p^k+1$. On the other hand the number of elements in $\mathfrak{A}_p(H_k) \cap \mathscr{B}^*$ is

$$k + \sum_{j=1}^{k-1} \{(k-j)(p-1)p^{k-j} + \sum_{i=j+1}^{k} (p-1)p^{k-i}\}$$

where the first term in $\{\cdots\}$ comes from counting the commutators with $m_j \neq p$ and the second term from the rest. The sum comes to $(k-1)p^k+1$ and the result follows.

Note that this proof implies that every element of $\mathscr{B}$ can be uniquely written in the way it is defined.

It follows that the commutator subgroup $N(0, 1)$ of H is a free $\mathfrak{A}_{p^\alpha}$-group freely generated by $\mathscr{B}$. The other terms $N(0, n)$ of the lower central series of H are a little more complicated to describe. This we do next after first defining weights for elements of $\mathscr{B}$.

The weight $\mathrm{wt}(b)$ of an element $b = [a_i, m_j a_j, \cdots, m_s a_s]$ of $\mathscr{B}$ is $1 + \sum_{k=j}^{s} m_k$. The weight $\mathrm{wt}(b, a_k)$ of b in the generator a_k is 'the number of occurrences of a_k in b', that is,

$$\operatorname{wt}(b, a_k) = \begin{cases} 0 & \text{if } k \notin \{j, \cdots, s\} \cup \{i\}, \\ m_k & \text{if } k \in \{j, \cdots, s\} \setminus \{i\}, \\ m_k+1 & \text{if } k = i \in \{j, \cdots, s\}, \\ 1 & \text{if } k = i \notin \{j, \cdots, s\}. \end{cases}$$

4.06 *For n in P the subgroup $N(0, n)$ is generated by the set $\mathscr{B}_n$ of elements of the form b^{p^k} where $b \in \mathscr{B}$, $k \in \{0, \cdots, \alpha-1\}$, $\operatorname{wt}(b)+k(p-1) > n$ and $k \operatorname{wt}(b) \neq 2$ unless $n = 1$.*

PROOF. A routine argument from 4.02 shows that $N(\tau, n)$ is generated by the $[a_{i_1}, \cdots, a_{i_m}]^{p^k}$ with $m > n$, $k \in \{\tau, \cdots, \alpha-1\}$ and $i_1 > i_2 \leqq i_3 \leqq \cdots \leqq i_m$. From this and 4.03 one gets immediately that $N(0, n)$ is generated by the set $\mathscr{S}$ of the elements $[a_{i_1}, \cdots, a_{i_m}]^{p^k}$ with $m \geqq 2$, $k \in \{0, \cdots, \alpha-1\}$, $km \neq 2$ unless $n = 1$, $m+(p-1)k > n$ and $i_1 > i_2 \leqq \cdots \leqq i_m$. Clearly $\mathscr{B}_n$ is a subset of $\mathscr{S}$. An induction on m using 4.02 shows that each element of $\mathscr{S}$ lies in the subgroup generated by $\mathscr{B}_n$. The result follows.

Note that if $b^{p^k} \in \mathscr{B}_n$ and $k < \alpha-1$, then $b^{p^{k+1}} \in \mathscr{B}_n$. Thus for all n *every element of $N(0, n)$ can be uniquely written (up to order) in the form $\prod_{i=1}^{t} b_i^{\beta(i)}$ where the b_i are distinct elements of $\mathscr{B}_n$ and the $\beta(i) \in \{1, \cdots, p-1\}$.*

Similar generating sets can be given for the $N(0, pr*)$.

4.07 *For $pr*$ in $P(p)$ the subgroup $N(0, pr*)$ is generated by the union $\mathscr{B}_{pr}$ of*

(i) $\mathscr{B}_{pr}$,

(ii) *the set of elements of the form $b^{p^{k+1}}$ where $b \in \mathscr{B}$, $k \in \{0, \cdots, \alpha-2\}$, $\operatorname{wt}(b)+(k+1)(p-1) = pr$, and $k \operatorname{wt}(b) \neq 2$, and*

(iii) *the set of elements of the form $\prod_{s=2}^{pr} b(s, pr)\psi$ where ψ is an endomorphism of H such that $a_j\psi = a_{i_j}$ where $i_1 \leqq i_2 \leqq \cdots \leqq i_{pr}$ and no p of $i_2, \cdots, i_{pr}$ are equal.*

PROOF. The argument is essentially the same as that in the proof of 4.06. It is routine to derive from 4.02 that $N(\tau, pr*)$ is generated by the $[a_{i_1}, \cdots a_{i_m}]^{p^k}$ with $m \geqq pr+1$, $k \in \{\tau, \cdots, \alpha-1\}$ and $i_1 > i_2 \leqq \cdots \leqq i_m$, and the

$$\prod_{s=2}^{pr} [a_{i_s}, a_{i_1}, a_{i_2}, \cdots, a_{i_{s-1}}, a_{i_{s+1}}, \cdots, a_{i_{pr}}]^{p^\tau} \quad \text{with } i_1 \leqq i_2 \leqq \cdots \leqq i_{pr}.$$

Hence, by 4.03 and 4.04, $N(0, pr*)$ is generated by the set of elements $[a_{i_1}, \cdots, a_{i_m}]^{p^k}$ with $m \geqq 2$, $k \in \{0, \cdots, \alpha-1\}$, $km \neq 2$, $(k-1)m\delta_{2r} \neq 2$, $m+(p-1)k \geqq pr+\delta_{k0}$ (where $\delta_{uv} = 1$ if $u = v$ and 0 if $u \neq v$), and $i_1 > i_2 \leqq i_3 \leqq \cdots \leqq i_m$, and the elements

$$\prod_{s=2}^{pr} [a_{i_s}, a_{i_1}, \cdots, a_{i_{s-1}}, a_{i_{s+1}}, \cdots, a_{i_{pr}}] \quad \text{with } i_1 \leqq i_2 \leqq \cdots \leqq i_{pr}.$$

An induction on m (using some ideas from the proof of 4.04) then yields the result.

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Observe that an element of $\mathscr{B}$ occurs in at most one of the products in (iii) above. It follows that *every element of* $N(0, pr*)$ *can be uniquely written* (*up to order*) *as* $\prod_{i=1}^{t} b_i^{\beta(i)}$ *where the* b_i *are distinct elements of* $\mathscr{B}_{pr*}$ *and each* $\beta(i) \in \{1, \cdots, p-1\}$.

It is a straight-forward consequence of 4.01 and the remarks after the proofs of 4.06 and 4.07 that *the* $N(\tau, \nu)$ *for* τ *in* $\{0, \cdots, \alpha-1\}$ *and* ν *in* $P(p)\setminus\{\omega\}$ *are non-trivial and distinct* and that the following relations between them hold:

4.08 *For all* τ,

$$N(\tau, n) \cap N(\tau+1, 1) = \begin{cases} N(\tau+1, n-p+1) & \text{for } 2p \leqq n \in P, \\ N(\tau+1, n-p+1)N(\tau+2, 1) & \text{for } p < n < 2p, \\ N(\tau+1, 2)N(\tau+2, 1) & \text{for } 2 \leqq n \leqq p; \end{cases}$$

$$N(\tau, pr*) \cap N(\tau+1, 1) = \begin{cases} N(\tau+1, p(r-1)) & \text{for } 1 \neq r \in P, \\ N(\tau+1, 2)N(\tau+2, 1) & \text{for } r = 1; \end{cases}$$

$$\bigcap_{n \in P} N(\tau, n)N(\tau+1, 1) = N(\tau+1, 1).$$

The next step is to prove that every fully invariant subgroup of H can be expressed in terms of the $B(\beta)$ and the $N(\tau, \nu)$.

4.09 *If* V *is a fully invariant subgroup of* H, *then there is a unique element, call it* $\boldsymbol{\beta}(V)$, *in* $\{0, \cdots, \alpha+1\}$ *such that* $V = B(\boldsymbol{\beta}(V))(V \cap N(0, 1))$.

Observe that if $\mathfrak{V}$ is a subvariety of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$, then $\boldsymbol{\beta}(\mathfrak{V})$ (see section 2) is the same as $\boldsymbol{\beta}(\mathfrak{V}(H))$.

Proof of 4.09. Recall that $N(0, 1)$ is the commutator subgroup of H. Clearly there is precisely one β in $\{0, \cdots, \alpha+1\}$ such that $VN(0, 1) = B(\beta)N(0, 1)$. Then $a_1^{p^\beta} = vd$ where $v \in V$, $d \in N(0, 1)$. Applying to this the endomorphism of H which maps a_1 to a_1 and all the other generators to the identity yields $a_1^{p^\beta} \in V$. Thus $B(\beta) \leqq V$ and the result follows.

4.10 *For* τ *in* $\{0, \cdots, \alpha-1\}$, *if* V *is a fully invariant subgroup of* H *contained in* $N(\tau, 1)$, *then there is just one element* $\boldsymbol{\nu}(\tau, V)$ *of* $P(p)$ *such that* $V = N(\tau, \boldsymbol{\nu}(\tau, V))(V \cap N(\tau+1, 1))$.

Observe that if $\mathfrak{V}$ is a subvariety of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ then

$$\boldsymbol{\nu}(\tau, \mathfrak{V}) = \boldsymbol{\nu}(\tau, \mathfrak{V} \vee (\mathfrak{N}(\tau, 1) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p))$$

so that

$$\boldsymbol{\nu}(\tau, \mathfrak{V}) = \boldsymbol{\nu}(\tau, \mathfrak{V}(H) \cap N(\tau, 1)).$$

4.10 is proved in two stages. The first will be stated as a separate result. The endomorphism of H which maps a_j to e and fixes the other generators will be denoted δ_j.

4.11 *Let* $n-1$ *be in* P *and* τ *in* $\{0, \cdots, \alpha-1\}$. *If* p *does not divide* n *or if* $p = n = 2$,

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there is no fully invariant subgroup of H strictly between $N(\tau, n)N(\tau+1, 1)$ and $N(\tau, n-1)N(\tau+1, 1)$. Otherwise $n = pr$ and $N(\tau, pr*)N(\tau+1, 1)$ is the only fully invariant subgroup of H strictly between them.

PROOF. Let V be a fully invariant subgroup of H such that $N(\tau, n)N(\tau+1, 1) < V \leqq N(\tau, n-1)N(\tau+1, 1)$. There are two cases.

(a) If V contains $w = \prod_{i=1}^{t} b_i^{\beta(i)}$ where the b_i are distinct elements of $\mathscr{B}$ (of 4.05) of weight n, each $\beta(i) \in \{p^\tau, 2p^\tau, \cdots, (p-1)p^\tau\}$ and $\mathrm{wt}(b_i, a_j) = p$ for some i and some j, then $V = N(\tau, n-1)N(\tau+1, 1)$.

Clearly it suffices to consider the case $\mathrm{wt}(b_1, a_j) = p$. Put $f(0) = 0$ and $f(k) = f(k-1)+\mathrm{wt}(b_1, a_k)$ and let θ be the endomorphism of H which maps a_k to $a_{f(k-1)+1} \cdots a_{f(k)}$ [to the identity if $f(k-1) = f(k)$]. Using 4.02 gives

$$w\theta \prod_{m=1}^{n} (\iota-\delta_m) = \prod_{s=1}^{p} b(f(j-1)+s, n)^r w'$$

where

$$r = \pm(p-1)! \prod_{k \neq j} \mathrm{wt}(b_1, a_k)!\beta(1),$$

$w' \in N(\tau, n)N(\tau+1, 1)$ and $b(1, n)$ is interpreted to be the identity. Hence V contains $\prod_{s=1}^{p} b(f(j-1)+s, n)^{p^\tau}$ because $p^{\tau+1}$ does not divide r. Applying $\iota-\pi_{1, f(j-1)+2}$ to this and using 4.02 yields that $b(f(j-1)+2, n)^{p^\tau}$ is in V and the result follows.

(b) The only products of the form $\prod_{i=1}^{t} b_i^{\beta(i)}$ where the b_i are distinct elements of $\mathscr{B}$ of weight n and the $\beta(i) \in \{p^\tau, \cdots, (p-1)p^\tau\}$ are such that $\mathrm{wt}(b_i, a_j) < p$ for all i, j.

For k in $P \cup \{0\}$ and m in $\{1, \cdots, p-1\}$ let $\Pi_{k,m}$ be the set of products of the above form in V in which $\mathrm{wt}(b_i, a_k) \leqq m$ for all i [take $\mathrm{wt}(b_i, a_0) = 0$], and for all j exceeding k the $\mathrm{wt}(b_i, a_j)$ are independent of i and equal to 0 or 1. Let $V_{k,m}$ be the fully invariant closure in H of $\Pi_{k,m}$ and $N(\tau, n)N(\tau+1, 1)$. Clearly $V_{k,p-1} \leqq V_{k+1,1}$ and $V_{k,m} \leqq V_{k,m+1}$ for all k and m in $\{1, \cdots, p-2\}$. If $w \in \Pi_{k+1,1}$, then both $w(\iota-\delta_{k+1})$ and $w\delta_{k+1}$ are in $\Pi_{k,p-1}$; hence w is in $V_{k,p-1}$ and $V_{k+1,1} = V_{k,p-1}$. The argument which follows establishes $V_{k,m} = V_{k,m+1}$. Let θ, ψ be the endomorphisms of H defined by:

$$a_j\theta = \begin{cases} a_j & \text{for } j < k, \\ a_k \cdots a_{k+m} & \text{for } j = k, \\ a_{j+m} & \text{for } j > k; \end{cases}$$

$$a_j\psi = \begin{cases} a_j & \text{for } j < k, \\ a_k & \text{for } j \in \{k, \cdots, k+m\}, \\ a_{j-m} & \text{for } j > k+m. \end{cases}$$

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It is easy to verify, using 4.02, that if $w \in \Pi_{k,m+1}$, then $w_1 = w\theta(\iota-\delta_k)\cdots(\iota-\delta_{k+m})$ is in $V_{k,1}$ and $w^{(m+1)!}(w_1\psi)^{-1}$ is in $V_{k,m}$, and hence that w is in $V_{k,m}$. From these equalities it follows that $V = V_{0,p-1}$; that is, V is the fully invariant closure of $N(\tau, n)N(\tau+1, 1)$ and the products of the form $\prod_{s=2}^{n} b(s, n)^{\beta(s)}$ (with $\beta(s) \in \{0, p^\tau, \cdots, (p-1)p^\tau\}$) which lie in it. If $w = \prod_{s=2}^{n} b(s, n)^{\beta(s)} \in V$, then $w(\iota-\pi_{s,t}) \in V$ for all s, t in $\{2, \cdots, n\}$. But $w(\iota-\pi_{s,t}) = [a_s, a_t, a_1, \cdots]^{\beta(s)-\beta(t)}$ by 4.02, so $V = N(\tau, n-1)N(\tau+1, 1)$ or $\beta(s) = \beta(t)$ for all s, t and all relevant w. In the latter case V is the fully invariant closure of $N(\tau, n)N(\tau+1, 1)$ and $x = \prod_{s=2}^{n} b(s, n)^{p^\tau}$. If $n = 2$, then $V = N(\tau, n-1)N(\tau+1, 1)$. If $n \neq 2$ and p divides n, then $V = N(\tau, n*)N(\tau+1, 1)$. If p does not divide n, then $x(\iota-\pi_{1,2}) = b(2, n)^{np^\tau} \in V$ and so $V = N(\tau, n-1)N(\tau+1, 1)$.

Proof of 4.10. It follows from 4.06 and 4.07 that if $\mu \neq \nu$ in $P(p)$, then $N(\tau, \nu)N(\tau+1, 1) \neq N(\tau, \mu)N(\tau+1, 1)$. Thus there is at most one ν in $P(p)$ such that $V = N(\tau, \nu)(V \cap N(\tau+1, 1))$. If $V \leqq N(\tau+1, 1)$, put $\nu(\tau, V) = \omega$. If $V \nleqq N(\tau+1, 1)$, then by 4.08 there is an n in P such that $V \leqq N(\tau, n-1)N(\tau+1, 1)$ but $V \nleqq N(\tau, n)N(\tau+1, 1)$, and it follows from 4.11 that $VN(\tau, n)N(\tau+1, 1)$ is either (a) $N(\tau, n-1)N(\tau+1, 1)$ or (b) $N(\tau, n*)N(\tau+1, 1)$.

Case (a): This implies $N(\tau, n-1) \leqq VN(\tau, n)N(\tau+1, 1)$. It follows that $N(\tau, m-1) \leqq VN(\tau, m)N(\tau+1, 1)$ for all m in P with $m \geqq n$. Hence $N(\tau, n-1) \leqq VN(\tau, pn)N(\tau+1, 1)$. Therefore $b(2, n)^{p^\tau} = v \prod_{i=1}^{t} b_i^{\beta(i)}$ where $v \in V$, the b_i are distinct elements of $\mathscr{B}$, p^τ divides each $\beta(i)$, and for each i either $\mathrm{wt}(b_i) > pn$ or $p^{\tau+1}$ divides $\beta(i)$. By a standard argument (applying in turn the mappings $\iota-\delta_1$, $\iota-\delta_2, \cdots$) it can be assumed that, for all i, $\mathrm{wt}(b_i, a_j) \geqq 1$ for $j \leqq n$ and $\mathrm{wt}(b_i, a_j) = 0$ for $j > n$. Hence $\mathrm{wt}(b_i) \geqq n$ and $p^{\tau+1}$ divides $\beta(i)$ for all i, because no element b of $\mathscr{B}$ satisfies $\mathrm{wt}(b) > pn$ and $\mathrm{wt}(b, a_j) = 0$ for all $j > n$. Thus $b(2, n)^{p^\tau} \in VN(\tau+1, n-1)$ and so $N(\tau, n-1) \leqq VN(\tau+1, n-1)$. It follows that $N(\rho, n-1) \leqq VN(\rho+1, n-1)$ for all $\rho \geqq \tau$. Therefore $N(\tau, n-1) \leqq V$. But $V \leqq N(\tau, n-1)N(\tau+1, 1)$ and so the result follows with $\nu(\tau, V) = n-1$.

Case (b): Now $n = pr \geqq 3$ and $N(\tau, pr*) \leqq VN(\tau, pr)N(\tau+1, 1)$. By 4.02, $[\prod_{s=2}^{pr} b(s, pr), a_{pr+1}](\iota-\pi_{2,pr+1}) = [a_2, a_{pr+1}, a_1, \cdots, a_{pr}]$. Hence $N(\tau, pr) \leqq VN(\tau, pr+1)N(\tau+1, 1)$ and so $N(\tau, pr*) \leqq VN(\tau, p^2r)N(\tau+1, 1)$. Therefore arguing as in (a) we obtain that

$$\prod_{s=2}^{pr} b(s, pr)^{p^\tau} = v \prod_{s=2}^{pr} b(s, pr)^{\mu(s)} \prod_{i=1}^{t} b_i^{\beta(i)}$$

where $v \in V$, the b_i are elements of $\mathscr{B}$ of weight at least $pr+1$ and $p^{\tau+1}$ divides each $\mu(s)$ and $\beta(i)$. Let π denote the automorphism of H which maps a_i to a_{i+1} if $2 \leqq i \leqq pr$, a_{pr} to a_2, and fixes all other generators. Apply the mapping $\sum_{m=0}^{pr-2} \pi^m$ to the last displayed relation above: since $pr-1$ is prime to p, it follows that $\prod_{s=2}^{pr} b(s, pr)^{p^\tau} \in VN(\tau+1, pr*)$. Hence $N(\tau, pr*) \leqq VN(\tau+1, pr*)$. Then arguing as in (a) shows that the result holds with $\nu(\tau, V) = pr*$.

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PROOF OF 2.1. Let $\mathfrak{V}$ be a subvariety of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. By 4.09 and repeated applications of 4.10,

$$\mathfrak{V}(H) = B(\boldsymbol{\beta}(\mathfrak{V}(H)))\prod_{\tau=0}^{\alpha-1} N(\tau, \boldsymbol{\nu}(\tau, \mathfrak{V}(H) \cap N(\tau, 1))).$$

It follows from the observations after 4.09 and 4.10 that

$$\mathfrak{V}(H) = B(\boldsymbol{\beta}(\mathfrak{V}))\prod_{\tau=0}^{\alpha-1} N(\tau, \boldsymbol{\nu}(\tau, \mathfrak{V})).$$

Going over to varieties gives the result.

PROOF OF 2.2. It follows from the argument in the proof of 2.1 that

$$(\mathfrak{U} \wedge \mathfrak{V})(H) = \mathfrak{U}(H)\mathfrak{V}(H) = B(\boldsymbol{\beta}(\mathfrak{U}))B(\boldsymbol{\beta}(\mathfrak{V}))\prod_{\tau=0}^{\alpha-1} N(\tau, \boldsymbol{\nu}(\tau, \mathfrak{U}))N(\tau, \boldsymbol{\nu}(\tau, \mathfrak{V})).$$

Since the $B(\ )$'s and the $N(\tau,\ )$'s are linearly ordered,

$$(\mathfrak{U} \wedge \mathfrak{V})(H) = B(\min\{\boldsymbol{\beta}(\mathfrak{U}), \boldsymbol{\beta}(\mathfrak{V})\})\prod_{\tau=1}^{\alpha-1} N(\tau, \min\{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}).$$

It follows from 4.09 that

$$\boldsymbol{\beta}(\mathfrak{U} \wedge \mathfrak{V}) = \min\{\boldsymbol{\beta}(\mathfrak{U}), \boldsymbol{\beta}(\mathfrak{V})\}$$

and

$$(\mathfrak{U} \wedge \mathfrak{V})(H) \cap N(0, 1) = \prod_{\tau=0}^{\alpha-1} N(\tau, \min\{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}).$$

An induction on ρ, using 4.10, yields

$$\boldsymbol{\nu}(\rho, (\mathfrak{U} \wedge \mathfrak{V})(H) \cap N(\rho, 1)) = \min\{\boldsymbol{\nu}(\rho, \mathfrak{U}), \boldsymbol{\nu}(\rho, \mathfrak{V})\}$$

and

$$(\mathfrak{U} \wedge \mathfrak{V})(H) \cap N(\rho+1) = \prod_{\tau=\rho+1}^{\alpha-1} N(\tau, \min\{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}).$$

Hence, by the remark after 4.10,

$$\boldsymbol{\nu}(\tau, \mathfrak{U} \wedge \mathfrak{V}) = \min\{\boldsymbol{\nu}(\tau, \mathfrak{U}), \boldsymbol{\nu}(\tau, \mathfrak{V})\}$$

as required.

Before proving 2.3 we need one more result.

4.12 *For* $\beta \geqq 1$

$$B(\beta) \cap N(0, 1) = \begin{cases} N(\beta-1, 1) & \text{for } p = 2, \\ N(\beta-1, p*)N(\beta, 1) & \text{for } p \text{ odd}. \end{cases}$$

PROOF. The result is an easy consequence of the case $\beta = 1$ so we only prove that. For $p = 2$ this is an immediate consequence of the well-known fact that all

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groups of exponent 2 are abelian. Let p be an odd prime. Since $N(0, 1) \geqq B(1) \cap N(0, 1) \geqq N(1, 1)$, it follows from 4.09 that there is a v in $P(p)$ such that $B(1) \cap N(0, 1) = N(0, v)N(1, 1)$. There are metabelian groups of exponent p and class precisely p (see [11] Satz 3 or [4] Example 3.2), so $v > p-1$. By 18.4.13 of [6], $[a_2, (p-1)a_1] \in (B(1) \cap N(0, 1))N(0, p)$. By 4.06, $[a_2, (p-1)a_1] \notin N(0, p)N(1, 1)$, so $v < p$. Thus $v = p*$ and the result follows.

PROOF OF 2.3. Clearly the set Σ is a sublattice of the direct product lattice Λ. It is a straight-forward matter to calculate using 4.08 and 4.12 that

$$\boldsymbol{\beta}(\mathfrak{B}(\beta) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p) = \min\{\beta, \alpha+1\},$$

$$\boldsymbol{v}(\tau, \mathfrak{B}(\beta) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p) = \begin{cases} \omega & \text{for } \tau < \beta-1, \\ p* & \text{for } \tau = \beta-1 \text{ and } p \text{ odd}, \\ 1 & \text{for } \tau = \beta-1 \text{ and } p = 2, \\ 1 & \text{for } \tau \geqq \beta; \end{cases}$$

and $\boldsymbol{\beta}(\mathfrak{N}(\tau, v) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p) = \alpha+1$,

$$\boldsymbol{v}(\rho, \mathfrak{N}(\tau, v) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p) = \begin{cases} \omega & \text{for } \rho < \tau, \\ v & \text{for } \rho = \tau, \\ \boldsymbol{v}(\rho, \mathfrak{N}(\tau+1, \bar{v}) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p) & \text{for } \rho > \tau \text{ and } v > 2p-1, \\ \bar{v} & \text{for } \rho = \tau+1 \text{ and } v \leqq 2p-1, \\ 1 & \text{for } \rho > \tau+1 \text{ and } v \leqq 2p-1, \end{cases}$$

where
$$\bar{v} = \begin{cases} v & \text{for } v \in \{1, \omega\} \\ v-p+1 & \text{for } v \in P \text{ and } v > p, \\ pr & \text{for } v = p(r+1)*, \\ 2 & \text{for } 2 \leqq v \leqq p. \end{cases}$$

Hence $(\mathfrak{B}(\beta) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p)\chi$ and $(\mathfrak{N}(\tau, v) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p)\chi$ belong to Σ and so the image of χ lies in Σ. Moreover it follows that if $(\beta, v_0, \cdots, v_{\alpha-1}) \in \Sigma$, then

$$(\mathfrak{B}(\beta) \wedge \bigwedge_{\tau=0}^{\alpha-1} \mathfrak{N}(\tau, v_\tau) \wedge \mathfrak{A}_{p^\alpha}\mathfrak{A}_p)\chi = (\beta, v_0, \cdots, v_{\alpha-1}).$$

5. Proof of Theorems 2 and 5

PROOF OF THEOREM 5. Since a group G is nilpotent if it has a nilpotent normal subgroup N such that $G/\mathfrak{A}(N)$ is nilpotent (P. Hall [7] Theorem 7), it suffices to prove the theorem for metabelian varieties. Let $\mathfrak{B}$ be a metabelian variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$; then there is a positive integer c such that $\mathfrak{B} \wedge \mathfrak{A}_p\mathfrak{A}_p \subsetneqq \mathfrak{N}_{c-1}$. We show that every p-group in $\mathfrak{B}$ lies in $\mathfrak{N}_{c-1}$. Suppose not; then there

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would be a finitely generated, and therefore finite, p-group in $\mathfrak{B}\setminus\mathfrak{N}_{c-1}$. Since all finite p-groups are nilpotent, it would follow that there is a p-group in $(\mathfrak{B}\wedge\mathfrak{N}_c)\setminus\mathfrak{N}_{c-1}$. The result is therefore a consequence of the following more precise lemma.

LEMMA. *If $\mathfrak{B}$ is a metabelian variety such that $\mathfrak{B}\wedge\mathfrak{A}_p\mathfrak{A}_p\subseteq\mathfrak{N}_\nu$ for some ν in $P(p)$, then for each μ in $P(p)\setminus\{\omega\}$ with $\mu>\nu$ there is a positive integer k not divisible by p such that $\mathfrak{B}\wedge\mathfrak{N}_\mu\subseteq\mathfrak{A}_k\mathfrak{N}_\nu$.*

PROOF. There is nothing to prove if $\nu=\omega$. If $\nu\neq\omega$, then it clearly suffices to prove the result when μ is the first positive integer exceeding ν – call it c. Let G be a free group of $\mathfrak{A}\mathfrak{A}\wedge\mathfrak{N}_c$ freely generated by $\{g_1,\cdots,g_c\}$, let $V=\mathfrak{B}(G)$ and $K=\mathfrak{N}_\nu(G)$. We will show there is an element y of K such that $y^p w\in V$ where $w=[g_1,\cdots,g_c]$ if $\nu=c-1$ and $w=\prod_{s=2}^{c}[g_s,g_1,\cdots,g_{s-1},g_{s+1},\cdots,g_c]$ if $\nu=c*$. Since K is finitely generated abelian and the fully invariant closure of w, it will follow that KV/V is a finitely generated abelian group in which every element has a p-th root; hence that KV/V is a finite abelian group of order k not divisible by p; and therefore $\mathfrak{B}\wedge\mathfrak{N}_c\subseteq\mathfrak{A}_k\mathfrak{N}_\nu$ as required. Since $\mathfrak{B}\wedge\mathfrak{A}_p\mathfrak{A}_p\subseteq\mathfrak{N}_\nu$ it follows that $K\leqq VD$ where $D=\mathfrak{A}_p\mathfrak{A}_p(G)$ and hence $w=v_0d_0$ with $v_0\in V$, $d_0\in D$. For each i in $\{1,\cdots,c\}$ let ε_i be the endomorphism of G which maps g_j to g_j for $j\neq i$ and g_i to e. We now define $v_1,\cdots,v_c\in V$ and $d_1,\cdots,d_c\in D$ by $v_i=v_{i-1}(v_{i-1}\varepsilon_i)^{-1}$ and $d_i=(d_{i-1}\varepsilon_i)^{-1}d_{i-1}$. It is easy to check for all i that $(v_{i-1}\varepsilon_i)(d_{i-1}\varepsilon_i)=e$, $w=v_id_i$ and $d_i\varepsilon_j=e$ for all $j\leqq i$. It follows ([12] 36.32) that d_c can be uniquely written in the form $\prod_{s=2}^{c}[g_s,g_1,\cdots,g_{s-1},g_{s+1},\cdots,g_c]^{\beta(s)}$. Let H be the free group of $\mathfrak{A}_p\mathfrak{A}_p$ defined in section 4. Let θ be the homomorphism of G into $H/N(0,c)$ defined by $g_i\theta=a_iN(0,c)$ for all i in $\{1,\cdots,c\}$. Then, as $D\theta=\{N(0,c)\}$,

$$\prod_{s=2}^{c}b(s,c)^{\beta(s)}N(0,c)=d_c\theta=N(0,c),$$

and so p divides $\beta(s)$ for all s. Therefore d_c has a p-th root b in $\mathfrak{N}_{c-1}(G)$ and $w=v_cb^p$. If $c=2$ or p does not divide c, then $\mathfrak{N}_{c-1}(G)=K$ and the proof is complete. Let π denote the automorphism of G which maps g_1 to g_1, g_i to g_{i+1} when $1<i<c$, and g_c to g_2; put $\psi=\sum_{m=0}^{c-2}\pi^m$. If p divides c and $c\geqq 3$, then applying ψ (cf. the last paragraph of the proof of 4.10) yields $w^{c-1}=v_c\psi(b\psi)^p$ and $b\psi\in K$. The result follows because p does not divide $c-1$.

PROOF OF THEOREM 2. Let $\mathfrak{U}$ be a nilpotent-by-abelian just-non-Cross variety. If $\mathfrak{A}\subseteq\mathfrak{U}$, then $\mathfrak{U}=\mathfrak{A}$. If $\mathfrak{A}\nsubseteq\mathfrak{U}$, then $\mathfrak{U}$ has finite exponent, t say. Hence $\mathfrak{U}$ is generated by its finite groups. We will show that there is a bound, t^t, on the order of chief factors of finite groups in $\mathfrak{U}$. By the Corollary in [8] applied to the class of finite groups in $\mathfrak{U}$, there is no bound on the class of finite nilpotent groups in $\mathfrak{U}$ and the result follows from Corollary 2 of Theorem 5. Let H/K be a chief factor of a finite group G in $\mathfrak{U}$. Clearly H/K is an elementary abelian p-group for some p dividing t. Let C be the centralizer of H/K in G; then G/C is an abelian group which has a faithful irreducible representation over the field of p elements. Hence G/C is cyclic and so has order dividing t. Therefore the order of H/K is at most t^t as required.

REMARK (added in proof, 9 December, 1970). The problem stated in section 2 has a negative solution on account of the results of Bachmuth, Mochizuki, and Walkup ['A nonsolvable group of exponent 5', *Bull. Amer. Math. Soc.* 76 (1970), 638–640] and O. Yu. Razmuslov [to appear]: for all primes $p \geqq 5$, there exist nonnilpotent locally finite varieties of exponent p. Our Theorem 2 has been superseded by results of J. M. Brady ['On the classification of just-non-Cross varieties of groups', *Bull. Austr. Math. Soc.* 3 (1970), 293–311; 'On soluble just-non-Cross varieties of groups', *ibid.* 313–323] and O. Yu. Ol'shanskij [to appear].

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