CORRESPONDENCE.

To the Editor of the Transactions of the Faculty of Actuaries.

SIR,

THEORETICAL BASIS OF WHITTAKER'S METHOD OF GRADUATION.

In opening the discussion (*T.F.A.*, vol. xi., p. 17) on Messrs. Davidson and Reid's Paper on "*A New Type of Summation* "*Graduation Formulæ*," I made the following remarks:—

> "The basis of the [*i.e.* Whittaker's] method in the theory of "probability, attractive as it is, is not altogether without "difficulty, and to some minds it may appear a little "artificial. It is not perhaps easy to see how a method can "be so based when it involves a constant ϵ which is assumed "to be known *a priori*, but which is in fact neither so known "nor determinable *a posteriori*. Then it does not seem true "without limitation that [*a priori*]* over the whole range of "a long series of values, such as we commonly have to deal "with, the nearer the whole series lies to a second difference "curve, with one uniform set of constants, the closer it is to "the truth, though that seems to be involved in the mathe-"matical expression of Whittaker's 'Hypothesis H.'"

In the course of the discussion, Professor Whittaker referred to these remarks and said:—

> "There is, however, one sentence which struck me, a "possible misapprehension which I should like to clear up. "The method of graduation under discussion does not "consist in taking a parabola to pass as simply as possible "through the ungraduated points. It is not a curve-fitting "method in that sense. We do not take a parabola and try "to fit the observations to it—what we say is that every "little bit of the graduated curve is to be nearly a bit of a "parabola, in fact we have really a continually varying

* These words do not appear in the Report, but were I think spoken and were certainly intended.

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"parabola, not the same one the whole way along. . . .
"You can see that a small but continuous break-away from
"a parabola can give rise to a very large break-away at
"more remote points."

Though these interesting remarks throw a good deal of light on the working of Professor Whittaker's method, I was not in fact, as he assumes, under the misapprehension to which he referred, and indeed in the latter part of my remarks (top of p. 17) I endeavoured to differentiate between his method and that of curve-fitting. My difficulty with regard to the basis of the method in the Theory of Probability therefore remains, and as I believe it is shared by some others it may be desirable to explain it a little more fully.

Let us in the first place consider the parallel case of the Method of Least Squares. In this case we have (1) a hypothesis, namely, that *a priori* the chance of an error of magnitude x is e^{-cx^2}; and (2) an undetermined constant c, involved in that hypothesis. The hypothesis is based (on certain assumptions) on the theory of probability, and its adoption is justified by experience. The constant c is shown to be determinable from the data to which the Method of Least Squares is to be applied.

In the case of Whittaker's Graduation Method, his underlying hypothesis may be shortly stated as follows. The *a priori* chance that the true values (which should have been yielded by the observations) are $u'_1, u'_2 \ . \ . \ . \ u'_n$ is proportional to $e^{-\lambda^2 S}$; where $S \equiv \sum_{t=1}^{n-3} (\Delta^3 u'_t)$, *i.e.* the sum of the squares of the 3rd differences of the u''s. Now this chance is greatest when $S=0$, *i.e.* when *each* 3rd difference vanishes. Thus the hypothesis involves the assumption that *a priori* the *most probable* set of u''s is a set all lying on *a single curve* of the 2nd degree; and as applied to a long series of values such as rates of mortality this assumption is simply not true, but rather one contradicted by general previous experience. (It does not appear that the assumption would be justified even if the order of differences involved in S were increased; for it is very doubtful whether rates of mortality or similar functions can be represented *over a long range* by any single parabolic curve of any reasonable order.) Further, the theory gives no means of determining the value of λ which leads to that of ϵ (equal to h^2/λ^2, which we may note is essentially positive). Thus the method starts with a hypothesis which is not in accordance with experience, and ends with a constant which is not determinable from the data.

It is of course true that the practical formula based on the hypothesis is found in the result to be very similar to some particular formula of the Summation Method, which itself gives results that sufficiently justify Professor Whittaker's description "that every little bit of the graduated curve is to be nearly "a bit of a parabola, in fact we have really a continually varying "parabola, not the same one the whole way along." Thus, in fact, the hypothesis does not lead us astray, but on the contrary leads to a new and brilliant method of graduation. But this does not in itself justify the hypothesis as such; and it still seems to me doubtful how far the method can be regarded as based on the Theory of Probability when it involves a hypothesis that is not true, and a constant that we have no practical means of determining. It is for this reason that it appears to me to be better (as it is simpler) to reach the expression $S+\epsilon F$ by more general and less theoretical considerations, as indicated in my remarks previously referred to. After this the whole of the work of Professor Whittaker and Dr. Aitken proceeds as before, and that being so it may be considered that the question I have raised is purely academic; but it seems important that the real philosophical basis of a new and important method should be clearly understood.

I am, Sir,

Your obedient Servant,

G. J. Lidstone.

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Edinburgh.

To the Editor of the Transactions of the Faculty of Actuaries.

Sir,

Let me thank you for allowing me to see Mr. Lidstone's letter.

If I may be allowed a few words of comment, it seems to me that he has introduced the term *most probable* without having defined the meaning of these words, and in a context where it is