ON UNITARY EQUIVALENCE OF MATRICES OVER THE RING OF CONTINUOUS COMPLEX-VALUED FUNCTIONS ON A STONIAN SPACE

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1. Introduction. This paper is a continuation of the earlier papers **(1, 5)** in which the author studied matrices with entries from the algebra $C(\mathfrak{X})$ of all continuous, complex-valued functions on an extremely disconnected, compact Hausdorff space $\mathfrak{X}$. (Such spaces are sometimes called Stonian, after M. H. Stone, who first considered them in **(8)**. They arise naturally as maximal ideal spaces of abelian W^*-algebras.) In this note, three theorems are proved. The first is that abelian *-subalgebras of the algebra $M_n(\mathfrak{X})$ of all $n \times n$ matrices over $C(\mathfrak{X})$ can be unitarily diagonalized. This result is then used to obtain in Theorem 2 a necessary and sufficient condition that a *-isomorphism between two W^*-subalgebras (AW^*-subalgebras) of a finite W^*-algebra (AW^*-algebra) of type I be implemented by a unitary element in the larger algebra. This can be regarded as a generalization for finite algebras of **(4**, Theorem 3**)**, and focuses attention on the question of whether the same theorem can be proved in W^*-algebras of type II_1. Finally, using Theorem 2, we prove that if A and B are matrices over $C(\mathfrak{X})$ and $A(t)$ is unitarily equivalent to $B(t)$ for each $t \in \mathfrak{X}$, then A and B are unitarily equivalent in the algebra $M_n(\mathfrak{X})$. This generalizes **(5**, Theorem 3**)** and enables us to give a "local" complete set of unitary invariants for certain operators on Hilbert space.

2. We denote by M_n the full ring of $n \times n$ complex matrices under the operator norm. Let $\mathfrak{X}$ be any Stonian space, and denote by $M_n(\mathfrak{X})$ the *-algebra of continuous functions from $\mathfrak{X}$ to M_n, where the algebraic operations in $M_n(\mathfrak{X})$ are defined pointwise. If one sets

$$||A|| = \sup_{t \in \mathfrak{X}} ||A(t)||$$

for $A \in M_n(\mathfrak{X})$, then $M_n(\mathfrak{X})$ becomes a C^*-algebra (identifiable with the C^*-algebra of all $n \times n$ matrices with entries from $C(\mathfrak{X})$), and in fact, an n-homogeneous AW^*-algebra **(4)**. We begin our programme with some structure theory in $M_n(\mathfrak{X})$. The reader is referred to **(4)** for the definition of an AW^*-subalgebra of $M_n(\mathfrak{X})$. A subalgebra $\mathbf{A}$ of $M_n(\mathfrak{X})$ is said to be diagonal if for each $A \in \mathbf{A}$ and each $t \in \mathfrak{X}$, the matrix $A(t)$ is diagonal.

THEOREM 1. *If* $\mathbf{A}$ *is any abelian *-subalgebra of* $M_n(\mathfrak{X})$, *then there is a unitary element* $U \in M_n(\mathfrak{X})$ *such that the algebra* $U\mathbf{A}U^*$ *is a diagonal subalgebra.*

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Proof. It suffices to prove the above in the case that **A** is an AW^*-subalgebra, since, in any event, the AW^*-subalgebra generated by **A** (the intersection of all AW^*-subalgebras containing **A**) will be abelian. (This can be gleaned from (**4**, Lemma 4).) Since linear combinations of the projections of an AW^*-subalgebra **A** are dense in **A**, it is clear that it suffices to find a unitary element $U \in M_n(\mathfrak{X})$ such that for every projection $E \in \mathbf{A}$, UEU^* is diagonal. To accomplish this, we consider collections $\{\mathfrak{U}_i\}$ of disjoint, non-empty, compact open sets $\mathfrak{U}_i \subset \mathfrak{X}$ such that if $\mathfrak{U}_i \in \{\mathfrak{U}_i\}$, then there is a unitary-valued function $U_i \in M_n(\mathfrak{U}_i)$ such that $U_i(t)E(t)U_i^*(t)$ is diagonal for each $t \in \mathfrak{U}_i$ and each projection $E \in \mathbf{A}$. Choose a maximal collection of this type $\{\mathfrak{U}_i\}_{i \in I}$, and let

$$\mathfrak{U} = \overline{\bigcup_{i \in I} \mathfrak{U}_i}.$$

In view of (**1**, Lemma 2.1), it suffices to prove $\mathfrak{U} = \mathfrak{X}$ to complete the argument. Thus, suppose $\mathfrak{X} - \mathfrak{U} \neq \emptyset$, and consider collections $\{E_j\}$ of projections in **A** with the property that at some point $t \in \mathfrak{X} - \mathfrak{U}$, the projections $\{E_j(t)\}$ are all distinct. Clearly there is at least one non-void collection of this type, and clearly any collection of this type can contain at most 2^n projections. Choose a collection $\{E_j\}_{j \in J}$ having a maximum number of elements. Then if $t_0 \in \mathfrak{X} - \mathfrak{U}$ is such that the projections $\{E_j(t_0)\}_{j \in J}$ are all distinct, it is clear that there is a compact open neighbourhood $\mathfrak{N} \subset \mathfrak{X} - \mathfrak{U}$ of t_0 such that for $t \in \mathfrak{N}$, the projections $\{E_j(t)\}_{j \in J}$ remain distinct. It follows from the maximality of the collection $\{E_j\}_{j \in J}$ that if E is any projection in **A** and $t \in \mathfrak{N}$, then $E(t)$ is some one of the projections $E_j(t)$. (Of course j can vary with t.) Thus to obtain a contradiction, it suffices to find some non-empty compact open subset $\mathfrak{M} \subset \mathfrak{N}$ and a unitary-valued function $V \in M_n(\mathfrak{M})$ which will simultaneously diagonialize the $\{E_j\}_{j \in J}$ on $\mathfrak{M}$. We do this as follows. For convenience, take J to be the collection of integers $\{1, 2, \ldots, k\}$. By applying (**1**, Corollary 3.3) to E_1 and changing notation, we can assume that E_1 is diagonal on $\mathfrak{N}$. Next choose a point $t_1 \in \mathfrak{N}$ where the rank of $E_1(t)$ is a maximum, and then choose a compact open neighbourhood $\mathfrak{P} \subset \mathfrak{N}$ of t_1 such that E_1 is constant on $\mathfrak{P}$. We can clearly assume that

$$E_1(t) = \begin{bmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & 0 & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \cdot & \\ & 0 & & & & & \cdot \\ & & & & & & 0 \end{bmatrix} \quad \text{for } t \in \mathfrak{P}.$$

Since E_2 commutes with E_1, it must be the case that, for $t \in \mathfrak{P}$, the matrix $E_2(t)$ has the form

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$$E_2(t) = \left[\begin{array}{c|c} G_1(t) & 0 \\ \hline 0 & G_2(t) \end{array}\right]$$

where G_1 and G_2 are projection-valued at each $t \in \mathfrak{P}$. Application of **(1,** Corollary 3.3**)** to G_1 and G_2 yields a unitary element $W \in M_n(\mathfrak{P})$ of the form

$$W(t) = \left[\begin{array}{c|c} W_1(t) & 0 \\ \hline 0 & W_2(t) \end{array}\right]$$

such that on $\mathfrak{P}$, WE_2W^* is diagonal. Since W commutes with E_1 on $\mathfrak{P}$, we have simultaneously diagonalized E_1 and E_2 on $\mathfrak{P}$, and the proof is completed by making an induction argument along the lines indicated above. We omit further details of the induction argument.

Notation. We denote by $\sigma(A)$ the trace in the usual sense of an $n \times n$ complex matrix A.

LEMMA 2.1. *Suppose that* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are abelian* AW^**-subalgebras of* $M_n(\mathfrak{X})$, *and that* ϕ *is an algebraic *-isomorphism of* $\mathbf{A}_1$ *onto* $\mathbf{A}_2$ *with the property that for each* $A \in \mathbf{A}_1$ *and each* $t \in \mathfrak{X}$, $\sigma[A(t)] = \sigma[\phi(A)(t)]$. *Then there is a unitary element* $U \in M_n(\mathfrak{X})$ *such that* $\phi(A) = UAU^*$ *for each* $A \in \mathbf{A}_1$*; i.e.,* ϕ *is implemented by* U.

Proof. Since ϕ is trace-preserving, it follows easily that if $A \in \mathbf{A}_1$ and $t \in \mathfrak{X}$, then

$$||A(t)||^2 = ||A^*(t)A(t)|| = ||\phi(A^*)(t)\phi(A)(t)|| = ||\phi(A)(t)||^2,$$

so that ϕ is actually norm-preserving also. For $t \in \mathfrak{X}$, let $\mathbf{A}_1(t)$ be the *-algebra of all matrices $A(t)$ where $A \in \mathbf{A}_1$, and let $\mathbf{A}_2(t)$ be defined similarly. It follows from the fact that ϕ is norm-preserving that for each $t \in \mathfrak{X}$, ϕ gives rise to a *-isomorphism $\tilde{\phi}_t$ of $\mathbf{A}_1(t)$ onto $\mathbf{A}_2(t)$ defined by $\tilde{\phi}_t: A(t) \to \phi(A)(t)$. These properties of ϕ are used several times in the course of the proof. Now consider collections $\{\mathfrak{U}_i\}$ of disjoint, non-empty, compact open subsets $\mathfrak{U}_i \subset \mathfrak{X}$ such that if $\mathfrak{U}_i \in \{\mathfrak{U}_i\}$, then there is a unitary-valued element $U_i \in M_n(\mathfrak{U}_i)$ such that for each $t \in \mathfrak{U}_i$ and each $A \in \mathbf{A}_1$, $\phi(A)(t) = U_i(t)A(t)U_i^*(t)$. Choose a maximal collection $\{\mathfrak{U}_i\}_{i\in I}$, and let

$$\mathfrak{U} = \overline{\bigcup_{i\in I} \mathfrak{U}_i}.$$

As before, it suffices to prove that $\mathfrak{U} = \mathfrak{X}$, so we suppose that $\mathfrak{X} - \mathfrak{U} \neq \emptyset$. Since ϕ is norm-preserving, and since the linear combinations of the projections in an AW^*-subalgebra are dense in the subalgebra, it is easy to see that to obtain a contradiction, it suffices to find a non-empty, compact open subset $\mathfrak{M} \subset \mathfrak{X} - \mathfrak{U}$ and a unitary-valued element $V \in M_n(\mathfrak{M})$ such that for each projection $E \in \mathbf{A}_1$ and for each $t \in \mathfrak{M}$, $\phi(E)(t) = V(t)E(t)V^*(t)$. We obtain such an $\mathfrak{M}$ and V as follows. By virtue of Theorem 1 we can assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are both diagonal subalgebras. We now choose a non-empty collection $\{E_j\}_{j\in J}$ of projections in $\mathbf{A}_1$, a point $t_0 \in \mathfrak{X} - \mathfrak{U}$, and a compact open neighbourhood $\mathfrak{N} \subset \mathfrak{X} - \mathfrak{U}$ of t_0 just as in the proof of Theorem 1; i.e., so that for $t \in \mathfrak{N}$, the projections $\{E_j(t)\}$ are all distinct, and furthermore if E is any projection in $\mathbf{A}_1$ and $t \in \mathfrak{N}$, then $E(t)$ is some one of the projections $\{E_j(t)\}_{j\in J}$. Just as before, we can drop down to a non-empty, compact open subset $\mathfrak{P}_1 \subset \mathfrak{N}$ such that on $\mathfrak{P}_1$ the projection E_{j_1} is constant, and by an obvious induction argument, we can eventually obtain a non-empty, compact open set $\mathfrak{P} \subset \mathfrak{P}_1 \subset \mathfrak{N}$ such that on $\mathfrak{P}$ the projections $\{E_j\}_{j\in J}$ are all constant. Going one step further and making a similar induction argument on the $\{\phi(E_j)\}_{j\in J}$, we can drop down to a non-empty, compact open subset $\mathfrak{M} \subset \mathfrak{P}$ such that the projections $\{\phi(E_j)\}_{j\in J}$ are also all constant on $\mathfrak{M}$. Note that to obtain a contradiction, it now suffices to find a unitary element $V \in M_n(\mathfrak{M})$ satisfying $\phi(E_j)(t) = V(t)E_j(t)V^*(t)$ for each $j \in J$ and $t \in \mathfrak{M}$, because then if E is any projection in $\mathbf{A}_1$ and $t \in \mathfrak{M}$, we have from the above that $E(t)$ is some $E_j(t)$, and thus $\phi(E)(t) = \phi(E_j)(t) = V(t)E_j(t)V^*(t) = V(t)E(t)V^*(t)$. To obtain such a V, choose any point $t_1 \in \mathfrak{M}$, and recall that $\tilde{\phi}_{t_1}$ is a trace-preserving *-isomorphism between the matrix algebras $\mathbf{A}_1(t_1)$ and $\mathbf{A}_2(t_1)$. It is an easy matter to obtain a unitary matrix W implementing $\tilde{\phi}_{t_1}$, and upon defining $V(t) \equiv W$ for $t \in \mathfrak{M}$, the desired unitary element $V \in M_n(\mathfrak{M})$ is obtained.

The above lemma can be extended to:

LEMMA 2.2. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are any* AW^**-subalgebras of* $M_n(\mathfrak{X})$ *and* ϕ *is an algebraic *-isomorphism of* $\mathbf{A}$ *onto* $\mathbf{B}$ *with the property that for each* $A \in \mathbf{A}$ *and each* $t \in \mathfrak{X}$, $\sigma[A(t)] = \sigma[\phi(A)(t)]$. *Then there is a unitary element* $U \in M_n(\mathfrak{X})$ *that implements* ϕ.

Proof. The mapping ϕ implements a trace-preserving *-isomorphism between the centres of the subalgebras $\mathbf{A}$ and $\mathbf{B}$. Thus by making an application of Lemma 2.1 and changing notation, we can assume that the algebras $\mathbf{A}$ and $\mathbf{B}$ have the common centre $\mathbf{Z}$ and that ϕ is constant on $\mathbf{Z}$. Now $\mathbf{A}$ and $\mathbf{B}$ must be finite AW^*-algebras of type I, and it follows from (**4**, Lemma 18) and (**3**, Lemma 4.10) that $\mathbf{A}$ and $\mathbf{B}$ are each finite C^*-sums of homogeneous algebras. Thus we write $\mathbf{A}$ as the C^*-sum $\mathbf{A} = \{\mathbf{A}_m\}_{m\in M}$, where each $\mathbf{A}_m$ is an m-homogeneous AW^*-subalgebra and M is some subset of the first n positive integers. Since m-homogeneity is an algebraic invariant, we must also have $\mathbf{B} = \{\mathbf{B}_m\}_{m\in M}$. It is clear that for each $m \in M$, ϕ gives rise to a trace-preserving *-isomorphism between the homogeneous algebras $\mathbf{A}_m$ and $\mathbf{B}_m$, so for the moment we fix m and consider the isomorphic algebras $\mathbf{A}_m$ and $\mathbf{B}_m$ with common centre $\mathbf{Z}_m$. If $m = 1$, we have done all we need to do; otherwise, let $\{E_{ij}\}$ be a set of matrix units for $\mathbf{A}_m$. (Thus each E_{ii} is an abelian projection in $\mathbf{A}_m$.) Then, of course, the corresponding collection

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$\{F_{ij} = \phi(E_{ij})\}$ is a set of matrix units for $\mathbf{B}_m$, and we consider the isomorphic abelian AW^*-subalgebras $E_{11}\,\mathbf{Z}_m$ and $F_{11}\,\mathbf{Z}_m$ of $\mathbf{A}_m$ and $\mathbf{B}_m$ respectively. Another application of Lemma 2.1 yields a unitary element $Y \in M_n(\mathfrak{X})$ such that $YE_{11}CY^* = F_{11}C$ for each $C \in \mathbf{Z}_m$. Define $V_1 = YE_{11}$, and for $i = 2, \ldots, m$, define $V_i = F_{i1}\,V_1\,E_{1i}$. Then define

$$V^{(m)} = \sum_{i=1}^{m} V_i.$$

Calculation yields $V_iV_i^* = F_{ii}$, $V_i^*V_i = E_{ii}$, and $V^{(m)*}V^{(m)} = V^{(m)}V^{(m)*} = I_m$, where I_m is the common unit of the algebras $\mathbf{A}_m$ and $\mathbf{B}_m$. Also for $i, j = 1, 2, \ldots, m$, one has $V^{(m)}E_{ij}V^{(m)*} = V_i\,E_{ij}\,V_j^* = F_{ij}$, and for each $C \in \mathbf{Z}_m$,

$$V^{(m)}CV^{(m)*} = \sum_{i,j} F_{i1}V_1E_{1i}CE_{j1}V_1^*F_{1j} = \sum_k F_{k1}V_1E_{11}CV_1^*F_{1k}$$

$$= \sum_k F_{k1}F_{11}CF_{k1} = C\sum_k F_{kk} = C.$$

Hence $V^{(m)}$ commutes with $\mathbf{Z}_m$, and since any element $A \in \mathbf{A}_m$ can be written as

$$A = \sum_{i,j} C_{ij}E_{ij},$$

where the $C_{ij} \in \mathbf{Z}_m$, we have

$$\phi(A) = \sum_{i,j} C_{ij}F_{ij} = \sum_{i,j} V^{(m)}C_{ij}V^{(m)*}V^{(m)}E_{ij}V^{(m)*} = V^{(m)}AV^{(m)*}.$$

Thus $V^{(m)}$ implements ϕ on $\mathbf{A}_m$ for each $m \in M$, and we define

$$W = \sum_{m \in M} V^{(m)}.$$

Clearly $W^*W = WW^* = I$, where I is the unit of $\mathbf{A}$, and it is also clear that W implements ϕ on $\mathbf{A}$. Finally define $U = (1 - I) + W$, where 1 is the unit of $M_n(\mathfrak{X})$. Then U is a unitary element in $M_n(\mathfrak{X})$ and if $A \in \mathbf{A}$, $UAU^* = WAW^* = \phi(A)$, so that the proof is complete.

Given the preceding lemma, the proof of Theorem 2 is easy. The reader is referred to (**2**, p. 260) for information concerning the unique Dixmier central trace on finite W^*-algebras and to (**9**) for information on the trace in AW^*-algebras.

THEOREM 2. *Suppose* $\mathbf{R}$ *is any finite* W^**-algebra* (AW^**-algebra*) *of type I,* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are any* W^**-subalgebras* (AW^**-subalgebras*) *of* $\mathbf{R}$, *and* $D(\cdot)$ *is the unique central trace on* $\mathbf{R}$. *If* ϕ *is an algebraic *-isomorphism of* $\mathbf{A}_1$ *onto* $\mathbf{A}_2$, *then there is a unitary element* $U \in \mathbf{R}$ *such that* $\phi(A) = UAU^*$ *for each* $A \in \mathbf{A}_1$ *if and only if* $D(A) = D(\phi(A))$ *for each* $A \in \mathbf{A}_1$.

Proof. Since $D(\cdot)$ is a unitary invariant, the "only if" half of the theorem is immediate. Turning to the proof of the other half of the theorem, one knows that $\mathbf{R}$ is a direct sum $\mathbf{R} = \{\mathbf{R}_i\}_{i \in I}$ of i-homogeneous algebras, and that $D(\cdot)$ is the sum of the unique central traces $D_i(\cdot)$ on the algebras $\mathbf{R}_i$. If E_i is the unit of $\mathbf{R}_i$, then $E_i\mathbf{A}_1$ and $E_i\mathbf{A}_2$ are W^*-subalgebras (AW^*-subalgebras) of $\mathbf{R}_i$, and the mapping $E_iA \to E_i\phi(A)$ is easily seen to be a *-isomorphism of $E_i\mathbf{A}_1$ onto $E_i\mathbf{A}_2$ which preserves the central trace $D_i(\cdot)$. Thus the problem is reduced to the case in which $\mathbf{R}$ is a homogeneous algebra, and the fact that this makes Lemma 2.2 applicable can be obtained from **(5, § 3)**.

The following lemma enables us to apply Theorem 2 to the question of unitary equivalence of elements of $M_n(\mathfrak{X})$.

LEMMA 2.3. *Let* $\mathbf{A}$ *be any *-subalgebra of* $M_n(\mathfrak{X})$, *and for each* $t \in \mathfrak{X}$ *denote by* $\mathbf{A}(t)$ *the *-algebra of matrices* $\{A(t) \mid A \in \mathbf{A}\}$. *Let* $\mathfrak{S}$ *be any compact open subset of* $\mathfrak{X}$ *with the property that for each* $t \in \mathfrak{S}$, *the algebra* $\mathbf{A}(t)$ *contains the same number* $k > 0$ *of linearly independent matrices, and define the subset* $\mathbf{R} \subset M_n(\mathfrak{S})$ *by:* $B \in \mathbf{R}$ *if and only if* $B \in M_n(\mathfrak{S})$ *and* $B(t) \in \mathbf{A}(t)$ *for each* $t \in \mathfrak{S}$. *Then the collection* $\mathbf{R}$ *is an* AW^**-subalgebra of* $M_n(\mathfrak{S})$.

Proof. It is clear that $\mathbf{R}$ is an algebraic *-subalgebra of $M_n(\mathfrak{S})$, and it follows from the fact that for $B \in \mathbf{R}$,

$$||B|| = \sup_{t \in \mathfrak{S}} ||B(t)||,$$

that $\mathbf{R}$ is a C^*-subalgebra of $M_n(\mathfrak{S})$. We separate out the next fact to be verified as a sublemma.

SUBLEMMA. *If* $\{E_\lambda \mid \lambda \in \Lambda\}$ *is any collection of mutually orthogonal projections in* $\mathbf{R}$, *and* $E = \sup_\lambda E_\lambda$ *(as calculated in* $M_n(\mathfrak{S})$*), then* $E \in \mathbf{R}$.

Proof. Suppose this sublemma is false. Then there is a point $r \in \mathfrak{S}$ such that $E(r) \notin \mathbf{A}(r)$. Let $\{A_1(r), \ldots, A_k(r) \mid A_i \in \mathbf{A}\}$ be a basis for $\mathbf{A}(r)$. Then the matrices $E(r), A_1(r), \ldots, A_k(r)$ are linearly independent, and by continuity there is a compact open neighbourhood $\mathfrak{N} \subset \mathfrak{S}$ of r such that for $t \in \mathfrak{N}$, $\{A_1(t), \ldots, A_k(t) \mid A_i \in \mathbf{A}\}$ remains a basis for $\mathbf{A}(t)$ and also the matrices $E(t), A_1(t), \ldots, A_k(t)$ remain linearly independent. Thus for $t \in \mathfrak{N}$, $E(t) \notin \mathbf{A}(t)$. Now for $t \in \mathfrak{N}$, let C_t be the collection of all $\lambda \in \Lambda$ such that $E_\lambda(t) \neq 0$. Note that for any t, C_t contains at most n elements, and choose $t_0 \in \mathfrak{N}$ with the property that C_{t_0} contains a maximum number of elements. Then, by continuity, there is a compact open neighbourhood $\mathfrak{P} \subset \mathfrak{N}$ of t_0 such that $C_t = C_{t_0}$ for each $t \in \mathfrak{P}$. Consider the projection $F \in M_n(\mathfrak{S})$ defined by

$$F(t) = \sum_{\lambda \in C_{t_0}} E_\lambda(t)$$

for $t \in \mathfrak{P}$ and $F(t) = E(t)$ for $t \in \mathfrak{S} - \mathfrak{P}$. Then F is an upper bound for the collection $\{E_\lambda \mid \lambda \in \Lambda\}$, and $F \leqslant E$. Thus $F = E$, and it follows that for $t \in \mathfrak{P}$,

$$E(t) = \sum_{\lambda \in C_{t_0}} E_\lambda(t),$$

which implies that for $t \in \mathfrak{P}$, $E(t) \in \mathbf{A}(t)$. This is a contradiction. (It is perhaps worth noting that implicit in the above argument is a new proof of **(3**, Lemma 4.11**)**.)

To show that $\mathbf{R}$ is an AW^*-subalgebra of $M_n(\mathfrak{S})$ there remains only one further fact to verify, and we also treat it as a sublemma.

SUBLEMMA. *If $B \in \mathbf{R}$, then the right projection* (rp) *of B (as calculated in $M_n(\mathfrak{S})$) is also an element of* $\mathbf{R}$.

Proof. Note that $\text{rp}[B] = \text{rp}[B^*B]$, so that B can be taken to be positive, and also that if $E = \text{rp}[B]$ then E can be characterized as the smallest projection in $M_n(\mathfrak{S})$ satisfying $BE = B$. Again we assume the sublemma false, i.e., that there is a point $r \in \mathfrak{S}$ such that $E(r) \notin \mathbf{A}(r)$. Then, just as before, it follows that there is a compact open neighbourhood $\mathfrak{N} \subset \mathfrak{S}$ of r such that for $t \in \mathfrak{N}$, $E(t) \notin \mathbf{A}(t)$. We proceed to a contradiction as follows. For each $t \in \mathfrak{S}$, consider the characteristic equation of $B(t)$. It follows from **(1**, Theorem 1**)** that there exist n functions $c_1, \ldots, c_n \in C(\mathfrak{S})$ with the property that for each $t \in \mathfrak{S}$, the numbers $c_1(t), \ldots, c_n(t)$ are exactly the eigenvalues (with correct multiplicities) of $B(t)$. For $t \in \mathfrak{N}$, let I_t be the set of integers i such that $c_i(t) \neq 0$. Choose $t_0 \in \mathfrak{N}$ such that I_{t_0} has a maximum number of elements. Then, by continuity, there is a compact open neighbourhood $\mathfrak{M} \subset \mathfrak{N}$ of t_0 such that for each $t \in \mathfrak{M}$, $I_t = I_{t_0}$. Let $\eta > 0$ be such that for each $t \in \mathfrak{M}$ and each $i \in I_{t_0}$, $c_i(t) > \eta$. Let f be any continuous function mapping the real line into itself such that $f(0) = 0$ and $f(s) = 1$ for $s > \eta/2$. Then $F = f[B] \in \mathbf{R}$ (recall that $\mathbf{R}$ is C^*), and it is easy to see that for each $t \in \mathfrak{S}$, $F(t) = f[B](t)$. Thus for $t \in \mathfrak{M}$, $F(t)$ is the projection on the range of $B(t)$, and as such, $F(t)$ is the smallest projection satisfying $B(t)F(t) = B(t)$. It follows that for $t \in \mathfrak{M}$ we must have $E(t) = F(t)$, which is a contradiction since $F \in \mathbf{R}$.

It now follows from the sublemmas and **(4**, Lemma 2**)** that $\mathbf{R}$ is an AW^*-subalgebra of $M_n(\mathfrak{S})$.

We are finally in a position to prove:

THEOREM 3. *If $A, B \in M_n(\mathfrak{X})$, and if $A(t)$ is unitarily equivalent to $B(t)$ for each $t \in \mathfrak{X}$, then there is a unitary element $U \in M_n(\mathfrak{X})$ such that $A = UBU^*$.*

Proof. We consider collections $\{\mathfrak{U}_i\}$ of disjoint, non-empty, compact open subsets $\mathfrak{U}_i \subset \mathfrak{X}$ such that if $\mathfrak{U}_i \in \{\mathfrak{U}_i\}$, then there is a unitary element $U_i \in M_n(\mathfrak{U}_i)$ such that for $t \in \mathfrak{U}_i$, $A(t) = U_i(t)B(t)U_i^*(t)$. If $\{\mathfrak{U}_i\}_{i \in I}$ is a maximal collection of this kind and

$$\mathfrak{U} = \overline{\bigcup_{i \in I} \mathfrak{U}_i},$$

then again in view of **(1**, Lemma 2.1**)**, it suffices to prove $\mathfrak{U} = \mathfrak{X}$. Suppose $\mathfrak{X} - \mathfrak{U} \neq \emptyset$, and taking $\mathbf{A}(t)$ as defined in Lemma 2.3, choose $r \in \mathfrak{X} - \mathfrak{U}$ so

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that the number of linearly independent matrices in the algebra $\mathbf{A}(t)$ is a maximum (over $\mathfrak{X} - \mathfrak{U}$) at r. Let $p_1(A(r), A^*(r)), \ldots, p_k(A(r), A^*(r))$, be a basis for $\mathbf{A}(r)$, and choose a compact open neighbourhood $\mathfrak{S} \subset \mathfrak{X} - \mathfrak{U}$ of r so that on $\mathfrak{S}$ the matrices $p_1(A(t), A^*(t)), \ldots, p_k(A(t), A^*(t))$ remain linearly independent. It follows from the hypothesis that for $t \in \mathfrak{S}$, the matrices $p_1(B(t), B^*(t)), \ldots, p_k(B(t), B^*(t))$ are a basis of the *-algebra $\mathbf{B}(t)$ generated by $B(t)$. Now let $\mathbf{R}(A)$ be the AW^*-subalgebra of $M_n(\mathfrak{S})$ corresponding to $\mathbf{A}(t)$, which Lemma 2.3 gives rise to, and let $\mathbf{R}(B)$ be the corresponding AW^*-subalgebra of $M_n(\mathfrak{S})$ for $\mathbf{B}(t)$.

It follows that each $C \in \mathbf{R}(A)$ can be written in the form

$$C(t) = \sum_{i=1}^{k} c_i(t) p_i(A(t), A^*(t))$$

for $t \in \mathfrak{S}$, and it is not difficult to see that the $c_i(\cdot)$ are uniquely determined continuous complex-valued functions on $\mathfrak{S}$. Elements of $\mathbf{R}(B)$ can be written similarly, and thus one can define a mapping

$$\phi\colon \sum_{i=1}^{k} c_i(\cdot) p_i(A(\cdot), A^*(\cdot)) \to \sum_{i=1}^{k} c_i(\cdot) p_i(B(\cdot), B^*(\cdot))$$

of $\mathbf{R}(A)$ onto $\mathbf{R}(B)$.

By virtue of Theorem 2, to complete the proof of the theorem it suffices to verify that ϕ is a trace-preserving *-isomorphism of $\mathbf{R}(A)$ onto $\mathbf{R}(B)$ which maps A to B. This one does pointwise, using the hypothesis to show that any polynomial $q(A(t), A^*(t))$ vanishes if and only if $q(B(t), B^*(t))$ does also. See **(5)** for further details of similar verifications.

3. We now briefly summarize some results of the author **(5)** on unitary equivalence, preparatory to obtaining a local complete set of unitary invariants for a certain class of operators on Hilbert space. Let W be the free multiplicative semi-group on the symbols x and y, and denote words in W by $w(x, y)$. Specht **(7)** showed that the collection of traces

$$\{\sigma[w(A, A^*)] \mid w(x, y) \in W\}$$

is a complete set of unitary invariants for $n \times n$ complex matrices. The author was able to improve this by showing in **(5)** that for n fixed but arbitrary, there is always a subset $W_n \subset W$ containing less than 4^{n^2} words such that the collection

$$\{\sigma[w(A, A^*)] \mid w(x, y) \in W_n\}$$

is already a complete set of unitary invairants for $n \times n$ complex matrices. Better results are known for $n = 2$ and $n = 3$ **(6)**. Now if A is an operator generating a finite W^*-algebra $\mathbf{R}(A)$ of type I, and $D_a(\cdot)$ is the unique Dixmier central trace on $\mathbf{R}(A)$, then **(5,** Theorem 5**)** A is unitarily equivalent to an operator B if and only if B generates a finite W^*-algebra $\mathbf{R}(B)$ of type I and there is a unitary isomorphism ϕ such that $\phi D_a[w(A, A^*)]\phi^{-1} = D_b[w(B, B^*)]$ for each $w(x, y) \in W$, where $D_b(\cdot)$ is the Dixmier trace on $\mathbf{R}(B)$. Thus a global set of unitary invariants for such operators A was provided.

However, in the case that A and B are operators in the same finite W^*-algebra $\mathbf{R}$ of type I, one might expect that the unitary equivalence of A and B relative to $\mathbf{R}$ would follow from the equations $D[w(A, A^*)] = D[w(B, B^*)]$, $w(x, y) \in W$. The author was unable to prove this in **(5)** except in the special case in which A generates $\mathbf{R}$, but we can now obtain this result easily from Theorem 3.

COROLLARY 3.1. *If* $\mathbf{R}$ *is a finite* W^**-algebra of type I,* $A, B \in \mathbf{R}$, *and* $D(\cdot)$ *is the unique central Dixmier trace on* $\mathbf{R}$, *then* A *is unitarily equivalent to* B *relative to* $\mathbf{R}$ *if and only if* $D[w(A, A^*)] = D[w(B, B^*)]$ *for each* $w(x, y) \in W$.

Proof. $\mathbf{R}$ is a direct sum of homogeneous algebras $\{\mathbf{R}_i\}$ and the Dixmier trace on $\mathbf{R}$ is the sum of the Dixmier traces on the homogeneous algebras. Thus the problem reduces to the case in which $\mathbf{R}$ is homogeneous, and the traces assumed equal above ensure that the hypotheses of Theorem 3 are satisfied. (For more detail in this connection, see **5.**)

4. Remarks.

1. Because of Specht's theorem mentioned above and the continuity of the functions $\sigma[w(A(t), A^*(t))]$, Theorem 3 remains true if it is assumed only that $A(t)$ is unitarily equivalent to $B(t)$ for t in any dense subset of $\mathfrak{X}$.

2. If in Corollary 3.1 $\mathbf{R}$ is assumed to be an n-homogeneous algebra, then one can obtain the same result by assuming only that $D[w(A, A^*)] = D[w(B, B^*)]$ for $w(x, y) \in W_n$, in view of **(5**, Theorem 1**)**.

3. The statements of Theorem 2 and Corollary 3.1 make sense in any W^*-algebra of type II_1, and the author conjectures that they are true there. However, he is unable to prove this except in one special case.

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