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INCOMPLETE SELF-ORTHOGONAL LATIN SQUARES

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Abstract

We show that for all $n \geqslant 3k + 1$, $n \neq 6$, there exists an incomplete self-orthogonal latin square of order n with an empty order k subarray, called an ISOLS(n; k), except perhaps when $(n; k) \in \{(6m + i; 2m) : i = 2, 6\}$.

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1. Notation and basic constructions

We study a particular family of incomplete orthogonal latin squares.

Let ILS(n; $b_1, b_2, \ldots, b_k$) be an order n array A with entries from an n-set B defined as follows, where the B_i, $1 \leqslant i \leqslant k$, are subsets of B so that $|B_i| = b_i$, and where $B_i \cap B_j = \varnothing$, $1 \leqslant i, j \leqslant k$.

(a) Each cell of A is empty or contains an element of B;

(b) the subarrays indexed by $B_i \times B_i$ are empty; and

(c) the elements in row or column b are exactly those of $B \setminus B_i$ if $b \in B_i$, and of B otherwise.

Two ILS(n; $b_1, b_2, \ldots, b_k$) are *orthogonal* if, on superposition, all ordered pairs $(B \times B) \setminus \bigcup_{i=1}^{k}(B_i \times B_i)$ result. Denote two such squares by IPOSL(n; $b_1, b_2, \ldots, b_k$). Similarly, r − IPOLS(n; $b_1, b_2, \ldots, b_k$) denotes a set of r ILS(n; $b_1, b_2, \ldots, b_k$) which are pairwise orthogonal. A pair of orthogonal ILS(n; $b_1, b_2, \ldots, b_k$) in which one is the transpose of the other (a self-orthogonal array) will be denoted ISOLS(n; $b_1, b_2, \ldots, b_k$). In the case when $b_i = 0$ for each i the arrays are all latin squares, and we denote them respectively by LS(n), POLS(n), r-POLS(n) and SOLS(n).

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We are interested in the existence of ISOLS(n; k) and will show that, except for two infinite families, an ISOLS(n; k) exists if and only if $n \geqslant 3k + 1$, $n \neq 6$. The infinite families are ISOLS($6m + 2; 2m$) and ISOLS($6m + 6; 2m$). While many of the second family can be constructed, none of the first is known and, in particular, computer search verifies that there is no ISOLS(8; 2).

This problem has been studied for quite some time. Simple counting shows that $n \geqslant 3k + 1$ is a necessary condition, and many ISOLS($3k + 1$; k) were constructed by Parker [17] (notably an ISOLS(10; 3)) and Hedayat [10]. Crampin and Hilton [5] showed that for every k there exists an $n(k)$ such that, for all $n > n(k)$, an ISOLS(n; k) can be constructed. This was greatly improved by Drake and Lenz [9] who gave constructions for ISOLS(n; k) whenever $k \geqslant 304$ and $n \geqslant 4k + 3$.

We will use two types of construction: generalized product constructions and a starter-adder type construction. The product constructions are adaptions of the Wilson-type constructions given by Brouwer and van Rees [4], although they give them in terms of orthogonal arrays. It is assumed that the reader is familiar with the usual product constructions: direct and semi-direct product.

First we state results on the existence of special sets of orthogonal latin squares.

LEMMA 1.1 (Lindner, Mullin, Stinson [16], Wang [19], Zhu [23]). *For all p, $p \notin E = \{2, 3, 6, 10, 14, 46, 54, 58, 62, 66, 70\}$ there exists an* SOLS(p) *with a symmetric orthogonal mate.*

LEMMA 1.2 (Wang [19]). *For all even p, $p \notin F = E \cup \{78, 82, 98, 102, 118, 142, 174, 194, 202, 214, 230, 258, 278, 282, 394, 398, 402, 422, 1322\}$, there exists an* SOLS($p$) *with a symmetric orthogonal mate which has only the entry* 1 *on the main diagonal.*

Call the above SOLS(p) P and the symmetric orthogonal mate P_1. Now, let P_0 denote the LS(p) defined by $e_{P_0}(s, t) \equiv t - s \pmod{p}$ and based on the elements $\{1, 2, \ldots, p\}$, where $e_X(s, t)$ is the entry in cell (s, t) of the array.

LEMMA 1.3. *For all p,* $\gcd(p, 6) = 1$, *there exist orthogonal latin squares P and P_1 of order p so that each square is orthogonal to P_0.*

PROOF. Let P be the SOLS(p) with $e_P(s, t) \equiv 2t - s \pmod{p}$, and let P_1 be defined by $e_{P_1}(s, t) \equiv t + s \pmod{p}$.

LEMMA 1.4. *For all odd prime powers p, $p \geqslant 5$, there exists a set of $(p - 1)$-*POLS(p) *which consists of P_0, a symmetric square P_1 and $(p - 3)/2$ self-orthgonal squares P, $P_2, \ldots, P_{(p-3)/2}$ and their transposes.*

PROOF. This result comes immediately from the construction of a complete set of latin squares of order p. (See, for example, [6, pp. 160–169].)

Note that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the p_i are distinct odd primes, and if $r = p_1^{e_1} < p_2^{e_2} < \cdots < p_k^{e_k}$, then there are (by direct product) squares P, P_0, P_1, $P_2, \ldots, P_{(r-3)/2}$ of order n.

The idea of the product construction is to take the square P and replace each cell by a $q \times q$ array; this array will in general either be one of a POLS(q) or, combined with additional rows and columns added to P, one of an IPOLS($q + x; x$).

If we have a symmetric transversal in P (determined say by the entry 1 in P_1, or p in P_0 if p is odd), then we add a_{11} new rows and columns to P. Replace the cells (i, j) and (j, i), $i \neq j$, of P determined by the transversal with one of an IPOLS($q + a_{11}; a_{11}$) and its transpose, respectively; any cell (i, i) of the transversal with an ISOLS($q + a_{11}; a_{11}$); and all other cells with either a POLS(q) and its transpose or an SOLS(q) (if the cell is on the main diagonal). This yields an IPOLS($pq + a_{11}; a_{11}$).

If we have a pair of symmetrically placed transversals T_1 and T_{p-1} in P, determined by say 1 and $p - 1$ in P_0, then we add $2a_{10}$ rows and columns to P. If (i, j) is a cell of T_1, then it is replaced by one of an IPOLS($q + a_{10}; a_{10}$) using columns $pq + 1, \ldots, pq + a_{10}$ and rows $pq + a_{10} + 1, \ldots, pq + 2a_{10}$, and cell (j, i) is then replaced by the transpose of the other using columns $pq + a_{10} + 1, \ldots, pq + 2a_{10}$ and rows $pq + 1, \ldots, pq +_{10}$. Other cells are replaced by POLS(q) and its transpose or by an SOLS(q). This yields an ISOLS($pq + 2a_{10}; 2a_{10}$).

Finally, if we have a pair of symmetrically placed transversals T and S in P, determined say by 1 in P_j and P_j^T, then we add $2a_{ij}$ new rows and columns to P. (Note that T and S intersect in cell $(1, 1)$.) Now, as before, if (i, j) is a cell of T, replace it by one of a POLS($q + a_j; a_j$) using columns $pq + 1, \ldots, pq + a_j$ and rows $pq + a_j + 1, \ldots, pq + 2a_j$. At the same time (j, i) is replaced by the transpose of the other of the IPOLS($q + a_j; a_j$) using columns $pq + a_j + 1, \ldots, pq + 2a_j$ and rows $pq + 1, \ldots, pq + a_j$. Other cells (except $(1, 1)$) are replaced by one of a POLS(q) and its transpose or an SOLS(q). Now using cell $(1, 1)$ we get an ISOLS($pq + 2a_j; q + 2a_j$).

In many cases these constructions can be combined, and this is the substance of the next lemmas. To begin with, let $a_j \in Z^+ \cup \{0\}$ be associated with the entry 1 of P_j and P_j^T, $2 \leqslant j \leqslant r$; that is, we will add a_j to each of these transversals in P. Let $a_{k,0} = a_{p-k,0} \in Z^+ \cup \{0\}$, $1 \leqslant k \leqslant (p - 1)/2$ (p is odd), be associated with the entries k and $p - k$ of P_0; let a_{p0} be associated with p on the main diagonal. Let $a_{k1} \in Z^+ \cup \{0\}$, $1 \leqslant k \leqslant p$, be associated with k in P_1. Then let $a_0 = \sum_{k=1}^{p} a_{k0}$, $a_1 = \sum_{k=1}^{p} a_{k1}$ and $a = a_0 + a_1 + 2\sum_{k=2}^{r} a_k$. Finally, let $A_0 = \{a_{10}, a_{20}, \ldots, a_{(p-1)/2,0}, a_{p0}\}$ and $A_1 = \{a_{11}, a_{21}, \ldots, a_{p1}\}$.

Most constructions will be given without proof. Also note that, although the premises of the lemma suppose the existence of certain squares, the constructions are still valid if we do not have P_1 (all $a_{j1} = 0$), P_0 (all $a_{j0} = 0$) or P_t ($a_t = 0$). So when the lemmas are later referred to, we may not have all the squares. Moreover, even without the square P_0 we can still use the transversal on the main diagonal of P.

LEMMA 1.5. *Suppose we have squares P, P_1 and P_0 of order p. Then, if there are* ISOLS$(q + a_{p0} + a_{j1}; a_{p0}, a_{j1})$ *and* IPOLS$(q + a_{i0} + a_{j1}; a_{i0}, a_{j1})$, *there is an* ISOLS$(pq + a; a_0, a_1)$. *Hence if an* SOLS$(a_0)$ (SOLS(a_1)) *exists, then we have an* ISOLS$(pq + a; a_1)$ (ISOLS$(pq + a; a_0)$).

PROOF. Add $a_0 + a_1$ rows and columns to P as in the earlier discussion. If (s, t), $s \neq t$, is a cell of P, and if $e_{P_1}:(s, t) = j$, $e_{P_0}(s, t) = i$, then by using the appropriate rows and columns (that is, we use the same a_{j1} rows as columns, but we use the first a_{i0} of $2a_{i0}$ rows and the second a_{i0} of $2a_{i0}$ columns), replace this cell by one of an IPOLS$(q + a_{i0} + a_{j1}; a_{i0}, a_{j1})$ and replace (t, s) by the transpose of its mate. On the main diagonal cells use the ISOLS$(q + a_{p0} + a_{j1}; a_{p0}, a_{j1})$ (with the same a_{p0} rows and columns, and the same a_{j1} rows and columns).

LEMMA 1.6. *If we have squares P, P_1 and P_0 of order p, and if we have* ISOLS$(q + a_{p0} + a_{j1}; x, a_{p0}, a_{j1})$, IPOLS$(q + a_{i0} + a_{j1}; x, a_{i0}, a_{j1})$, ISOLS$(a_0)$ *and* ISOLS(a_1), *then we have an* ISOLS$(pq + a; px)$. *Moreover, if we also have an* ISOLS$(q + a_{i0} + a_{j1}; x, a_{i0}, a_{j1})$, *then we have* ISOLS$(pq + a; p^{q-x}, px, a_0, a_1)$, *where p^{q-x} indicates $q - x$ order p arrays.*

Note that in the constructions of Lemmas 1.5 and 1.6 there are also other subsquares. For example, if in Lemma 1.5, $a_{p0} = a_{j1} = 0$ for some j, where $e_{P_1}(s,s) = j$, and we have both an SOLS(a_0) and an SOLS(a_1), then we get an ISOLS$(pq + a; q)$.

LEMMA 1.7. *Suppose we have squares P, $P_2, \ldots, P_r$ of order p. Then if we have* IPOLS$(q + a_j; q_j)$, $2 \leqslant j \leqslant r$, *and* IPOLS$(q)$, *we can construct an* ISOLS$(pq + a; q + a)$.

PROOF. Simply add rows and columns and extend using each of P_j and P_j^T in turn. Ignore the cell $(1, 1)$ of P, which is contained in all transversals. Leaving this cell and the corresponding rows and columns empty yields the ISOLS$(pq + a; q + a)$.

We can now combine the constructions of Lemmas 1.5 and 1.7.

LEMMA 1.8. *Given squares* P, P_0, P_1, $P_2, \ldots, P_r$ *of order* p, *given* IPOLS($a + a_{i0} + a_{j1} + a_k; a_{i0}, a_{j1}, a_k$), ISOLS($q + a_{p0} + a_{j1}; a_{p0}, a_{j1}$), ISOLS($a_0; a_{p0}$) *and* ISOLS($a_1; a_{11}$), *we have an* ISOLS($pq + a; q + a_{p0} + a_{11} + 2(a_2 + \cdots + a_r)$).

PROOF. This is apparent upon noting that the empty subarray comes from the cell $(1,1)$ in P through which most transversals pass (see Figure 1.1). Again, this lemma allows for many more subsquares, as can be seen from Figure 1.1 below. In particular, note that if we have no $P_2, \ldots, P_r$, then we get ISOLS($pq + a; q + a_{p0} + a_{11}$).

LEMMA 1.9. *Given squares* P, P_0, P_1, $P_2, \ldots, P_r$ *of order* p, *given* IPOLS($q + a_{i0} + a_{j1} + a_k; x, a_{i0}, a_{j1}, a_k$), ISOLS($q + a_{p0}, a_{j1}; x, a_{p0}, a_{j1}$), ISOLS($a_0; a_{p0}$), ISOLS($a_1; a_{11}$) *and* ISOLS($q + a_{p0} + a_{11} + 2(a_2 + \cdots + a_r); x$), *we have an* ISOLS($pq + a; px$).

Finally, we note a trivial result.

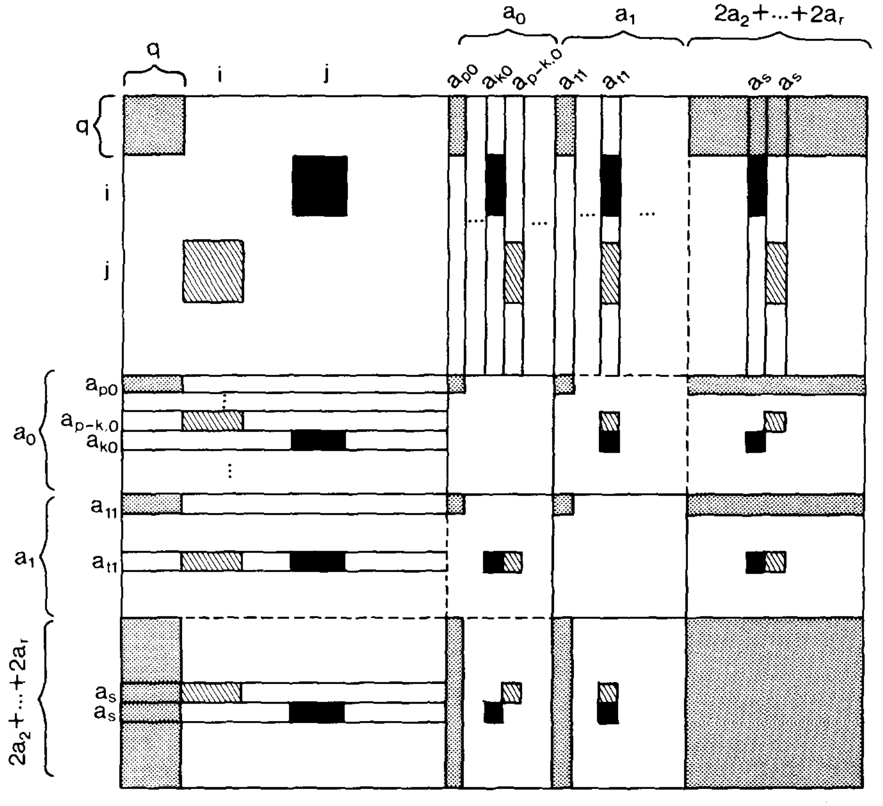

FIGURE 1.1

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LEMMA 1.10. *Given an* ISOLS(a; b) *and an* ISOLS(b; y), *we have an* ISOLS(a; y).

Special cases of the next lemma have been given by Dinitz and Stinson [7].

LEMMA 1.11. *Given an* ISOLS(n; $b_1, b_2, \dots, b_k$), *where* $n = \sum_{i=1}^{k} b_i$, *and an* ISOLS($b_i + x$; x) *for each* $i \neq j$, *we have an* ISOLS($n + x$; $b_j + x$).

Our last technique is a starter-adder type construction. This idea has been described by several authors including Horton [15], Hedayat and Seiden [11], and Zhu [20]. The plan is to construct an ISOLS(n; k) A from its first row (given by $\mathbf{e} = (e_A(1,1), \dots, e_A(1, n-k))$ and $\mathbf{f} = (e_A(1, n-k+1), \dots, e_A(1, n)))$ and from the last k entries of the first column (given by $\mathbf{g} = (e_A(n-k+1, 1), \dots, e_A(n, 1)))$. The entries of the array are $\{1, 2, \dots, n-k\} \cup X$, where $X = \{x_1, x_2, \dots, x_k\}$. The arrary is constructed modulo $n - k$ using a fixed $d \in \{1, 2, \dots, n-k\}$, where the x_i act as "infinity" elements according to the following rules.

$$\text{(a)} \quad \left.\begin{array}{l} e_A(s+1, t+1) = e_A(s,t) \text{ if } e_A(s,t) = x_i \\ e_A(s+1, t+1) \equiv e_A(s,t) + d \pmod{n-k} \text{ otherwise} \end{array}\right\}, \quad 1 \leqslant s, t \leqslant n-k,$$

$$\text{(b)} \quad e_A(s+1, n-k+t) \equiv e_A(s, n-k+t) + d \pmod{n-k}, \quad 1 \leqslant t \leqslant k, 1 \leqslant s \leqslant n-k,$$

$$\text{(c)} \quad e_A(n-k+t, s+1) \equiv e_A(n-k+t, s) + d \pmod{n-k}, \quad 1 \leqslant t \leqslant k, 1 \leqslant s \leqslant n-k.$$

Note that in case (a) all cell labels are determined modulo $n - k$, but in cases (b) and (c) this applies only to the row and column labels respectively.

It is not difficult to determine the conditions that $\mathbf{e}, \mathbf{f}, \mathbf{g}$ and d must satisfy, but we shall not concern ourselves with that. When needed, the value of d and of the vectors $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ will be given. Simple calculations verify that they work. For example, an ISOLS(20; 3) can be constructed from

$$\mathbf{e} = (1, 14, 10, 6, 2, x_1, 11, 7, 3, 16, 12, 8, 4, 17, 13, x_2, x_3),$$
$$\mathbf{f} = (5, 9, 15), \quad \mathbf{g} = (13, 2, 3) \quad \text{and} \quad d = 14.$$

2. Main result

We shall now apply the constructions described earlier to construct ISOLS(n; k), $n \geqslant 3k + 1$. Throughout, x, y, a_{i0}, a_{i1} and a_j are given only when they are non-zero. The entries of A_0 and A_1 will be given, and, although it will not be indicated to which non-zero a_{i0} or a_{i1} we refer, it is always easy to decide. Also, as mentioned earlier, we abbreviate notation by writing i^j when i occurs j times in a set.

The following results are crucial to the constructions.

THEOREM 2.1. *For all* $n \geqslant 3k$, $(n; k) \neq (6; 1)$, *there exists an* IPOLS$(n; k)$.

The proof is given in a series of papers ([14, 18, 21, 22]), but can be found in its entirety in [13].

COROLLARY 2.2. *For all* $n \geqslant 3k + 1$ *with* $(n; k) \neq (6; 1)$, *there is an* IPOLS$(n; 1, k)$.

THEOREM 2.3 (Brayton, Coppersmith and Hoffman [3]). *For all* $n \neq 2, 3, 6$, *there exists an* SOLS(n).

COROLLARY 2.4. *For all* $n \neq 2, 3, 6$, *there exists a* POLS(n).

COROLLARY 2.5. *For all* $n \neq 2, 3, 6$, *there exists an* ISOLS$(n; 1^r)$, $1 \leqslant r \leqslant n$, (*and hence an* IPOLS$(n; 1^r)$).

LEMMA 2.6 (Heinrich [12]). *There exists an* ISOLS$(3k + 1; k)$ *for all* k.

PROOF. Apply the starter-adder technique with

$$\mathbf{e} = (1, x_1, x_2, \ldots, x_k, 2, 4, 6, \ldots, 2k), \quad \mathbf{f} = (3, 5, 7, \ldots, 2k + 1),$$
$$\mathbf{g} = (2, 3, 4, \ldots, k + 1) \quad \text{and} \quad d = 1.$$

LEMMA 2.7. *There exists an* ISOLS$(n; 2)$ *when* $7 \leqslant n \leqslant 50$ *and* $n \neq 8$.

PROOF. For $9 \leqslant n \leqslant 21$, $n \neq 18$, vectors for the starter adder technique are given in Table 2.1. (The case $n = 7$ was given in Lemma 2.6.)

When $n = 25$, apply Lemmas 1.8 and 1.10 with $p = 7$, $q = 3$, $a_{p0} = a_{11} = a_2 = 1$ and $y = 2$. For $n = 31, 32$, apply Lemmas 1.9 and 1.10 with $p = 7$, $q = 4$, $x = 1$, $y = 2$ and, respectively, $a_{p0} = a_2 = 1$ and $a_{p0} = a_{11} = a_2 = 1$. The remaining orders n are given in Table 2.2.

TABLE 2.1

n	d	e	f	g
9	1	$(1, x_1, x_2, 5, 7, 2, 6)$	(3, 4)	(6, 5)
10	1	$(1, 3, x_1, 6, 4, x_2, 2, 5)$	(7, 8)	(5, 7)
11	1	$(1, 3, 5, 7, 9, 2, 6, x_1, x_2)$	(4, 8)	(8, 7)
12	1	$(1, 4, x_1, 8, x_2, 5, 3, 6, 10, 7)$	(2, 9)	(4, 6)
13	1	$(1, 4, 8, 3, 6, 10, 2, 11, x_1, x_2, 9)$	(5, 7)	(9, 8)
14	1	$(1, 3, 8, 10, x_1, x_2, 6, 11, 5, 12, 9, 4)$	(2, 7)	(10, 8)
15	1	$(1, x_1, x_2, 5, 8, 11, 3, 2, 13, 7, 6, 10, 12)$	(4, 9)	(3, 7)
16	1	$(1, 14, 10, 5, 13, 3, 12, 7, 11, x_1, x_2, 4, 2, 9)$	(6, 8)	(5, 11)
17	1	$(1, 3, 8, 6, 11, 14, 2, 5, x_1, x_2, 10, 15, 7, 12, 4)$	(9,13)	(12, 8)
19	5	$(1, 5, 9, 13, x_1, 4, 8, 12, 16, 3, 7, 11, 15, 2, 6, 10, x_2)$	(14, 17)	(2, 14)
20	1	$(1, x_1, x_2, 2, 6, 12, 16, 18, 13, 9, 8, 5, 15, 17, 4, 3, 11, 14)$	(7, 10)	(14, 9)
21	1	$(1, x_1, x_2, 2, 4, 7, 16, 15, 19, 13, 6, 14, 9, 18, 8, 5, 3, 12, 11)$	(10, 17)	(17, 7)

TABLE 2.2

n	p	q	A_0	A_1	Lemma(s)
18	4	4	–	$\{1^2\}$	1.5
22	5	4	$\{1^2\}$	–	1.5
23	7	3	$\{1\}$	$\{1\}$	1.6 and 1.10 $x = 1$ $y = 2$
24	7	3	$\{1\}$	$\{1^2\}$	1.5
26	7	3	$\{1^5\}$	–	1.6 and 1.10 $x = 1$ $y = 2$
27	7	3	$\{1^5\}$	$\{1\}$	1.6 and 1.10 $x = 1$ $y = 2$
28	7	4	–	–	1.6 and 1.10 $x = 1$ $y = 2$
29	7	4	$\{1\}$	–	1.6 and 1.10 $x = 1$ $y = 2$
30	7	4	$\{1^2\}$	–	1.5
33	7	4	$\{1^5\}$	–	1.6 and 1.10 $x = 1$ $y = 2$
34	4	8	–	$\{2\}$	1.5
35	7	5	–	–	1.6 and 1.10 $x = 1$ $y = 2$
36	9	4	–	–	1.6 and 1.10 $x = 1$ $y = 2$
37	9	4	$\{1\}$	–	1.6 and 1.10 $x = 1$ $y = 2$
38	9	4	$\{1^2\}$	–	1.5
39	5	7	$\{1^4\}$	–	1.8 and 1.10 $y = 2$
40	4	10	–	–	1.8 and 1.10 $y = 2$
41	4	10	–	$\{1\}$	1.8 and 1.10 $y = 2$
42	4	10	–	$\{2\}$	1.5
43	5	7	$\{2^4\}$	–	1.8 and 1.10 $y = 2$
44	4	11	–	–	1.8 and 1.10 $y = 2$
45	5	9	–	–	1.8 and 1.10 $y = 2$
46	5	9	–	$\{1\}$	1.8 and 1.10 $y = 2$
47	5	9	$\{2\}$	–	1.5
48	4	12	–	–	1.8 and 1.10 $y = 2$
49	4	12	–	$\{1\}$	1.8 and 1.10 $y = 2$
50	4	12	–	$\{2\}$	1.5

THEOREM 2.8. *There exists an* ISOLS(n; 2) *for all* $n \geqslant 7$ *and* $n \neq 8$.

PROOF. By Lemma 2.7 we need only consider the case $n \geqslant 51$, and the proof is given in Table 2.3, where, in each case, $p = 7$ and $q = k$ $(k \geqslant 7)$.

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TABLE 2.3

$n-2$	A_0	A_1	Lemma(s)
$7k$	$\{1^2\}$	–	1.5
$7k+1$	$\{1^2\}$	$\{1\}$	1.5
$7k+2$	$\{1^4\}$	–	1.6 and 1.10 $x=1$ $y=2$
$7k+3$	$\{1^5\}$	$\{1\}$	1.6 and 1.10 $x=1$ $y=2$
$7k+4$	$\{1^2\}$	$\{1^4\}$	1.5
$7k+5$	$\{1^2\}$	$\{1^5\}$	1.5
$7k+6$	$\{1^4\}$	$\{1^4\}$	1.6 and 1.10 $x=1$ $y=2$

We shall now construct ISOLS(n; k) for all $n \geqslant 3k+1$ and $k \geqslant 7$, except when $k = 2m$ and $n = 3k+2$ or $3k+6$. This leaves only finitely many values of n for each k, $k \in \{3,4,5,6\}$ to be considered.

THEOREM 2.9. *If $n = mk + \varepsilon$, $0 \leqslant \varepsilon < k$, $k \geqslant 7$, $m \geqslant 7$, then there is an* ISOLS(n; k).

PROOF. The case $\varepsilon = 0$ is immediate by product.

First we consider the cases $k \notin E$.

If $\varepsilon \neq 2, 3$, or 6, apply Lemma 1.6 with $p = k$, $q = m$, $x = 1$ and $A_1 = \{1^\varepsilon\}$.

If $\varepsilon = 2, 3$ or 6, use Lemma 1.6 with $p = k$, $q = m - 1$, $x = 1$ and $A_1 = \{1^{k-\varepsilon}, 2^\varepsilon\}$. If k is even, there are at least $k/2$ symmetric transversals (determined from P_1) which contain no cells of the main diagonal of P, and we can thus avoid the need for an ISOLS($m+1$; 2). But for odd k this is not possible unless $m \neq 7$. When $m = 7$ and $\varepsilon = 2, 3$, put $p = k$, $q = 7$, $x = 1$, $a_{p0} = \varepsilon$ and apply Lemma 1.9; and when $\varepsilon = 6$, repeat but with $a_{p0} = a_{11} = 3$.

Now, if $k \in E$, then $k \geqslant 10$ and $k - 1 \notin F$. We consider separately the cases m odd and m even. Let $m = 2h + 1$. Applying Lemma 1.6 with $p = k - 1$, $q = m$, $x = 1$ and $A_1 = \{1^\varepsilon, h^2\}$, provided that $\varepsilon + 2 \leqslant k - 1$, in conjunction with Corollary 2.5 and Lemma 2.6, we obtain an ISOLS($km + \varepsilon - 1$; $(k-1)^m, m + \varepsilon - 1$). Now Lemma 1.11, applied to this with $x = 1$, gives an ISOLS($km + \varepsilon$; k). However, we still have $\varepsilon \in \{k-2, k-1\}$ to consider. In these cases we repeat the above argument, but with $A_1 = \{1^{k-4}, 2, h^2\}$ and $A_1 = \{1^{k-5}, 2^2, h^2\}$, respectively.

Now let $m = 2h$. Assuming that $\varepsilon \leqslant k - 6$, apply Lemma 1.6 (also using Corollary 2.5, Lemma 2.6 and Theorem 2.8) with $p = k - 1$, $q = 2h - 1$, $x = 1$ and $A_1 = \{2^{(k+\varepsilon)/2}, (h-1)^2\}$ if ε is even, and $A_1 = \{1, 2^{(k+\varepsilon-1)/2}, (h-1)^2\}$ if ε is odd. This can be seen to yield an ISOLS($km + \varepsilon - 1$; $(k-1)^{m-1}, m + k + \varepsilon - 2$), and so by Lemma 1.11 with $x = 1$, an ISOLS($km + \varepsilon$; k).

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We still have to consider the cases $n = km + (k - \delta)$, m even, $1 \leqslant \delta \leqslant 5$. These are all covered by Lemma 1.6 with $p = k/2$ (which is odd), $q = 2m + 1$, $x = 2$, $a_{p0} = 1$ and $A_1 = \{1^{(k/2-\delta-1)}\}$, where $k/2 - \delta - 1 \geqslant 0$ and $k/2 - \delta - 1 \neq 2, 3, 6$. This leaves the cases $k = 10$ and $\delta = 1, 2, 5$; and $k = 14$, $\delta = 3, 4$.

For $k = 10$ and $\delta = 1, 5$, apply Lemma 1.6 with $p = 5$, $q = 2m + 1$, $x = 2$ and $A_0 = \{1^4\}$, A_0 empty, respectively. For $\delta = 2$, apply Lemma 1.8 with $p = m - 1$, $q = 10$ and $A_1 = \{2^9\}$, provided that $m \geqslant 11$. When $m = 8$, put $p = 8$, $q = 10$ and $A_1 = \{2^4\}$ in Lemma 1.8; and when $m = 10$, put $p = 7$, $q = 14$ and $A_0 = \{2^5\}$ in Lemma 1.5.

Finally, for $k = 14$ and $\delta = 3$, apply Lemma 1.6 with $p = 7$, $q = 2m + 1$, $x = 2$ and $A_0 = \{1^4\}$; and for $k = 14$ and $\delta = 4$, apply Lemma 1.6 with $p = 7$, $q = 2m$, $x = 2$ and $A_0 = A_1 = \{1^5\}$.

Theorem 2.10. *If $n = 5k + \varepsilon, 0 \leqslant \varepsilon < 2k$, $k \geqslant 7$, then there is an* ISOLS$(n; k)$.

Proof. Again, the case $\varepsilon = 0$ is immediate by product.

Suppose ε is odd and $\varepsilon \neq 3, 7$. Apply Lemma 1.8 with $p = 5$, $q = k$ and $A_0 = \{[\varepsilon/2]^2\}$, $A_1 = \{1\}$ if $0 < \varepsilon \leqslant k$; $A_0 = \{[k/2]^2, [(\varepsilon - k)/2]^2\}$, $A_1 = \{1\}$ if $k < \varepsilon < 2k$. If ε is even and $\varepsilon \neq 2$, repeat the procedure but with A_1 empty and, if $k < \varepsilon < 2k$, $A_0 = \{[k/2]^2, [(\varepsilon - k + 1)/2]^2\}$. Should $\varepsilon \in \{2, 3, 7\}$, use Lemma 1.8 with $p = 5$, $q = k - 1$ and $A_0 = \{1^3, 2^2\}$, $A_0 = \{1^3, 2^2\}$ and $A_1 = \{1\}$, and $A_0 = \{1, 2^2, 3^2\}$ and $A_1 = \{1\}$, respectively.

Theorem 2.11. *If $n = 4k + \varepsilon, 0 \leqslant \varepsilon < k$, $k \geqslant 7$, then there is an* ISOLS$(n; k)$.

Proof. When $\varepsilon = 0$, product yields the result. If $\varepsilon \neq 2, 3, 6$, apply Lemma 1.8 with $p = 4$, $q = k$, $A_1 = \{[\varepsilon/2]^2\}$ if ε is even, and $A_1 = \{[\varepsilon/2]^2, 1\}$ if ε is odd. For $\varepsilon = 3, 6$, apply Lemma 1.8 with $p = 4$, $q = k - 1$ and $A_1 = \{1, [(\varepsilon + 2)/2]^3\}$. Finally, we consider the case $\varepsilon = 2$. Provided that $k \neq 7, 8$, the same lemma, but with $p = 4$, $q = k - 2$ and $A_1 = \{2^2, 3^2\}$, will work. An ISOLS(30; 7) comes from Lemma 1.6 with $p = 7$, $q = 3$, $x = 1$, $A_0 = \{1^5\}$ and $A_1 = \{1^4\}$. The ISOLS(34; 8) found by Wang [19] is given by $d = 1$ and the vectors $\mathbf{e} = (1, 26, 24, 23, 22, 19, 18, 16, 21, 6, 17, 13, 15, 2, 20, 25, x_1, 12, x_2, 10, x_3, x_4, x_5, x_6, x_7, x_8)$, $\mathbf{f} = (14, 8, 5, 3, 4, 7, 11, 9)$ and $\mathbf{g} = (4, 16, 19, 24, 8, 26, 5, 11)$.

The final case $n = 3k + \varepsilon$, $1 \leqslant \varepsilon < k$, is more difficult and will be done in a series of lemmas.

Lemma 2.12. *There exists an* ISOLS$(3k + 3; k)$ *for $k \geqslant 3$.*

PROOF. Consider the residue class of k modulo 4. Write k as $4t+8$, $4t+9$, $4t+6$ and $4t+3$, $t \geqslant 0$. The "starter-adder" technique will be used. In all four cases $d=1$ and $\mathbf{e}=(1, x_1, x_2, \ldots, x_k, k+3, k+2, \ldots, 2)$, $\mathbf{f}=(k+4, k+5, \ldots, 2k+3)$, and the first $4t$ positions of $\mathbf{g}$ are given by $\mathbf{g}'=(8,6,4,10,\ldots,8j+8, 8j+6, 8j+4, 8j+10, \ldots, 8t, 8t-2, 8t-4, 8t+2)$. Respectively, the last 8, 9, 6, 3 positions of $\mathbf{g}$ are given by $\mathbf{g}''$, where

$$\mathbf{g}''=(8t+6, 8t+10, 8t+4, 8t+8, 8t+12, 8t+18, 8t+16, 8t+14),$$
$$\mathbf{g}''=(8t+8, 8t+6, 8t+4, 8t+12, 8t+10, 8t+16, 8t+20, 8t+14, 8t+18),$$
$$\mathbf{g}''=(8t+8, 8t+6, 8t+4, 8t+14, 8t+10, 8t+12), \text{ and}$$
$$\mathbf{g}''=(8t+8, 8t+6, 8t+4).$$

The cases $k=4,5$ remain. When $k=5$, $\mathbf{g}=(6,10,4,8)$, and when $k=5$, $\mathbf{g}=(10,12,8,6,4)$.

LEMMA 2.13 (Heinrich [12]). *There exists an* ISOLS$(3k+2; k)$ *for* $k \geqslant 3$ *and* k *odd.*

PROOF. Again, we simply give d and vectors $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$. Let $k=2m+1$ and put $d=1$. Then

$$\mathbf{e}=(1, x_1, 2, x_2, \ldots, m+1, x_{m+1}, 4m+4, m+2, x_{m+2}, m+3, x_{m+3}, \ldots, 2m+2),$$
$$\mathbf{f}=(3m+3, 2m+3, 4m+3, 2m+4, 4m+2, \ldots, 3m+2, 3m+4), \text{ and}$$
$$\mathbf{g}=(3m+4, 2m+1, 2, 2m, 3, \ldots, m+2, m+1).$$

LEMMA 2.14. *There exists an* ISOLS$(3k+4; k)$ *for* k *odd*, $k \geqslant 3$.

PROOF. First we do the cases $k=3,5,15$. When $k=3$, put $d=1$, $\mathbf{e}=(1,5,10,9,3,2,8,x_1,x_2,x_3)$, $\mathbf{f}=(4,6,7)$ and $\mathbf{g}=(10,3,5)$; and when $k=5$, put $d=1$, $\mathbf{e}=(1,4,6,8,12,5,3,2,14,x_1,x_2,x_3,x_4,x_5)$, $\mathbf{f}=(7,9,10,11,13)$, and $\mathbf{g}=(2,13,12,10,7)$. For $k=15$, use Lemma 1.6 with $p=5$, $q=9$, $x=3$, $a_{p0}=1$ and $a_{11}=3$. (Note that product yields an IPOLS$(12;3,3)$, and that we have slightly varied the lemma by using an ISOLS$(13;3)$.)

Now, write $k=3^\alpha 5^\beta K$, where $\gcd(K,30)=1$, can assume that $k \neq 3,5,15$. There are four cases to consider, and, in each, Lemma 1.9 is applied with $a_{p0}=a_{11}=a_2=1$. If both $\alpha \neq 1$ and $\beta \neq 1$, put $p=k$, $q=3$ and $x=1$; if $\alpha \neq 1$ but $\beta=1$, put $p=k/5$, $q=15$ and $x=5$; if $\alpha=1$ but $\beta \neq 1$, put $p=k/3$, $q=9$ and $x=3$; and if $\alpha=\beta=1$, put $p=k/15$, $q=45$ and $x=15$.

LEMMA 2.15. *There exists an* ISOLS($3k+7$; k) *when k is odd and* $k \geqslant 3$.

PROOF. The proof is essentially the same as that of Lemma 2.14. We first construct ISOLS($3k+7$; k) for $k = 3, 5$ and 15 and then proceed as before, applying Lemma 1.9 with $a_{p0} = a_2 = 1$ and $A_1 = \{1^4\}$. (Note that we also need Lemmas 2.13 and 2.14.)

An ISOLS(16; 3) and an ISOLS(22; 5) are given by $d = 1$, $\mathbf{e} = (1, 4, x_1, 10, 12, 9, 8, 7, 6, 2, x_2, 3, x_3)$, $\mathbf{f} = (5, 11, 13)$, $\mathbf{g} = (9, 12, 10)$ and by $d = 14$, $\mathbf{e} = (1, x_1, x_2, x_3, x_4, 15, x_5, 7, 3, 16, 12, 8, 4, 17, 13, 9, 5)$, $\mathbf{f} = (2, 6, 10, 11, 14)$ and $\mathbf{g} = (14. 16, 12, 15, 17)$, respectively. To construct an ISOLS(52; 15), apply Lemma 1.6 (as in Lemma 2.14) with $p = 5$, $q = 9$, $x = 3$ and $A_0 = \{1, 3^2\}$.

LEMMA 2.16. *If* $n = 3k + \varepsilon$, $1 \leqslant \varepsilon < k$, $k \notin F \setminus \{3\}$ *and if, when k is even,* $\varepsilon \neq 2, 6$, *then there is an* ISOLS($3k+\varepsilon$; k).

PROOF. Suppose $k \geqslant 5$ and k is odd. Then, for $\varepsilon \neq 3, 4, 7$, we apply Lemma 1.6 with $p = k$, $q = 3$, $x = 1$, $a_{p0} = 1$ and $A_1 = \{1^{\varepsilon-1}\}$. If k is even, $k \notin F$ and $\varepsilon \neq 2, 3, 6$, apply Lemma 1.6 with $p = k$, $q = 3$, $x = 1$ and $A_1 = \{1^\varepsilon\}$ (with $a_{11} = 1$).

All other cases in the statement of the lemma were dealt with in earlier lemmas.

LEMMA 2.17. *If* $k \in F \setminus \{2, 3, 6\}$, *then there exists an* ISOLS($3k+\varepsilon$; k), $1 \leqslant \varepsilon < k$ *and* $\varepsilon \neq 2, 6$.

PROOF. We may assume that $\varepsilon \geqslant 4$, as the other cases have been dealt with. There are twenty-seven values of k to consider.

For $k = 10$, there are five cases: $n \in \{34, 35, 37, 38, 29\}$. When $n = 35$, use Lemma 1.6 with $p = 5$, $q = 7$ and $x = 2$; and for the others use Lemma 1.8 with $p = 4$ and $q = 8$, $a_{11} = 2$; $q = 9$, $a_{11} = 1$; $q = 7$, $A_1 = \{3^3, 1\}$; and $q = 8$, $A_1 = \{2^3, 1\}$; respectively.

For $k \in G = \{14, 46, 54, 58, 62, 74, 82, 98, 118, 142, 194, 202, 214, 278, 394, 398, 422, 1322\}$, $k/2$ is an odd prime power. Apply Lemma 1.9 with $p = k/2$, $q = 6$ and $x = 2$. If $\varepsilon \leqslant k - 5$ is odd, then, if $\varepsilon \leqslant k/2$, put $A_0 = \{1^\varepsilon\}$; and if $\varepsilon > k/2$, put $A_0 = \{1^{k/2}\}$ and $a_2 = \cdots a_b = 1$, where $b = \frac{1}{2}(\varepsilon - k/2) + 1$. If ε is even and $\varepsilon \leqslant k - 4$, then, if $\varepsilon \leqslant k/2$, put $A_0 = \{1^{\varepsilon-3}\}$, $a_{11} = a_2 = 1$; and if $\varepsilon > k/2$, put $A_0 = \{1^{k/2}\}$, $a_{11} = a_2 = \cdots = a_b = 1$ (if $\varepsilon = k/2 + 1$ we just have $a_{11} = 1$), where $b = \frac{1}{2}(\varepsilon - k/2 - 1) + 1$.

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For $k \in H = \{66, 102, 258, 282, 402\}$, $k/3 \notin F$. In Lemma 1.6 put $p = k/3$, $q = 9$, $x = 3$ and $A_1 = \{1^u, 3^v\}$, where $1 \leqslant u + v \leqslant k/3$, $a_{11} = 1$ and $u + 3v = \varepsilon$. Clearly u and v can always be found, provided that $\varepsilon \neq k - 1, k - 3$.

When $k \in G \cup H \setminus \{14, 46, 258, 402\}$, $(k+6)/4 \notin F$ and $(k+6)/4 \geqslant 7$. From the preceding lemmas there is an ISOLS$(n; (k+6)/4)$ for $n \geqslant 3((k+6)/4) + 7$. Using Lemma 1.8, put $p = 4$, $q = 3(k-2)/4$, and $A_1 = \{a_{11} = (k+6)/4, a_{21}, a_{31}, a_{41}\}$, so that $0 \leqslant a_{j1} \leqslant [3(k-2)/8]$, $j = 2, 3, 4$, and $a_{21} + a_{31} + a_{41} = \varepsilon - ((k+6)/4) \geqslant 2((k+6)/4) + 7$; all of these are possible for $k \geqslant 58, 54, 50$ when $\varepsilon = k - 3, k - 2, k - 1$, respectively. This leaves $k = 54$, $\varepsilon = 51$. In this case apply Lemma 1.5 with $p = 5$, $q = 31$, $A_0 = \{13^2, 14^2\}$ and $A_1 = \{1^4\}$.

The cases $k = 14, 46$ and $\varepsilon = k - 3, k - 2, k - 1$; and $k = 258, 402$ and $\varepsilon = k - 3, k - 1$ are shown in Table 2.4. This leaves only the cases $k = 70, 174$ and 230.

TABLE 2.4

k	ε	p	q	A_0	A_1	a_2	Lemma
14	11	7	6	$\{1^5\}$	$\{1^4\}$	1	1.9 $x = 2$
14	12	8	5	–	$\{2^7\}$	–	1.5
14	13	7	6	$\{1^7\}$	$\{1^4\}$	1	1.9 $x = 2$
46	43	5	27	$\{11^2, 12^2\}$	–	–	1.5
46	44	5	27	$\{11^2, 12^2\}$	$\{1\}$	–	1.5
46	45	7	19	$\{7^2, 8^4\}$	$\{1^4\}$	–	1.5
258	255	77	10	$\{1^2, 2^3, 5^{50}\}$	$\{1\}$	–	1.5
258	257	45	17	$\{2, 8^{32}\}$	$\{1^8\}$	–	1.5
402	399	149	8	$\{2, 4^{100}\}$	$\{1^{11}\}$	–	1.5
402	401	149	8	$\{2, 4^{100}\}$	$\{1^{13}\}$	–	1.5

We first look at $k = 174, 230$. Since $k \notin E$, there are at least $k/2$ disjoint symmetric transversals in P, determined by P_1, all of which avoid the main diagonal. Use these in Lemma 1.6 with $p = k$, $q = 3$, $x = 1$ and $A_1 = \{1^\varepsilon\}$ for $4 \leqslant \varepsilon \leqslant k/2 + 1$. (Note that we also use the main diagonal of P, which is a transversal.)

For $k = 174$ and $89 \leqslant \varepsilon \leqslant 169$, write $3k + \varepsilon = 29 \cdot 21 + (\varepsilon - 87)$, where $2 \leqslant \varepsilon - 87 \leqslant 82$, and then in Lemma 1.9 put $p = 29$, $q = 21$, $x = 6$, $A_0 = \{1^t\}$, $A_1 = \{1^s\}$ and $a_2 = \cdots = a_u = 1$, so that $0 \leqslant s, t \leqslant 29$, $s, t \neq 2, 3, 6$, $u \leqslant 13$ and $2 \leqslant t + s + 2(u - 1) \leqslant 82$. (Any ISOLS$(n; 6)$ required which have not already been constructed are given in Lemma 2.22). For the remaining four cases, $170 \leqslant \varepsilon \leqslant 173$, use Lemma 1.8 with $p = 4$, $q = 117$ and $A_1 = \{a_{11} = 57, a_{12}, a_{13}, a_{14}\}$ where $171 \geqslant a_{12} + a_{13} + a_{14} = \varepsilon + 54 - 57 = \varepsilon - 3 \geqslant 115$.

The case $k = 230$ is handled similarly. When $117 \leqslant \varepsilon \leqslant 156$, write $3k + \varepsilon = 23 \cdot 35 + (\varepsilon - 115)$, and in Lemma 1.9 put $p = 23$, $q = 35$, $x = 10$, $A_0 = \{1^t\}$, $A_1 = \{1^s\}$, $a_2 = \cdots = a_u = 1$, so that $t \neq 2, 6$ is even, $0 \leqslant t \leqslant 22$, $s = 0$ or 1,

TABLE 2.5

p	q	ε	A_0	A_1	a_2
7	30	4	$\{1\}$	$\{1\}$	1
7	30	5	$\{1^5\}$	–	–
7	30	7	$\{1^7\}$	–	–
7	30	8	$\{1^5\}$	$\{1\}$	1
7	30	9	$\{1^7\}$	–	1
7	30	10	$\{1^7\}$	$\{1\}$	1
7	30	11	$\{1^4\}$	$\{1^7\}$	–
7	30	12	$\{10, 1^2\}$	–	–
7	30	13	$\{1^4\}$	$\{1^7\}$	–
7	31	14	$\{1^4\}$	$\{1\}$	1
7	32	15	$\{1\}$	–	–
7	32	16	$\{1\}$	$\{1\}$	–
7	32	17	$\{1\}$	–	1
7	32	19	$\{1\}$	$\{1^4\}$	–
7	32	20	$\{1\}$	$\{1^5\}$	–
7	32	21	$\{1\}$	$\{1^4\}$	1
7	32	22	$\{1\}$	$\{1^5\}$	1
7	32	23	$\{1^5\}$	$\{1^4\}$	–
7	32	24	$\{1^5\}$	$\{1^5\}$	–
7	32	25	$\{1^5\}$	$\{1^4\}$	1
7	32	26	$\{1^5\}$	$\{1^5\}$	1
7	33	27	$\{1\}$	$\{1^5\}$	–
7	33	28	$\{1^7\}$	–	–
7	33	29	$\{1^4\}$	$\{1^4\}$	–
7	33	30	$\{1^5\}$	$\{1^4\}$	–
7	33	31	$\{1^5\}$	$\{1^5\}$	–
7	34	32	–	$\{1^4\}$	–
7	34	33	–	$\{1^5\}$	–
7	34	34	–	$\{1^4\}$	1
7	34	35	$\{1^5\}$	–	1
7	34	36	$\{1^5\}$	$\{1\}$	1
7	34	37	–	–	1
7	35	38	–	$\{1\}$	1
7	35	39	$\{1^4\}$	–	–
7	35	41	$\{1^4\}$	–	1
7	35	42	$\{1^4\}$	$\{1\}$	1
7	36	43	$\{1\}$	–	–
7	36	44	$\{1\}$	$\{1\}$	–
7	36	45	$\{1\}$	–	1
7	36	46	$\{1\}$	$\{1\}$	1
7	36	47	$\{1^5\}$	–	–

and, if $s = 1$, then at least one a_j is non-zero, and $u \leqslant 10$. It follows that $2 \leqslant s + t + 2(u - 1) \leqslant 41$. (All the restrictions on s and t are to ensure that an ISOLS(36; 10) is not required.) For $157 \leqslant \varepsilon \leqslant 229$, we use Lemma 1.8 with $p = 4$, $q = 155$ and $A_1 = \{a_{11} = 75, a_{12}, a_{13}, a_{14}\}$, where $231 \geqslant a_{12} + a_{13} + a_{14} = \varepsilon + 70 - 75 = \varepsilon - 5 \geqslant 152$.

This leaves only the case $k = 70$. The values given in Table 2.5 used in Lemma 1.6 or 1.9 with $x = 10$ cover all cases $\varepsilon \leqslant 47$ except $\varepsilon = 18$, when we set $p = 5$,

$q = 42$, $x = 14$ and $A_0 = \{14, 1^4\}$ in Lemma 1.6, and $\varepsilon = 40$, when we put $p = 20$, $q = 9$ and $A_1 = \{4^{17}, 1^2\}$ in Lemma 1.5. When $48 \leqslant \varepsilon \leqslant 70$, use Lemma 1.8 with $p = 4$, $q = 47$ and $A_1 = \{a_{11} = 23, a_{12}, a_{13}, a_{14}\}$, where $69 \geqslant a_{12} + a_{13} + a_{14} = \varepsilon + 22 - 23 = \varepsilon - 1 \geqslant 47$.

THEOREM 2.18. *If $n = 3k + \varepsilon$, $1 \leqslant \varepsilon < k$ and $\varepsilon \neq 2, 6$ if k is even, then there exists an* ISOLS(n; k).

PROOF. Combine the last six lemmas.

Although many infinite families of ISOLS($6m + 6$; $2m$) can be constructed, we know of no ISOLS($6m + 2$; $2m$) and computer search verifies that the smallest of these, an ISOLS(8; 2), does not exist.

In order to complete our work it remains only to construct ISOLS(n; k), $n \geqslant 3k + 1$ and $3 \leqslant k \leqslant 6$. It follows from the existence of ISOLS(n; m) for $n \geqslant 3m + 1$ when m is odd, and for $n \geqslant 3m + 7$ when m is even, and of an ISOLS($3k + 1$; k), $k \in \{3, 4, 5, 6\}$, that an ISOLS(n; k), $k \in \{3, 5\}$, exists for all $n \geqslant 9k + 10$, and that ISOLS(n; k), $k \in \{4, 6\}$, exist for all $n \geqslant 9k + 4$. Many of the small orders have been constructed by other authors (for example Bennett [1], Bennett and Mendelsohn [2], Drake and Larson [8] and Wang [19]).

THEOREM 2.19. *There exists an* ISOLS(n; 3) *for all $n \geqslant 10$.*

PROOF. We have only to consider integers $n \in \{m\colon 14 \leqslant m \leqslant 37\} \setminus \{16\}$. First, the cases $n \in \{14, 17, 20, 21, 22, 26, 32\}$ are given by d and by the vectors $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ in Table 2.6. An ISOLS(33; 3) exists as there exists an ISOLS(33; 10). Applying Lemma 1.5 with $q = 3$, and with the other variables as in Table 2.7, we obtain the remainder.

THEOREM 2.20. *There exists an* ISOLS(n; 4) *for all $n \geqslant 13$, except perhaps for $n = 14$.*

PROOF. Only n in the range $16 \leqslant n \leqslant 39$ need be considered. If $n = 4t$, $t \neq 6$, product gives the result, and if $n = 4t + 1$, $t \neq 6$, we use Lemma 1.8 with $p = t$, $q = 4$ and $A_1 = \{1\}$ ($a_{11} = 0$). The cases $n = 18, 26, 30$ were found by Wang [19] and are constructed using the starter-added technique, as also are $n = 19, 23, 27$ (see Table 2.8). The remaining constructions are given in Table 2.9.

THEOREM 2.21. *There exists an* ISOLS(n; 5) *for all $n \geqslant 16$.*

TABLE 2.6

n	d	**e**	**f**	**g**
14	9	$(1, x_1, x_2, 3, 11, x_3, 5, 2, 10, 7, 4)$	(6, 8, 9)	(10, 11, 7)
17	1	$(1, 3, 5, 8, x_1, 13, 12, 7, x_2, x_3, 6, 9, 2, 10)$	(4, 11, 14)	(7, 13, 9)
20	14	$(1, x_1, x_2, 6, 2, 15, 11, 7, 3, 16, 12, 8, x_3, 17, 13, 9, 5)$	(4, 10, 14)	(16, 17, 6)
21	1	$(1, x_1, x_2, x_3, 2, 4, 8, 13, 16, 9, 15, 7, 6, 5, 3, 18, 11, 10)$	(12, 14, 17)	(9, 4, 15)
22	5	$(1, x_1, x_2, 13, 17, 2, 6, 10, x_3, 18, 3, 7, 11, 15, 19, 4, 8, 12, 16)$	(5, 9, 14)	(18, 12, 19)
26	1	$(1, x_1, x_2, x_3, 2, 4, 6, 9, 12, 18, 22, 16, 15, 21, 20, 8, 10, 7, 5, 3, 11, 17, 14)$	(13, 19, 23)	(11, 18, 20)
32	1	$(1, x_1, x_2, x_3, 24, 6, 9, 11, 13, 15, 26, 8, 21, 28, 27, 25, 23, 29, 16, 14,$ $12, 10, 7, 5, 3, 19, 22, 7)$	(18, 20, 24)	(19, 16, 21)

TABLE 2.7

n	p	A_0	A_1
15	4	–	$\{1^3\}$ $(a_{11} = 1)$
18	5	$\{1^3\}$	–
19	5	$\{1^3\}$	$\{1\}$
23	5	$\{1^3\}$	$\{1^5\}$
24	7	$\{1^3\}$	–
25	7	$\{1^3\}$	$\{1\}$
27	8	–	$\{1^3\}$ $(a_{11} = 1)$
28	7	$\{1^3\}$	$\{1^4\}$
29	7	$\{1^3\}$	$\{1^5\}$
30	9	$\{1^3\}$	–
31	9	$\{1^3\}$	$\{1\}$
34	9	$\{1^3\}$	$\{1^4\}$
35	9	$\{1^3\}$	$\{1^5\}$
36	11	$\{1^3\}$	–
37	11	$\{1^3\}$	$\{1\}$

TABLE 2.8

n	d	
18	1	$\mathbf{e} = (1, 5, x_1, x_2, x_3, x_4, 2, 9, 11, 6, 8, 10, 12, 7)$ $\mathbf{f} = (3, 4, 13, 14)$ $\mathbf{g} = (7, 6, 5, 9)$
19	1	$\mathbf{e} = (1, x_1, x_2, x_3, 11, 3, x_4, 4, 8, 14, 6, 2, 7, 15, 13)$ $\mathbf{f} = (5, 9, 10, 12)$ $\mathbf{g} = (4, 3, 8, 9)$
23	7	$\mathbf{e} = (1, x_1, 13, x_2, 6, x_3, 18, 5, 11, 17, x_4, 10, 16, 3, 9, 15, 2, 8, 14)$ $\mathbf{f} = (4, 7, 12, 19)$ $\mathbf{g} = (10, 19, 17, 15)$
26	1	$\mathbf{e}(1, 19, x_1, x_2, x_3, x_4, 12, 15, 17, 2, 22, 13, 7, 5, 3, 6, 10, 14, 16, 18, 20, 9)$ $\mathbf{f} = (4, 8, 11, 21)$ $\mathbf{g} = (3, 5, 4, 7)$
27	18	$\mathbf{e} = (1, x_1, 12, 6, x_2, 17, 11, 5, 22, 16, 10, x_3, 21, 15, 9, 3, x_4, 14, 8, 2, 19, 13, 7)$ $\mathbf{f} = (4, 8, 20, 23)$ $\mathbf{g} = (20, 13, 23, 8)$
30	1	$\mathbf{e} = (1, 11, x_1, x_2, x_3, x_4, 17, 13, 26, 23, 4, 6, 21, 15, 3, 2, 12, 10, 8, 5, 25, 18, 20, 22, 24, 7)$ $\mathbf{f} = (14, 9, 16, 19)$ $\mathbf{g} = (3, 4, 7, 17)$

TABLE 2.9

n	p	q	A_0	A_1	Lemma
22	7	3	$\{1\}$	–	1.8
24	5	4	$\{1^4\}$	–	1.5
25	5	4	$\{2^2\}$	$\{1\}$	1.5
31	10	3	$\{1\}$	–	1.8
34	11	3	$\{1\}$	–	1.8
35	4	7	–	$\{1, 3^2\}$	1.6 $x = 1$
38	11	3	$\{1\}$	$\{1^4\}$	1.5
39	5	7	$\{1^4\}$	–	1.5

TABLE 2.10

n	d	
28	17	$\mathbf{e} = (1, x_1, 10, x_2, 19, 12, 5, 21, 14, 7, 23, 16, 9, 2, x_3, 11, 4, 20, 13, x_4, 22, x_5, 8)$ $\mathbf{f} = (3, 6, 15, 17, 18)$ $\mathbf{g} = (10, 5, 23, 3, 21)$
30	8	$\mathbf{e} = (1, x_1, 15, 22, 4, x_2, 18, 25, 7, 14, 21, x_3, x_4, 17, 24, 6, 13, 20, 2, 9, 16, 23, x_5, 12, 19)$ $\mathbf{f} = (3, 5, 8, 10, 11)$ $\mathbf{g} = (25, 21, 14, 15, 4)$
34	6	$\mathbf{e} = (1, x_1, 11, 16, 21, 26, 2, 7, 12, 17, 22, 27, x_2, x_3, x_4, 18, 23, 28, 4, 9, x_5, 19, 24, 29, 5, 10, 15, 20, 25)$ $\mathbf{f} = (3, 6, 8, 13, 14)$ $\mathbf{g} = (10, 29, 18, 17, 16)$

TABLE 2.11

n	p	q	A_0	A_1	Lemma
23	4	4	–	$\{1, 2^3\}$	1.8
24	4	5	–	$\{2^2\}$	1.8
26	7	3	$\{1^5\}$	–	1.5
27	5	5	–	$\{2\}$	1.8
29	7	4	$\{1\}$	–	1.8
31	7	4	$\{1\}$	$\{2\}$	1.8
32	9	3	$\{1^5\}$	–	1.5
33	8	4	–	$\{1\}$	1,8
37	5	7	–	$\{2\}$	1.6 $x = 1$
38	5	7	$\{1\}$	$\{2\}$	1.6 $x = 1$
39	5	7	$\{2^2\}$	–	1.6 $x = 1$
42	5	8	–	$\{2\}$	1.6 $x = 1$
43	5	8	–	$\{3\}$	1.6 $x = 1$
44	5	8	$\{2^2\}$	–	1.6 $x = 1$
47	5	9	–	$\{2\}$	1.6 $x = 1$
48	5	9	–	$\{3\}$	1.6 $x = 1$
49	5	9	–	$\{4\}$	1.6 $x = 1$
52	5	10	$\{1\}$	$\{1\}$	1.6 $x = 1$
53	5	10	–	$\{3\}$	1.6 $x = 1$
54	5	10	$\{2^2\}$	–	1.6 $x = 1$

TABLE 2.12

n	p	q	A_0	A_1	a_2	Lemma
22	4	4	–	$\{2^3\}$	–	1.5
24	7	3	$\{1\}$	–	1	1.8
25	7	3	$\{1\}$	$\{1\}$	1	1.8
26	4	5	–	$\{2^3\}$	–	1.5
27	5	4	$\{1\}$	$\{2^3\}$	–	1.5
28	7	3	$\{1\}$	$\{1^6\}$	–	1.5
29	7	3	$\{1\}$	$\{1^5\}$	1	1.8
30	8	3	–	$\{1^6\}$	–	1.5
31	5	5	$\{2^3\}$	–	–	1.5
32	7	3	$\{1^5\}$	$\{1^6\}$	–	1.5
34	4	7	–	$\{2^3\}$	–	1.5
35	7	4	$\{1\}$	$\{2^3\}$	–	1.5
36	11	3	$\{1\}$	–	1	1.8
37	11	3	$\{1\}$	$\{1\}$	1	1.8
38	4	8	–	$\{2^3\}$	–	1.5
39	7	4	$\{1^5\}$	$\{2^3\}$	–	1.5
40	9	3	$\{1^7\}$	$\{1^6\}$	–	1.5
41	7	5	$\{2^3\}$	–	–	1.5
42	13	3	$\{1\}$	–	1	1.8
43	13	3	$\{1\}$	$\{1\}$	1	1.8
44	11	3	$\{1^5\}$	$\{1^6\}$	–	1.5
45	5	7	$\{1^4\}$	$\{2^3\}$	–	1.5
46	4	10	–	$\{2^3\}$	–	1.5
47	9	4	$\{1^5\}$	$\{2^3\}$	–	1.5
48	11	3	$\{1^9\}$	$\{1^6\}$	–	1.5
49	9	4	$\{1^7\}$	$\{2^3\}$	–	1.5
50	11	4	$\{1^6\}$	–	–	1.5
51	11	4	$\{1\}$	$\{2^3\}$	–	1.5
52	5	9	$\{1\}$	$\{2^3\}$	–	1.5
53	13	3	$\{1^{11}\}$	$\{1\}$	1	1.8
54	4	12	–	$\{2^3\}$	–	1.5
55	7	7	$\{2^3\}$	–	–	1.5
56	5	10	$\{2^3\}$	–	–	1.5
57	5	10	$\{1\}$	$\{2^3\}$	–	1.5

PROOF. As before we need only consider n in the range $20 \leqslant n \leqslant 54$, but with $n \neq 5t$ or $5t + 1$ unless $n = 26, 30, 31$, as in the first case, product suffices, and, in the second, Lemma 1.6 applies with $p = 5$, $q = t$, $x = 1$ and $A_0 = \{1\}$. The cases $n = 28, 30, 34$ are constructed from vectors as in Table 2.10, and the remainder are given in Table 2.11. Recall that an ISOLS(22; 5) was given in Lemma 2.15.

THEOREM 2.22. *There exists an* ISOLS(n; 6) *for all* $n \geqslant 19$, *except perhaps for* $n = 20$.

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PROOF. For $n = 19$ and 21, ISOLS(n; 6) have been given. Only two, $n = 23$ and $n = 33$ are constructed via the "starter-adder" method. For $n = 23$, put $d = 15$, $\mathbf{e} = (1, x_1, x_2, x_3, x_4, x_5, 17, 14, x_6, 8, 5, 2, 16, 13, 10, 7, 4)$, $\mathbf{f} = (3, 6, 9, 11, 12, 15)$ and $\mathbf{g} = (16, 15, 10, 14, 17, 13)$; and for $n = 33$, put $d = 1$, $\mathbf{e} = (1, x_1, x_2, x_3, x_4, x_5, x_6, 2, 4, 6, 8, 10, 12, 20, 22, 21, 19, 9, 7, 5, 3, 25, 27, 11, 18, 15, 13)$, $\mathbf{f} = (14, 16, 17, 23, 24, 26)$ and $\mathbf{g} = (12, 11, 18, 9, 20, 2)$. The remainder are given in Table 2.12.

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