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COMPOSITION OPERATORS AND SEVERAL COMPLEX VARIABLES

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Let U^n be the open unit polydisc, and let T be a mapping from U^n into itself. Then the composition transformation C_T is a mapping on the Hardy space $H^2(U^n)$ into the space of complex functions on U^n defined as $C_T f = f \circ T$ for every $f \in H^2(U^n)$. An attempt is made to study some properties of C_T in this note. A partial generalization of a result of Schwartz, and a relation between intertwining analytic Toeplitz operators and composition operators are reported.

1. Introduction

Let U be the open unit disc in the complex plane and ∂U its boundary. Let U^n and $(\partial U)^n$ denote the Cartesian products of n copies of U and ∂U respectively. If $H^2(U^n)$ is the Hilbert space of functions f holomorphic in U^n for which

$$\|f\|^2 = \sup_{0<r<1} \left\{ \int_{(\partial U)^n} |f(rw)|^2 dm_n(w) \right\} < \infty ,$$

where m_n is the normalized Lebesgue measure on $(\partial U)^n$, and $T : U^n \to U^n$

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is a holomorphic mapping (for definitions, see [2] and [4]), then the composition transformation C_T on $H^2(U^n)$ is defined as

$$C_T f = f \circ T \quad \text{for every} \quad f \in H^2(U^n) .$$

If C_T maps $H^2(U^n)$ into itself, then by an application of the closed graph theorem C_T is a bounded linear operator on $H^2(U^n)$. In case C_T is bounded, we call it a composition operator induced by T. The composition operators have been studied on various function spaces including the classical Hardy space (see, for example, [5], [6] and [7]). A study of these operators on $H^2(U^n)$, in the case when $n = 2$, is made in this paper. For $L \in H^\infty(U^n)$, the Banach algebra of bounded holomorphic functions on U^n, the Toeplitz operator M_L on $H^2(U^n)$ is defined by

$$(M_L f)(z) = L(z) \cdot f(z) .$$

NOTATIONS. If $0 < r \le 1$, then U_r will stand for the open disc $\{z \in \mathbb{C} : |z| < r\}$; in particular $U = U_1$ and we write U_r^n for the Cartesian product of n copies of U_r. Similarly, by $(\partial U)_r^n$ we mean the Cartesian product of n copies of $(\partial U)_r$, where $(\partial U)_r = \{z \in \mathbb{C} : |z| = r\}$ and $(\partial U)_1 = \partial U$. If $z = (z_1, z_2) \in U^2$ and $\alpha = (\alpha_1, \alpha_2) \in Z_+^2 \; (= Z_+ \times Z_+)$, we write z^α for the monomial $z_1^{\alpha_1} \cdot z_2^{\alpha_2}$, where Z_+ is the set of all non-negative integers. By $B(H^2(U^n))$ we denote the Banach algebra of all bounded linear operators on $H^2(U^n)$.

DEFINITIONS. A continuous complex valued function f on an open subset of $\mathbb{C}^n$ is n-harmonic if f is harmonic in each variable separately [4, p. 16].

Suppose T is a holomorphic map from U^n into itself. Then T is said to be proper if $T^{-1}(E)$ is a compact subset of U^n for every compact subset E of U^n.

A biholomorphic mapping of a domain onto itself is called an automorphism [8, p. 47].

2. Boundedness and norm estimates

We begin this section with the following lemmas.

LEMMA 2.1. *Let* f *be a holomorphic function from* U^2 *onto a domain* D *of* U *, and let* $g : D \to \mathbb{C}$ *be real harmonic. Then* $g \circ f$ *is 2-harmonic.*

The proof of the lemma follows from the facts that every real harmonic function of a complex variable is the real part of a holomorphic function and that real parts of holomorphic functions of two complex variables are 2-harmonic.

LEMMA 2.2. *Let* $0 < s \le 1$ *and let* f *be 2-harmonic on* U_s^2 *. Then*

$$f(0, 0) = \int_{(\partial U)^2} f(rw)\,dm_2(w) ,$$

where $w = (w_1, w_2)$ *and* $0 < r < s$ *.*

Proof.

$$\begin{aligned} f(0, 0) &= \int_{\partial U} f(rw_1, 0)\,dm_1(w_1) \\ &= \int_{\partial U}\int_{\partial U} f(rw_1, rw_2)\,dm_1(w_1)\,dm_1(w_2) . \end{aligned}$$

From Fubini's theorem, we have

$$f(0, 0) = \int_{(\partial U)^2} f(rw)\,dm_2(w) .$$

We now give the main theorem of this section.

THEOREM 2.1. *If* $T : U^2 \to U^2$ *is a holomorphic function such that* $T_1(z) = a(|a| < 1)$ *, where* $T(z) = (T_1(z), T_2(z))$ *for every* $z = (z_1, z_2) \in U^2$ *, then* $C_T \in B(H^2(U^2))$ *and*

$$\|C_T\| \le ((1+\delta)/(1-\delta))^{\frac{1}{2}} \cdot ((1+|a|)/(1-|a|))^{\frac{1}{2}} ,$$

where $\delta = |T_2(0, 0)|$.

Proof. Let us choose r sufficiently close to 1 such that T maps the closed polydisc $\overline{U}_q^2$ $(q < 1)$ into U_r^2 Then for $z \in U^2$ and $f \in H^2(U^2)$ we have by Poisson's integral,

$$\begin{aligned}[f(T(z))]^2 &= [f(T_1(z), T_2(z))]^2 \\ &= \int_{(\partial U)^2} [f(rw)]^2 \\ &\quad \cdot\left\{\left(r^2-|T_1(z)|^2\right)/\left(|rw_1-T_1(z)|^2\right)\right\}\cdot\left\{\left(r^2-|T_2(z)|^2\right)/\left(|rw_2-T_2(z)|^2\right)\right\}dm_2(w) .\end{aligned}$$

Taking absolute values and using the fact that T_1 is constant we get

$$|f(T(z))|^2 \le ((r+|a|)/(r-|a|)) \int_{(\partial U)^2} |f(rw)|^2 \cdot\left\{\left(r^2-|T_2(z)|^2\right)/\left(|rw_2-T_2(z)|^2\right)\right\}dm_2(w) .$$

Integrating on $(\partial U)_q^2$, we have

$$\text{(2.1)}\quad \int_{(\partial U)^2} |f(T(qw'))|^2 dm_2(w') \le ((r+|a|)/(r-|a|)) \int_{(\partial U)^2}\int_{(\partial U)^2} |f(rw)|^2 \cdot\left\{\left(r^2-|T_2(qw')|^2\right)/\left(|rw_2-T_2(qw')|^2\right)\right\}dm_2(w)dm_2(w') .$$

We claim that

$$\int_{(\partial U)^2} \left\{\left(r^2-|T_2(qw')|^2\right)/\left(|rw_2-T_2(qw')|^2\right)\right\}dm_2(w') = \left(r^2-|T_2(0, 0)|^2\right)/\left(|rw_2-T_2(0, 0)|^2\right) .$$

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Since we know that the Poisson kernel $\left(s^2-|z|^2\right)/\left(|se^{i\theta}-z|^2\right)$ is harmonic for $|z| < s$, by Lemma 2.1,

$$\left(r^2-|T_2(z)|^2\right)/\left(|rw_2-T_2(z)|^2\right) \text{ is 2-harmonic in } \bar{U}_q^2 .$$

Hence an application of Lemma 2.2 completes the proof of the required claim.

In light of the above claim, (2.1) yields

$$\int_{(\partial U)^2} |f(T(qw'))|^2 dm_2(w')$$
$$\le \left((r+|a|)/(r-|a|)\right)\left\{(r+|T_2(0, 0)|)/(r-|T_2(0, 0)|)\right\} \int_{(\partial U)^2} |f(rw)|^2 dm_2(w) .$$

If r tends to 1, then the above inequality becomes

$$\|C_T f\| \le \left((1+|a|)/(1-|a|)\right)^{\frac{1}{2}}\left((1+\delta)/(1-\delta)\right)^{\frac{1}{2}} f .$$

This proves that C_T is bounded, and

$$\|C_T\| \le \left((1+|a|)/(1-|a|)\right)^{\frac{1}{2}}\left((1+\delta)/(1-\delta)\right)^{\frac{1}{2}} .$$

REMARKS. 1. If T_2 is constant, say b, in the statement of the above theorem, then a similar result can be proved. In this case

$$\|C_T\| \le \left((1+|b|)/(1-|b|)\right)^{\frac{1}{2}}\left((1+|T_1(0, 0)|)/(1-|T_1(0, 0)|)\right)^{\frac{1}{2}} .$$

2. It would be nice to prove the above theorem in a more general form; that is, when both T_i's are non-constant.

Schwartz [6] proved that if T is a holomorphic function from U into itself, then $C_T \in B\left(H^2(U)\right)$. We give a partial generalization of this result to $H^2\left(U^2\right)$ and finally to $H^2\left(U^n\right)$ with $n > 2$ in the following theorems.

THEOREM 2.2. *Let t_1 and t_2 be two holomorphic functions from U into itself and let $T : U^2 \to U^2$ be such that*

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$T(z_1, z_2) = (t_1(z_1), t_2(z_2))$ for every $(z_1, z_2) \in U^2$. Then $C_T \in B(H^2(U^2))$ and

$$\|C_T\| \le ((1+|t_1(0)|)/(1-|t_1(0)|))^{\frac{1}{2}} \cdot ((1+|t_2(0)|)/(1-|t_2(0)|))^{\frac{1}{2}} .$$

Proof. We have, as in the proof of Theorem 2.1, for $f \in H^2(U^2)$,

$$\int_{(\partial U)^2} |f(T(qw'))|^2 dm_2(w')$$

$$\le \int_{(\partial U)^2} \int_{(\partial U)^2} |f(rw)|^2 \cdot \left[\left(r^2-|t_1(qw_1')|^2\right)/\left(|rw_1-t_1(qw_1')|^2\right)\right] \cdot \left[\left(r^2-|t_2(qw_2')|^2\right)/\left(|rw_2-t_2(qw_2')|^2\right)\right] \cdot dm_2(w)dm_2(w')$$

$$= \int_{\partial U}\int_{\partial U}\int_{\partial U}\int_{\partial U} |f(rw)|^2 \cdot \left[\left(r^2-|t_1(qw_1')|^2\right)/\left(|rw_1-t_1(qw_1')|^2\right)\right] \cdot \left[\left(r^2-|t_2(qw_2')|^2\right)/\left(|rw_2-t_2(qw_2')|^2\right)\right] \cdot dm_1(w_1)dm_1(w_1')dm_1(w_2)dm_1(w_2') .$$

The rest of the proof is similar to that of Theorem 1 of Ryff [5].

COROLLARY 2.1. *If $t_1 = t_2 = t$, then*

$$\|C_T\| \le (1+|t(0)|)/(1-|t(0)|) .$$

Theorem 2.2 can easily be generalized to $H^2(U^n)$ with $n > 2$. We state the result without proof in the following theorem.

THEOREM 2.3. *Let $(i_1, \ldots, i_n)$ be a permutation of $(1, \ldots, n)$ and let $t_k : U \to U$ be holomorphic for $1 \le k \le n$. If $T : U^n \to U^n$ is defined by $T(z_1, \ldots, z_n) = (t_1(z_{i_1}), \ldots, t_n(z_{i_n}))$, then $C_T \in B(H^2(U^n))$ and*

$$\|C_T\| \le \prod_{k=1}^{n} ((1+|t_k(0)|)/(1-|t_k(0)|))^{\frac{1}{2}} .$$

COROLLARY 2.2. *Let* $T : U^n \to U^n$ *be a proper holomorphic map. Then* C_T *is a composition operator on* $H^2(U^n)$.

Proof. By Theorem 4.3.3 of [4], there exist n holomorphic functions $t_1, \ldots, t_n$ such that

$$T(z_1, \ldots, z_n) = \left(t_1\left(z_{i_1}\right), \ldots, t_n\left(z_{i_n}\right)\right)$$

for $z = (z_1, \ldots, z_n) \in U^n$ and a suitable permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$. Hence, by Theorem 2.3, C_T is bounded.

COROLLARY 2.3. *If* T *is an automorphism of* U^n *, then* C_T *is a composition operator on* $H^2(U^n)$.

The proof follows from the corollary to Theorem 7.3.3 of [4] and Theorem 2.3 above.

The family of functions $e_\alpha(z) = z^\alpha$ for $\alpha = (\alpha_1, \alpha_2) \in Z_+^2$ is an orthonormal basis for the Hilbert space $H^2(U^2)$. If $y = (y_1, y_2) \in U^2$, then the reproducing kernel k_y of $H^2(U^2)$ is given by the relation

$$\langle f, k_y \rangle = f(y) \text{ , for every } f \in H^2(U^2) .$$

Using Problem 30 of [3], it can be shown that

$$k_y(z) = \left(1-z_1\bar{y}_1\right)^{-1} \cdot \left(1-z_2\bar{y}_2\right)^{-1} \text{ for } z = (z_1, z_2) \in U^2 .$$

Furthermore, the function k_y is itself in $H^2(U^2)$ and its norm is given by

$$\begin{aligned} \|k_y\|^2 = \langle k_y, k_y \rangle &= k_y(y) \\ &= \left(1-|y_1|^2\right)^{-1} \cdot \left(1-|y_2|^2\right)^{-1} . \end{aligned}$$

In the following theorem these functions are used as effective tools to obtain a lower bound for the norm of a composition operator.

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THEOREM 2.4. *If* C_T *is a composition operator on* $H^2(U^2)$ *, then*

$$\sup_{z=(z_1,z_2)\in U^2} \left\{\left(\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)\right)/\left(\left(1-|T_1(z)|^2\right)\left(1-|T_2(z)|^2\right)\right)\right\} \le \|C_T\|^2 .$$

We require some additional machinery to prove the theorem.

Let f be holomorphic in U^2 . Then $f(z) = \sum c(\alpha)z^\alpha$, $\alpha = (\alpha_1 \cdot \alpha_2) \in Z_+^2$. The function f is in $H^2(U^2)$ if and only if $\sum_\alpha |c(\alpha)|^2 < \infty$. In fact

$$\|f\| = \left\{\sum_\alpha |c(\alpha)|^2\right\}^{\frac{1}{2}}$$

[4, p. 50].

LEMMA 2.3. *Let* $f \in H^2(U^2)$. *Then*

$$|f(z)| \le \|f\|\left(1-|z_1|^2\right)^{-\frac{1}{2}}\left(1-|z_2|^2\right)^{-\frac{1}{2}} \quad \textit{for} \quad z \in U^2 . \tag{2.2}$$

Proof. Let $f \in H^2(U^2)$. Then we have

$$f(z) = \sum_\alpha c(\alpha)z^\alpha .$$

Therefore

$$\begin{aligned} |f(z)| &\le \sum_\alpha |c(\alpha)|\cdot|z^\alpha| \\ &\le \left\{\sum_\alpha |c(\alpha)|^2\right\}^{\frac{1}{2}}\cdot\left\{\sum_\alpha |z_1|^{2\alpha_1}\cdot|z_2|^{2\alpha_2}\right\}^{\frac{1}{2}} \\ &= \|f\|\cdot\left(1-|z_1|^2\right)^{-\frac{1}{2}}\cdot\left(1-|z_2|^2\right)^{-\frac{1}{2}} . \end{aligned}$$

Hence the proof of the lemma is completed.

Proof of Theorem 2.4. For a fixed $z \in U^2$, we have

$$\|k_{T(z)}\|^2 = k_{T(z)}(T(z)) \tag{2.3}$$
$$= \left(1-|T_1(z)|^2\right)^{-1} \cdot \left(1-|T_2(z)|^2\right)^{-1} .$$

Applying (2.2) to $k_{T(z)} \circ T$ and using (2.3), we have

$$\|k_{T(z)}\|^2 = k_{T(z)}(T(z))$$
$$\leq \|C_T k_{T(z)}\| \cdot \left(1-|z_1|^2\right)^{-\frac{1}{2}}\left(1-|z_2|^2\right)^{-\frac{1}{2}}$$
$$\leq \|C_T\|\|k_{T(z)}\| \cdot \left(1-|z_1|^2\right)^{-\frac{1}{2}} \cdot \left(1-|z_2|^2\right)^{-\frac{1}{2}} .$$

Thus

$$\|k_{T(z)}\| \leq \|C_T\| \cdot \left(1-|z_1|^2\right)^{-\frac{1}{2}} \cdot \left(1-|z_2|^2\right)^{-\frac{1}{2}} ,$$

which implies that

$$\left(\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)\right) / \left(\left(1-|T_1(z)|^2\right)\left(1-|T_2(z)|^2\right)\right) \leq \|C_T\|^2 .$$

Since $z \in U^2$ is arbitrary, the result follows.

COROLLARY 2.4. *If* C_T *is a composition operator on* $H^2(U^2)$ *and* $T(0, 0) = (0, 0)$ *, then* $\|C_T\| = 1$.

The proof follows from Theorems 2.1 and 2.4.

3. Intertwining analytic Toeplitz operators on $H^2(U^2)$

DEFINITION. Let A and B be bounded linear operators on a Hilbert space H . We say that a bounded linear operator X intertwines A and B if $XA = BX$.

Let $t \in H^\infty(U)$ be univalent. Then define T on U^2 as $T(z_1, z_2) = t(z_1)$ for $(z_1, z_2) \in U^2$. Clearly $T \in H^\infty(U^2)$. Also let $L \in H^\infty(U^2)$. In this section we give a sufficient condition for existence of a non-zero bounded linear operator X which intertwines M_T and M_L .

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THEOREM 3.1. *Let* t, T *and* L *be as above. If the range of* L *is a subspace of the range of* t *, then there exists a non-zero bounded linear operator* X *on* $H^2(U^2)$ *satisfying the condition* $XM_T = M_L X$.

Proof. Since t is one-to-one and the range of L is a subspace of the range of t , the function $F_1(z_1, z_2) = t^{-1}(L(z_1, z_2))$ is a holomorphic mapping from U^2 into U . Then

$$F(z_1, z_2) = (F_1(z_1, z_2), 0)$$

is a holomorphic mapping from U^2 into U^2 . Define X on $H^2(U^2)$ by

$$\begin{aligned}(Xf)(z_1, z_2) &= f(F_1(z_1, z_2), 0) \\ &= f(F(z_1, z_2)) .\end{aligned}$$

Then, by Theorem 2.1, X is a bounded linear operator on $H^2(U^2)$ and

$$\begin{aligned}(XM_T)(f)(z_1, z_2) &= (X(T\cdot f))(z_1, z_2) \\ &= (T\cdot f)(F(z_1, z_2)) \\ &= T(F_1(z_1, z_2), 0)\cdot f(F(z_1, z_2)) \\ &= L(z_1, z_2)\cdot (Xf)(z_1, z_2) \\ &= (M_L X)(f)(z_1, z_2) ,\end{aligned}$$

for every $(z_1, z_2) \in U^2$ and hence

$$(XM_T)(f) = (M_L X)(f)$$

for every $f \in H^2(U^2)$, which implies that

$$XM_T = M_L X .$$

This completes the proof of the theorem.

EXAMPLE. Let $t : U \to U$ be the identity map and let $L(z_1, z_2) = z_1 z_2$ for $(z_1, z_2) \in U^2$. Then $L \in H^\infty(U^2)$ and also the function T defined as

$$T(z_1, z_2) = t(z_1)$$

is in $H^{\infty}(U^2)$. Clearly the bounded linear operator X on $H^2(U^2)$ defined by the relation

$$(Xf)(z_1, z_2) = f(z_1 z_2, 0)$$

intertwines M_T and M_L.

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