SOME RESULTS INVOLVING HYPERGEOMETRIC AND E-FUNCTIONS

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1. Recently R. S. Varma [**11**] gave the generalisation

$$\phi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} W_{k,m}(px)\, h(x)\, dx \quad \text{.............(1)}$$

for the Laplace transform

$$f(p) = p\int_0^\infty e^{-px} h(x)\, dx. \quad \text{.............(2)}$$

Since

$$W_{-m+\frac{1}{2},m}(x) \equiv x^{-m+\frac{1}{2}} e^{-\frac{1}{2}x},$$

(1) gives (2) when $k = -m + \frac{1}{2}$.

We shall represent (1) by

$$\phi(p) \overset{\vee}{\underset{\vee}{=}} h(x)$$

and as usual, (2) will be denoted by

$$f(p) \doteqdot h(x).$$

The transform defined by (1) possesses some interesting properties and is capable of yielding very general results. The object of this paper is to prove two theorems for this generalised transform and to obtain a few results involving the hypergeometric function and the E-function.

2. We begin by establishing a close relationship that exists between (1) and (2).

THEOREM 1. If

$$\phi(p) \overset{\vee}{\underset{\vee}{=}} h(x)$$

and

$$f(p) \doteqdot x^l h(x),$$

then

$$\phi(p) = \frac{p^{\frac{1}{2}+m+k}}{\Gamma(\frac{1}{2}+l-m-k)} \int_0^\infty x^{l-m-k-\frac{1}{2}} (x+p)^{-1} f(x+p) \times {}_2F_1\left(\tfrac{1}{2}-k-m,\ \tfrac{1}{2}-k+m;\ \tfrac{1}{2}+l-k-m;\ -\frac{x}{p}\right) dx,$$

provided that $R(p)>0$, $R(\frac{1}{2}+l-m-k)>0$, and the integral is convergent.

PROOF: We know that [**7**, p. 11] if

$$x^l h(x) \doteqdot f(p)$$

then

$$e^{-ax} x^l h(x) \doteqdot p(p+a)^{-1} f(p+a),\ R(a)>0. \quad \text{.............(3)}$$

Also,

$$e^{\frac{1}{2}ap} p^{\frac{1}{2}-l+m} W_{k,m}(ap) \doteqdot \frac{a^k x^{l-m-k-\frac{1}{2}}}{\Gamma(\frac{1}{2}+l-m-k)}\, {}_2F_1\left(\tfrac{1}{2}-k-m,\ \tfrac{1}{2}-k+m;\ \tfrac{1}{2}+l-k-m;\ -\frac{x}{a}\right), \quad \text{.........(4)}$$

$R(\frac{1}{2}+l-m-k)>0,\ R(p)>0,\ R(a)>0.$

Now Goldstein [2] has given Parseval's theorem of Operational Calculus in the form : If

$$f_1(p) \doteqdot h_1(x) \text{ and } f_2(p) \doteqdot h_2(x)$$

then

$$\int_0^\infty f_1(x) h_2(x) x^{-1}\, dx = \int_0^\infty f_2(x) h_1(x) x^{-1}\, dx. \qquad (5)$$

Using the relations (3) and (4) in (5) and then replacing a by p, the theorem follows immediately.

3. To illustrate the application of the theorem, we evaluate two infinite integrals.

(i) From the integral [**10**, p. 171]

$$\int_0^\infty e^{-\frac{1}{2}u} u^{m+\gamma-\frac{3}{2}} W_{k,m}(u)\, E\left(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : \frac{p}{u}\right) du$$
$$= E(\alpha_1, \ldots, \alpha_r, \gamma, \gamma+2m : \beta_1, \ldots, \beta_s, \gamma+m-k+\tfrac{1}{2} : p), \qquad (6)$$

$R(\gamma)>0$, $R(\gamma+2m)>0$, $R(p)>0$,

we find that if

$$h(x) = x^{\gamma-1} E\left(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : \frac{1}{x}\right)$$

then

$$\phi(p) = p^{1-\gamma} E(\alpha_1, \ldots, \alpha_r, \gamma, \gamma+2m : \beta_1, \ldots, \beta_s, \gamma+m-k+\tfrac{1}{2} : p),$$

$R(\gamma)>0$, $R(\gamma+2m)>0$, $R(p)>0$.

Also, [**6**, p. 255]

$$x^l h(x) = x^{l+\gamma-1} E\left(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : \frac{1}{x}\right)$$
$$\doteqdot p^{1-\gamma-l} E(\alpha_1, \ldots, \alpha_r, l+\gamma : \beta_1, \ldots, \beta_s : p)$$
$$= f(p),\ R(l+\gamma)>0,\ R(p)>0.$$

Applying the theorem and replacing $\frac{1}{2}-k-m$ by λ, $\frac{1}{2}-k+m$ by μ, and $\frac{1}{2}+l-k-m$ by ν, we get

$$\int_0^\infty x^{\nu-1}(p+x)^{\lambda-\nu-\gamma} E(\alpha_1, \ldots, \alpha_r, \nu-\lambda+\gamma : \beta_1, \ldots, \beta_s : p+x)\, {}_2F_1\left(\lambda, \mu ; \nu ; -\frac{x}{p}\right) dx$$
$$= \Gamma(\nu)\, p^{\lambda-\gamma} E(\alpha_1, \ldots, \alpha_r, \gamma, \gamma+\mu-\lambda : \beta_1, \ldots, \beta_s, \gamma+\mu : p) \qquad (7)$$

valid, by analytic continuation, for $R(\nu)>0$, $R(\gamma)>0$, $R(\gamma+\mu-\lambda)>0$, $R(p)>0$.

If we take $\lambda=\nu$ or $\mu=\nu$ in (7) we get a result given by MacRobert [**6**, p. 256].

(ii) Since

$$\int_0^\infty e^{-\frac{u}{2}(1-z)} u^{k-\lambda-2} W_{k,m}(u) W_{\lambda,\mu}(zu)\, du$$
$$= \left\{\frac{\Gamma(k-\lambda+m+\mu)\Gamma(k-\lambda-m+\mu)\Gamma(k-\lambda+m-\mu)\Gamma(k-\lambda-m-\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)\Gamma(\frac{1}{2}-\lambda-\mu)\Gamma(2k-2\lambda)}\right\}$$
$$\times z^{\lambda+m-k+\frac{1}{2}}\, {}_2F_1\left(k-\lambda-m+\mu, k-\lambda-m-\mu ; 2k-2\lambda ; 1-\frac{1}{z}\right), \qquad (8)$$

$R(k-\lambda\pm m\pm\mu)>0$, $R(z)>\frac{1}{2}$,

we find that, if

$$h(x) = e^{\frac{1}{2}x} x^{k-\lambda-m-\frac{3}{2}} W_{\lambda,\mu}(x)$$

then

$$\phi(p) = \left\{\frac{\Gamma(k-\lambda+m+\mu)\Gamma(k-\lambda-m+\mu)\Gamma(k-\lambda+m-\mu)\Gamma(k-\lambda-m-\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)\Gamma(\frac{1}{2}-\lambda-\mu)\Gamma(2k-2\lambda)}\right\}$$
$$\times p\, {}_2F_1(k-m-\lambda+\mu, k-m-\lambda-\mu ; 2k-2\lambda ; 1-p),$$

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$R(k-\lambda \pm m \pm \mu)>0$, and [7, p. 51]

$$x^l h(x) = e^{\frac{1}{2}x} x^{l+k-\lambda-m-\frac{3}{2}} W_{\lambda,\mu}(x)$$

$$\doteqdot \left\{\frac{\Gamma(l+k-\lambda-m+\mu)\Gamma(l+k-\lambda-m-\mu)}{\Gamma(l-k-2\lambda-m+\frac{1}{2})}\right\} \times p\, {}_2F_1(l+k-\lambda-m+\mu, l+k-\lambda-m-\mu;\ l-k-2\lambda-m+\tfrac{1}{2};\ 1-p)$$

$$= f(p), \quad R(l+k-\lambda-m\pm\mu)>0.$$

Applying the theorem and replacing $\frac{1}{2}-k-m$ by α, $\frac{1}{2}-k+m$ by β, $k-m-\lambda+\mu$ by a, $k-m-\lambda-\mu$ by b, and $1-p$ by z, we get

$$\int_0^\infty x^{l+\alpha-1}{}_2F_1\left(\alpha, \beta;\ l+\alpha;\ \frac{x}{z-1}\right){}_2F_1(l+a, l+b;\ l+a+b+\beta;\ z-x)\,dx$$

$$= \left\{\frac{\Gamma(a)\Gamma(b)\Gamma(a+\beta-\alpha)\Gamma(b+\beta-\alpha)\Gamma(l+a+b+\beta)\Gamma(l+\alpha)}{\Gamma(a+\beta)(b+\beta)\Gamma(a+b+\beta-\alpha)\Gamma(a+l)\Gamma(b+l)}\right\} \times (1-z)^\alpha {}_2F_1(a, b;\ a+b+\beta-\alpha;\ z), \qquad (9)$$

valid, by analytic continuation, for

$$R(l+\alpha)>0,\ R(a)>0,\ R(b)>0,\ R(a+\beta-\alpha)>0,\ R(b+\beta-\alpha)>0,\ |z|<1.$$

THEOREM II. If

$$\phi(p) \overset{\vee}{\underset{\vee}{=}} h(x)$$

$$p^{2-\lambda} h(p) \doteqdot g(x)$$

and

$$p^{2-\mu} g(p) \doteqdot f(x),$$

then

$$\phi(p) = p^{1-\lambda}\int_0^\infty x^{-\mu} E(\lambda, \lambda+2m, \mu :\ \lambda+m-k+\tfrac{1}{2} :\ px) f(x)\,dx,$$

provided that $R(\lambda)>0$, $R(\lambda+2m)>0$, $R(\mu)>0$, and the integral is convergent.

Proof: We know that if [9, p. 234]

$$\phi(p) \overset{\vee}{\underset{\vee}{=}} h(x)$$

and

$$p^{2-\lambda} h(p) \doteqdot g(x)$$

then

$$\phi(p) = \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p^{1-\lambda}\int_0^\infty {}_2F_1\left(\lambda, \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{t}{p}\right) g(t)\,dt.$$

But since

$$g(t) = t^{\mu-1}\int_0^\infty e^{-xt} f(x)\,dx,$$

we have

$$\phi(p) = \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p^{1-\lambda}\int_0^\infty {}_2F_1\left(\lambda, \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{t}{p}\right) \times t^{\mu-1}\left\{\int_0^\infty e^{-xt} f(x)\,dx\right\} dt$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p^{1-\lambda}\int_0^\infty f(x)\left\{\int_0^\infty e^{-xt} t^{\mu-1}{}_2F_1\left(\lambda, \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{t}{p}\right) dt\right\} dx$$

$$= p^{1-\lambda}\int_0^\infty x^{-\mu} E(\lambda, \lambda+2m, \mu :\ \lambda+m-k+\tfrac{1}{2} :\ px) f(x)\,dx,$$

on using the integral [5, p. 10]

$$\int_0^\infty e^{-xt}\,t^{\gamma-1}\,{}_2F_1(\alpha,\beta;\ \delta;\ \ -t)\,dt=\frac{\Gamma(\delta)x^{-\gamma}}{\Gamma(\alpha)\Gamma(\beta)}E(\alpha,\beta,\gamma:\ \delta:\ x). \qquad \ldots\ldots\ldots\ldots(10)$$

The change in the order of integration in the above process is permissible by virtue of de la Vallee Poussin's theorem, when the integrals involved are absolutely convergent [1, p. 457].

When $k=-m+\frac{1}{2}$, we have

COROLLARY: If

$$\phi(p) \doteqdot h(x)$$
$$p^{2-\lambda}\,h(p) \doteqdot g(x)$$

and

$$p^{2-\mu}\,g(p) \doteqdot f(x)$$

then

$$\phi(p)=p^{1-\lambda}\int_0^\infty x^{-\mu}E(\lambda,\mu::px)f(x)\,dx,$$

when $R(\lambda)>0$, $R(\mu)>0$, and the integral is convergent.

5. We give below some applications of the theorem. The following result [9, p. 233] will be useful.

$$x^{\nu-1}\,{}_rF_s(\alpha_1,\ldots,\alpha_r;\ \beta_1,\ldots,\beta_s;\ \pm x)$$
$$\underset{\vee}{\overset{\vee}{=}}\frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})}p^{1-\nu}\,{}_{r+2}F_{s+1}\left\{\begin{matrix}\alpha_1,\ldots,\alpha_r,\nu,\nu+2m;\\ \beta_1,\ldots,\beta_s,\nu+m-k+\frac{1}{2};\end{matrix}\ \ \pm\frac{1}{p}\right\}, \quad \ldots\ldots(11)$$

$R(\nu)>0$, $R(\nu+2m)>0$, $R(p)>0\,(r<s)$; $R(p)>1\,(r=s)$.

Example 1. Take

$$h(x)=\frac{\Gamma(\gamma+\alpha)\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)\Gamma(\gamma)}x^{\lambda-1}\,{}_1F_2(\beta;\ \gamma,1-\alpha;\ x)+x^{\lambda+\alpha-1}\Gamma(-\alpha)\,{}_1F_2(\beta+\alpha;\ \gamma+\alpha,1+\alpha;\ x)$$
$$\underset{\vee}{\overset{\vee}{=}}\frac{\Gamma(\gamma+\alpha)\Gamma(\beta)\Gamma(\alpha)\,\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\beta+\alpha)\Gamma(\gamma)\Gamma(\lambda+m-k+\frac{1}{2})}p^{1-\lambda}{}_3F_3\left(\lambda,\lambda+2m,\beta;\lambda+m-k+\tfrac{1}{2},\gamma,1-\alpha;\ \frac{1}{p}\right)$$
$$+\frac{\Gamma(-\alpha)\Gamma(\lambda+\alpha)\Gamma(\lambda+\alpha+2m)}{\Gamma(\lambda+\alpha+m-k+\frac{1}{2})}p^{1-\lambda-\alpha}$$
$$\times\,{}_3F_3\left(\lambda+\alpha,\lambda+\alpha+2m,\beta+\alpha;\ \lambda+\alpha+m-k+\tfrac{1}{2},\gamma+\alpha,1+\alpha;\ \frac{1}{p}\right)$$
$$=\phi(p),\quad R(\lambda)>0,\quad R(\lambda+2m)>0,\ R(\lambda+\alpha)>0,\ R(\lambda+\alpha+2m)>0,$$

on interpretation with the help of (11).

We then have [3, p. 148]

$$p^{2-\lambda}\,h(p)=\frac{\Gamma(\gamma+\alpha)\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)\Gamma(\gamma)}p\,{}_1F_2(\beta;\ \gamma,1-\alpha;\ p)+p^{1+\alpha}\Gamma(-\alpha)\,{}_1F_2(\beta+\alpha;\ \gamma+\alpha,1+\alpha;\ p)$$
$$\doteqdot x^{-\alpha-1}\,{}_1F_1\left(\beta+\alpha;\ \gamma+\alpha;\ -\frac{1}{x}\right)$$
$$=g(x),\quad R(\beta)>0,$$

and

$$p^{2-\mu}\,g(p)=p^{1-\mu-\alpha}\,{}_1F_1\left(\beta+\alpha;\ \gamma+\alpha;\ -\frac{1}{p}\right)$$
$$\doteqdot\frac{x^{\mu+\alpha-1}}{\Gamma(\mu+\alpha)}\,{}_1F_2(\beta+\alpha;\ \gamma+\alpha,\mu+\alpha;\ -x)$$
$$=f(x),\quad R(\mu+\alpha)>0.$$

Applying the theorem and replacing λ by a, $\lambda+2m$ by b, $\lambda+m-k+\frac{1}{2}$ by c, we get

$$\int_0^\infty x^{\alpha-1}E(a,b,\mu:c:px)\,{}_1F_2(\beta+\alpha;\ \gamma+\alpha,\mu+\alpha;\ -x)\,dx$$
$$=\frac{\Gamma(\gamma+\alpha)\Gamma(\beta)\Gamma(\alpha)\Gamma(a)\Gamma(b)\Gamma(\mu+\alpha)}{\Gamma(\beta+\alpha)\Gamma(\gamma)\Gamma(c)}\,{}_3F_3\left(a,b,\beta;\ c,\gamma,1-\alpha;\ \frac{1}{p}\right)$$
$$+\frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(-\alpha)\Gamma(\mu+\alpha)}{\Gamma(c+\alpha)}p^{-\alpha}\,{}_3F_3\left(a+\alpha,b+\alpha,\beta+\alpha;\ c+\alpha,\gamma+\alpha,1+\alpha;\ \frac{1}{p}\right),\ \ldots\ldots(12)$$

valid, by analytic continuation, for $R(a+\alpha)>0$, $R(b+\alpha)>0$, $R(\mu+\alpha)>0$, $R(\beta)>0$, $R(\beta+\alpha-\gamma-\mu)<\frac{1}{2}$, $R(p)>0$. [See note after formula (17)].

In particular, if we take $\beta=\mu$, $\gamma+\alpha=\nu+1$, put $\frac{1}{4}y^2t^2$ for x, $\frac{2}{y^2}$ for p and use the relation

$$J_\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}\,{}_0F_1(\nu+1;\ -\tfrac{1}{4}x^2)$$

we find that

$$\int_0^\infty \sqrt{yt}\,J_\nu(yt)(\tfrac{1}{2}t^2)^{\alpha-\frac{1}{2}\nu-\frac{3}{4}}E(a,b,\mu:c:\tfrac{1}{2}t^2)\,dt$$
$$=\frac{\Gamma(\mu)\Gamma(\alpha)\Gamma(a)\Gamma(b)}{\Gamma(\nu-\alpha+1)\Gamma(c)}(\tfrac{1}{2}y^2)^{\frac{1}{2}\nu-\alpha+\frac{1}{4}}\,{}_3F_3(a,b,\mu;\ c,\nu-\alpha+1,1-\alpha;\ \tfrac{1}{2}y^2)$$
$$+\frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(\mu+\alpha)\Gamma(-\alpha)}{\Gamma(\nu+1)\Gamma(c+\alpha)}(\tfrac{1}{2}y^2)^{\frac{1}{2}\nu+\frac{1}{4}}\,{}_3F_3(a+\alpha,b+\alpha,\mu+\alpha;\ c+\alpha,\nu+1,1+\alpha;\ \tfrac{1}{2}y^2),\ (13)$$

$R(a+\alpha)>0$, $R(b+\alpha)>0$, $R(\mu+\alpha)>0$, $R(2\alpha-\nu-\frac{3}{2})<0$.

From this result it is easy to infer that the functions

$$(\tfrac{1}{2}x^2)^{\alpha-\frac{1}{2}\nu-\frac{3}{4}}E(a,b,\mu:c:\tfrac{1}{2}x^2)$$

and

$$\frac{\Gamma(a)\Gamma(b)\Gamma(\mu)\Gamma(\alpha)}{\Gamma(\nu-\alpha+1)\Gamma(c)}(\tfrac{1}{2}x^2)^{\frac{1}{2}\nu-\alpha+\frac{1}{4}}\,{}_3F_3(a,b,\mu;\ c,\nu-\alpha+1,1-\alpha;\ \tfrac{1}{2}x^2)$$
$$+\frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(\mu+\alpha)\Gamma(-\alpha)}{\Gamma(\nu+1)\Gamma(c+\alpha)}(\tfrac{1}{2}x^2)^{\frac{1}{2}\nu+\frac{1}{4}}\,{}_3F_3(a+\alpha,b+\alpha,\mu+\alpha;\ c+\alpha,\nu+1,\alpha+1;\ \tfrac{1}{2}x^2)\quad(14)$$

are Hankel transforms of each other of order ν, provided that $R(a+\alpha)>0$, $R(b+\alpha)>0$, $R(\mu+\alpha)>0$, $R(2\alpha-\nu-\frac{3}{2})<0$, $R(\nu-\alpha+1)>0$, $R(\nu+1)>0$, $R(\nu-2\alpha-2a+\frac{1}{2})<0$, $R(\nu-2\alpha-2b+\frac{1}{2})<0$, and $R(\nu-2\alpha-2\mu+\frac{1}{2})<0$.

Further when $\mu=c$, this gives the pair of functions

$$(\tfrac{1}{2}x^2)^{\alpha-\frac{1}{2}\nu-\frac{3}{4}}E(a,b::\tfrac{1}{2}x^2)$$

and

$$\frac{\Gamma(a)\Gamma(b)\Gamma(\alpha)}{\Gamma(\nu-\alpha+1)}(\tfrac{1}{2}x^2)^{\frac{1}{2}\nu-\alpha+\frac{1}{4}}\,{}_2F_2(a,b;\ \nu-\alpha+1,1-\alpha;\ \tfrac{1}{2}x^2)$$
$$+\frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(-\alpha)}{\Gamma(\nu+1)}(\tfrac{1}{2}x^2)^{\frac{1}{2}\nu+\frac{1}{4}}\,{}_2F_2(a+\alpha,b+\alpha;\ \nu+1,1+\alpha;\ \tfrac{1}{2}x^2),\ \ldots\ldots(15)$$

which are Hankel transforms of each other of order ν, provided that $R(a+\alpha)>0$, $R(b+\alpha)>0$, $R(2\alpha-\nu-\frac{3}{2})<0$, $R(\nu-\alpha+1)>0$, $R(\nu+1)>0$, $R(\nu-2\alpha-2a+\frac{1}{2})<0$, $R(\nu-2\alpha-2b+\frac{1}{2})<0$.

When $\alpha=\nu$, (15) yields the pair of functions investigated by Ram Kumar [**8**, p. 79].

Since

$$E(\alpha,\beta::x)=\Gamma(\alpha)\Gamma(\beta)\,x^{\frac{1}{2}(\alpha+\beta-1)}e^{\frac{1}{2}x}\,W_{\frac{1}{2}(1-\alpha-\beta),\frac{1}{2}(\beta-\alpha)}(x)$$
$$=\sum_{\alpha,\beta}\Gamma(\beta-\alpha)\Gamma(\alpha)x^\alpha\,{}_1F_1(\alpha;\ \alpha-\beta+1;\ x),\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

(15) also yields when a or b is equal to $\nu-\alpha+1$, the pair of functions given by Hari Shanker [**4**, p. 63].

Example 2. Now take

$$\begin{aligned} h(x) &= \Gamma(\alpha)x^{\lambda-\alpha-1}\,{}_2F_2(a, b\,;\; c, 1-\alpha\,;\; x) \\ &\quad + \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(c)\Gamma(-\alpha)}{\Gamma(a)\Gamma(b)\Gamma(c+\alpha)} x^{\lambda-1}\,{}_2F_2(a+\alpha, b+\alpha\,;\; c+\alpha, 1+\alpha\,;\; x) \\ &\doteqdot \frac{\Gamma(\alpha)\Gamma(\lambda-\alpha)\Gamma(\lambda-\alpha+2m)}{\Gamma(\lambda-\alpha+m-k+\frac{1}{2})} p^{1+\alpha-\lambda}\,{}_4F_3\left\{\begin{matrix} a, b, \lambda-\alpha, \lambda-\alpha+2m\,; & \frac{1}{p} \\ \lambda-\alpha+m-k+\frac{1}{2}, c, 1-\alpha\,; & \end{matrix}\right\} \\ &\quad + \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(c)\Gamma(-\alpha)\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(a)\Gamma(b)\Gamma(c+\alpha)\Gamma(\lambda+m-k+\frac{1}{2})} p^{1-\lambda}\,{}_4F_3\left\{\begin{matrix} a+\alpha, b+\alpha, \lambda, \lambda+2m\,; & \frac{1}{p} \\ \lambda+m-k+\frac{1}{2}, c+\alpha, 1+\alpha\,; & \end{matrix}\right\} \\ &= \phi(p), \quad R(\lambda-\alpha)>0, \quad R(\lambda-\alpha+2m)>0, R(\lambda)>0, R(\lambda+2m)>0, R(p)>1. \end{aligned}$$

Then [**9**, p. 240]

$$\begin{aligned} p^{2-\lambda}h(p) &= \Gamma(\alpha)p^{1-\alpha}\,{}_2F_2(a, b\,;\; c, 1-\alpha\,;\; p) \\ &\qquad + \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(c)\Gamma(-\alpha)}{\Gamma(a)\Gamma(b)\Gamma(c+\alpha)} p\,{}_2F_2(a+\alpha, b+\alpha\,;\; c+\alpha, 1+\alpha\,;\; p) \\ &\doteqdot x^{\alpha-1}\,{}_2F_1\left(a, b\,;\; c\,;\; -\frac{1}{x}\right) \\ &= g(x), \; R(a+\alpha)>0, \quad R(b+\alpha)>0, \end{aligned}$$

and

$$\begin{aligned} p^{2-\mu}g(p) &= p^{1+\alpha-\mu}\,{}_2F_1\left(a, b\,;\; c\,;\; -\frac{1}{p}\right) \\ &\doteqdot \frac{x^{\mu-\alpha-1}}{\Gamma(\mu-\alpha)}\,{}_2F_2(a, b\,;\; c, \mu-\alpha\,;\; -x) \\ &= f(x), \; R(\mu-\alpha)>0. \end{aligned}$$

Applying the theorem and replacing λ by ξ, $\lambda+2m$ by η, and $\lambda+m-k+\frac{1}{2}$ by ζ, we have

$$\begin{aligned} &\int_0^\infty x^{-\alpha-1}E(\xi, \eta, \mu : \zeta : px)\,{}_2F_2(a, b\,; c, \mu-\alpha\,;\; -x)\,dx \\ &= \frac{\Gamma(\alpha)\Gamma(\xi-\alpha)\Gamma(\eta-\alpha)\Gamma(\mu-\alpha)}{\Gamma(\zeta-\alpha)} p^{\alpha}\,{}_4F_3\left\{\begin{matrix} a, b, \xi-\alpha, \eta-\alpha\,; & \frac{1}{p} \\ \zeta-\alpha, c, 1-\alpha\,; & \end{matrix}\right\} \\ &\qquad + \frac{\Gamma(\mu-\alpha)\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(c)\Gamma(-\alpha)\Gamma(\xi)\Gamma(\eta)}{\Gamma(a)\Gamma(b)\Gamma(c+\alpha)\Gamma(\zeta)}\,{}_4F_3\left\{\begin{matrix} a+\alpha, b+\alpha, \xi, \eta\,; & \frac{1}{p} \\ \zeta, c+\alpha, 1+\alpha\,; & \end{matrix}\right\}, \quad \ldots\ldots(17) \end{aligned}$$

valid, by analytic continuation, for $R(\xi-\alpha)>0$, $R(\eta-\alpha)>0$, $R(\mu-\alpha)>0$, $R(a+\alpha)>0$, $R(b+\alpha)>0$, $R(p)>1$.

Note. Formulae (12) and (17) are special cases of a formula [**12**, (2)] given by Ragab.

In particular, taking $b=c$ and $a=\mu-\alpha$, we have

$$\begin{aligned} &\int_0^\infty x^{-\alpha-1}E(\xi, \eta, \mu : \zeta : px)e^{-x}\,dx \\ &\qquad = \frac{\Gamma(\mu-\alpha)\Gamma(\alpha)\Gamma(\xi-\alpha)\Gamma(\eta-\alpha)}{\Gamma(\zeta-\alpha)} p^{\alpha}\,{}_3F_2\left\{\begin{matrix} \mu-\alpha, \xi-\alpha, \eta-\alpha\,; & \frac{1}{p} \\ \zeta-\alpha, 1-\alpha\,; & \end{matrix}\right\} \\ &\qquad\qquad + \frac{\Gamma(\mu)\Gamma(-\alpha)\Gamma(\xi)\Gamma(\eta)}{\Gamma(\zeta)}\,{}_3F_2\left\{\begin{matrix} \mu, \xi, \eta\,; & \frac{1}{p} \\ \zeta, 1+\alpha\,; & \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(18) \end{aligned}$$

from which we get the operational image

$$x^{-\alpha-1}E(\xi, \eta, \mu : \zeta : x)$$
$$\doteqdot \frac{\Gamma(\mu-\alpha)\Gamma(\alpha)\Gamma(\xi-\alpha)\Gamma(\eta-\alpha)}{\Gamma(\zeta-\alpha)} p\, {}_3F_2\left\{\begin{matrix}\mu-\alpha, \xi-\alpha, \eta-\alpha ; \\ \zeta-\alpha, 1-\alpha ;\end{matrix}\ p\right\}$$
$$+\frac{\Gamma(\mu)\Gamma(-\alpha)\Gamma(\xi)\Gamma(\eta)}{\Gamma(\zeta)} p^{\alpha+1}\, {}_3F_2\left\{\begin{matrix}\mu, \xi, \eta ; \\ \zeta, 1+\alpha ;\end{matrix}\ p\right\}. \qquad \text{.............(19)}$$

If in (17) we take $\alpha=\frac{1}{2}$, $\xi=a+1$, $\eta=b+1$, $\zeta=c+1$ and use the relation

$${}_2F_1(2a, 2b ;\ 2c ;\ v) = {}_4F_3(a, a+\tfrac{1}{2}, b, b+\tfrac{1}{2} ;\ \tfrac{1}{2}, c, c+\tfrac{1}{2} ;\ v^2)$$
$$+\frac{2abv}{c}\, {}_4F_3(a+\tfrac{1}{2}, a+1, b+\tfrac{1}{2}, b+1 ;\ \tfrac{3}{2}, c+\tfrac{1}{2}, c+1 ;\ v^2), \quad \text{......(20)}$$

we get

$$\int_0^\infty x^{-\frac{3}{2}} E(a+1, b+1, \mu : c+1 : px)\, {}_2F_2(a, b ;\ c, \mu-\tfrac{1}{2} ;\ -x)\, dx$$
$$=\frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(\mu-\frac{1}{2})}{\Gamma(c+\frac{1}{2})}\sqrt{\pi p}\, {}_2F_1\left(2a, 2b ;\ 2c ;\ -\frac{1}{\sqrt{p}}\right), \quad \text{..............(21)}$$

$R(p)>1$, $R(a+\frac{1}{2})>0$, $R(b+\frac{1}{2})>0$. $R(\mu-\frac{1}{2})>0$.

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