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FINITE GROUPS OF DEFICIENCY ZERO INVOLVING THE LUCAS NUMBERS

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In this paper, we investigate a class of 2-generator 2-relator groups $G(n)$ related to the *Fibonacci groups* $F(2,n)$, each of the groups in this new class also being defined by a single parameter n, though here n can take negative, as well as positive, values. If n is odd, we show that $G(n)$ is a finite soluble group of derived length 2 (if n is coprime to 3) or 3 (otherwise), and order $|2n(n+2)g_n f_{(n,3)}|$, where f_n is the *Fibonacci number* defined by $f_0=0$, $f_1=1$, $f_{n+2}=f_n+f_{n+1}$ for $n\geqq 0$, and g_n is the *Lucas number* defined by $g_0=2$, $g_1=1$, $g_{n+2}=g_n+g_{n+1}$ for $n\geqq 0$. On the other hand, if n is even then, with three exceptions, namely the cases $n=2,4$ or -4, $G(n)$ is infinite; the groups $G(2)$, $G(4)$ and $G(-4)$ have orders 16, 240 and 80 respectively.

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1. Introduction

The groups defined by the presentations

$$\langle a,b: a^2=b^n=ab^2ab^{-2}ab^{-1}ab=1\rangle$$

were studied in [**2**], and shown to be finite of order $2ng_n$ if n is odd, but infinite if n is even with $n\geqq 6$. (The groups with $n=2$ and $n=4$ have orders 4 and 40 respectively). Here g_n denotes the *Lucas* numbers defined by $g_0=2$, $g_1=1$, $g_{n+2}=g_n+g_{n+1}$, which are related to the *Fibonacci* numbers f_n, where $f_0=0$, $f_1=1$, $f_{n+2}=f_n+f_{n+1}$, via the relation $g_n=f_{n-1}+f_{n+1}$. Note that, if $n<0$, then $f_n>0$ if and only if n is odd, whereas $g_n>0$ if and only if n is even.

The purpose of this paper is to examine the related class of deficiency zero groups $G(n)$ defined by

$$\langle a,b: a^2=1, ab^2ab^{-2}ab^{-1}ab=b^n\rangle.$$

We show that, among these groups, there is an infinite subclass of non-metabelian finite groups, thus adding to the small number of known classes of such groups of deficiency zero; a general survey of finite groups of deficiency zero is given in [**5**]. The notation used here is standard, and is consistent with that of [**2**]. Our result is:

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Theorem A. *Let* $G=G(n)$. *Then:*

(i) *If* $n=0, G'$ *is free of rank* 2 *and* G/G' *is isomorphic to* $C_2 \times C_\infty$.

(ii) *If* n *is odd, then* G *is a finite soluble group of order* $|2n(n+2)g_n f_{(n,3)}|$, *and*:

$$[G:G'] = 2|n|, \quad [G':G''] = |(n+2)g_n|,$$

$$[G'':G'''] = f_{(n,3)}, \quad |G'''| = 1.$$

(iii) $G(2)$ *is semi-dihedral of order* 16; $G(-2)$ *is the infinite dihedral group*; $G(4)$ *is metabelian of order* 240; $G(-4)$ *is metabelian of order* 80; *if* n *is even,* $|n| \geqq 6$, *then* $[G:G'] = 2|n|$, $[G':G''] = |n+2|(g_n - 2)$, *and* G'' *is infinite.* □

Evidence for this result originated from the computer programs mentioned in Section 6, and these were used to prove the results concerning the finite groups $G(n)$ with n even.

2. First reduction

For the rest of this paper, let G denote the group defined by

$$\langle a, b : a^2 = 1, ab^2ab^{-2}ab^{-1}ab = b^n \rangle,$$

where $n \in \mathbf{Z}$, and let $x = ab^{-1}ab$, $y = abab^{-1}$, $z = b^n$. We start with an elementary lemma:

Lemma 1. (i) $axa = x^{-1}$, (ii) $aya = y^{-1}$, (iii) $aza = xz^{-1}x^{-1}$, (iv) $bxb^{-1} = y^{-1}$, (v) $byb^{-1} = y^{-1}zx^{-1}$, (vi) $bzb^{-1} = z$, (vii) $b^{-1}yb = x^{-1}$, (viii) $b^{-1}zb = z$, (ix) $b^{-1}xb = y^{-1}xz$.

Proof. The proofs of (i), (ii), (iv), (vi) and (viii) are immediate.

(iii) $aza = b^2ab^{-2}ab^{-1}aba = ab^{-1}ab.b^{-1}abab^2ab^{-2}a.b^{-1}aba = xz^{-1}x^{-1}$.

(v) $byb^{-1} = babab^{-2} = bab^{-1}a.ab^2ab^{-2}ab^{-1}ab.b^{-1}aba = y^{-1}zx^{-1}$.

(viii) From (iv), $x = b^{-1}y^{-1}b$, and so $x^{-1} = b^{-1}yb$.

(ix) From (v), $byb^{-1} = y^{-1}zx^{-1}$, and thus

$$y = b^{-1}y^{-1}bb^{-1}zbb^{-1}x^{-1}b = xzb^{-1}x^{-1}b$$

from (vii) and (viii), giving that $b^{-1}xb = y^{-1}xz$. □

Corollary 2. $G' = \langle x, y, z \rangle$.

Proof. This follows immediately, since $x=ab^{-1}ab$, $y=abab^{-1}$ and $z=ab^2ab^{-2}.ab^{-1}ab$ all lie in G', and $N=\langle x,y,z\rangle$ is normal in G by Lemma 1. □

3. Proof of Theorem A (i)

If $n=0$, then $z=1$, and the next result follows using Corollary 2:

Lemma 3. *G/G' is isomorphic to $C_2\times C_\infty$; $G'=\langle x,y\rangle$.*

Given this, let $c=aba$, and let N be the normal subgroup $\langle b,c\rangle$ of index 2 in G. Then N has presentation

$$\langle b,c: c^2b^{-2}c^{-1}b=b^2c^{-2}b^{-1}c=1\rangle.$$

The second relation is redundant, so introducing $d=c^{-1}b$ and deleting $c=bd^{-1}$ yields

$$\langle b,d: b^{-2}db^2=d.b^{-1}db\rangle.$$

Introducing $e=b^{-1}db$ gives

$$\langle b,d,e: b^{-1}db=e, b^{-1}eb=de\rangle.$$

The normal subgroup $\langle d,e\rangle$ is now seen to be free of rank 2. Since d and e lie in G', $\langle d,e\rangle=G'$, and the result follows. □

4. Further reductions

From now on, assume that $n\neq 0$. First we have:

Lemma 4. (i) $[x,z]=1$, (ii) $aza=z^{-1}$, (iii) $[y,z]=1$, (iv) $bab^2ab^{-1}=z(ab)^2$, (v) $yx^{-1}y^{-1}xz^{n+2}=1$ *when n is odd*, (vi) $z^{n+2}=1$ *when n is even.*

Proof. (i) $yzy^{-1}=abab^{-1}zbab^{-1}a$

$=abazab^{-1}a$ by Lemma 1 (viii)

$=abxz^{-1}x^{-1}b^{-1}a$ by Lemma 1 (iii)

$=abxb^{-1}bz^{-1}b^{-1}bx^{-1}b^{-1}a$

$=ay^{-1}z^{-1}ya$ by Lemma 1 (iv) & (vi)

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$$=yxzx^{-1}y^{-1} \text{by Lemma 1 (ii) \& (iii)},$$

so that $z=xzx^{-1}$ as required.

(ii) This follows immediately from (i) and Lemma 1 (iii).

(iii)
$$
\begin{aligned}
z &= xzx^{-1} && \text{by (i)}\\
&= ab^{-1}abzb^{-1}aba\\
&= ab^{-1}azaba && \text{by Lemma 1 (vi)}\\
&= ab^{-1}xz^{-1}x^{-1}ba && \text{by Lemma 1 (iii)}\\
&= ay^{-1}xzz^{-1}z^{-1}x^{-1}ya && \text{by Lemma 1 (viii) \& (ix)}\\
&= ay^{-1}z^{-1}ya && \text{by (i)}\\
&= yxzx^{-1}y^{-1} && \text{by Lemma 1 (ii) \& (iii)}\\
&= yzy^{-1} && \text{by (i).}
\end{aligned}
$$

(iv) From $ab.bab^{-2}ab^{-1}.ab=z$, we have

$$
\begin{aligned}
bab^{-2}ab^{-1} &= b^{-1}azb^{-1}a\\
&= b^{-1}ab^{-1}za && \text{by Lemma 1 (viii)}\\
&= b^{-1}ab^{-1}az^{-1} && \text{by (ii)}
\end{aligned}
$$

and hence $bab^2ab^{-1}=z(ab)^2$.

(v)
$$
\begin{aligned}
b^{-1}ab^{-1}azabab &= b^{-1}ab^{-1}z^{-1}bab && \text{by (ii)}\\
&= b^{-1}az^{-1}ab && \text{by Lemma 1 (viii)}\\
&= b^{-1}zb && \text{by (ii)}\\
&= z && \text{by Lemma 1 (viii),}
\end{aligned}
$$

so that $[z,(ab)^2]=1$. Then we have

$$
\begin{aligned}
z^n(ab)^{2n} &= bab^{2n}ab^{-1} && \text{by (iv)}\\
&= baz^2ab^{-1}
\end{aligned}
$$

$$=z^{-2} \text{ by (ii) and Lemma 1 (vi)},$$

so that $z^{n+2}(ab)^{2n}=1$. Assume n is odd, and let $m=(n+1)/2$. Similar to the above, we have

$$\begin{aligned} z^m(ab)^{2m} &= bab^{2m}ab^{-1} \\ &= bazbab^{-1} \\ &= z^{-1}babab^{-1} \quad \text{by (ii) and Lemma 1 (viii)}, \end{aligned}$$

and hence $babab^{-1}=z^{m+1}(ab)^{2m}$, so that $babab^{-2}a=z^{m+1}(ab)^n$. We then have

$$\begin{aligned} (babab^{-2}a)^2 &= z^{m+1}(ab)^n z^{m+1}(ab)^n \\ &= (ab)^{2n} \end{aligned}$$

by (ii) and Lemma 1 (viii) since n is odd. Since $z^{n+2}(ab)^{2n}=1$, this gives that $(babab^{-2}a)^2=z^{-(n+2)}$, so that

$$\begin{aligned} z^{-(n+2)} &= (abab^{-2}ab)^2 \quad \text{by Lemma 1 (viii)} \\ &= (yax)^2 \\ &= yaxaayax \\ &= yx^{-1}y^{-1}x \quad \text{by Lemma 1 (i) \& (ii)}, \end{aligned}$$

which yields the result.

(vi) Assume n is even, say $n=2m$. As in (v), we have

$$\begin{aligned} z^m(ab)^{2m} &= bab^{2m}ab^{-1} \\ &= bazab^{-1} \\ &= z^{-1} \quad \text{by (ii) and Lemma 1 (vi)}, \end{aligned}$$

so that $z^{m+1}=(ab)^{-2m}$. Conjugating by a gives

$$\begin{aligned} z^{-(m+1)} &= (ba)^{-2m} \\ &= b(ab)^{-2m}b^{-1} \end{aligned}$$

$$=bz^{m+1}b^{-1}$$

$$=z^{m+1} \text{ by Lemma 1 (vi).}$$

So $z^{2m+2}=1$, i.e. $z^{n+2}=1$ as required. □

Notation. For convenience in the following, we let u denote z^{n+2}.

Lemma 5. (i) $u=1$ *for n even,*
(ii) $u^2=1$ *for n odd.*

Proof. (i) This is just a restatement of Lemma 4 (vi).

(ii) Assume n is odd. By Lemma 4 (v), $yx^{-1}y^{-1}xu=1$. Conjugating by b and using Lemma 1 (iv), (v) and (vi) gives that $y^{-1}zx^{-1}yxz^{-1}yy^{-1}u=1$, which, on using Lemma 4 (i) and (iii), gives that $y^{-1}x^{-1}yxu=1$, i.e. $x^{-1}yxuy^{-1}=1$, i.e. $uy^{-1}=x^{-1}y^{-1}x$. But, by Lemma 4 (iii) and (v) we have $u^2=yuy^{-1}u=y.x^{-1}y^{-1}x.u=1$ as required. □

Lemma 6. *If s and t are integers, then:*

(i) $y^sx^t=x^ty^su^{st}$,

(ii) $by^sb^{-1}=x^{-s}y^{-s}z^su^{s(s+1)/2}$.

Proof. (i) By Lemma 4 (v), $yx^{-1}y^{-1}xu=1$, and, by Lemma 4 (i) and (iii), $[x,u]=[y,u]=1$, so that $yx=xyu$, and the result follows.

(ii) We first consider the case where s is positive, and proceed by induction on s, the result being clear for $s=0$. So assume that

$$by^ib^{-1}=x^{-i}y^{-i}z^iu^{i(i+1)/2}$$

for $0\leqq i\leqq s$. Then

$$\begin{aligned} by^{s+1}b^{-1} &= byb^{-1}.x^{-s}y^{-s}z^su^{s(s+1)/2} \\ &= x^{-1}y^{-1}zu.x^{-s}y^{-s}z^su^{s(s+1)/2} && \text{by (i) and Lemma 1 (v)} \\ &= x^{-1}y^{-1}x^{-s}y^{-s}z^{s+1}u^{s(s+1)/2+1} && \text{by Lemma 4 (i) \& (iii)} \\ &= x^{-1}x^{-s}y^{-1}u^sy^{-s}z^{s+1}u^{s(s+1)/2+1} && \text{by (i)} \\ &= x^{-(s+1)}y^{-(s+1)}z^{s+1}u^{(s+1)(s+2)/2} && \text{by Lemma 4 (i) \& (iii)} \end{aligned}$$

as required.

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If $s<0$, let $t=-s$ and apply (i) to $by^sb^{-1}=(by^tb^{-1})^{-1}$. □

Notation. Let

$$e_i=\begin{cases}1 & \text{if } i\equiv 4 \text{ or } 5 \pmod 6,\\ 0 & \text{otherwise.}\end{cases}$$

Note that, for the Fibonacci numbers f_n:

$$f_i\equiv\begin{cases}0 \pmod 4 & \text{if } i\equiv 0 \pmod 6,\\ 1 \pmod 4 & \text{if } i\equiv 1, 2 \text{ or } 5 \pmod 6,\\ 2 \pmod 4 & \text{if } i=3 \pmod 6,\\ 3 \pmod 4 & \text{if } i\equiv 4 \pmod 6.\end{cases}$$

The following result is easily checked:

Lemma 7. *For $i\geqq 0$, we have:*

(i) $f_i(f_i+(-1)^i)/2\equiv e_{i+2} \pmod 2$,

(ii) $f_if_{i+1}\equiv e_{i+1}+e_{i+2}+e_{i+3} \pmod 2$.

We then have:

Lemma 8. *For $i\geqq 0$, $b^ixb^{-i}=x^jy^kz^lu^m$, where*

$$j=(-1)^if_{i-1},\quad k=(-1)^if_i,\quad l=(-1)^{i+1}f_{i-2}-1,\quad m=e_i.$$

Proof. If $i=0$, the result is clear. Assume the result is true for i; we then have

$$\begin{aligned}b^{i+1}xb^{-(i+1)}&=b(x^jy^kz^lu^m)b^{-1}\\ &=bx^jb^{-1}.by^kb^{-1}.z^l.u^m && \text{by Lemma 1 (vi)}\\ &=y^{-j}.x^{-k}y^{-k}z^ku^{k(k+1)/2}.z^l.u^m && \text{by Lemmas 1 (iv) \& 6 (ii)}\\ &=x^{-k}y^{-j}u^{kj}y^{-k}z^{k+l}u^{m+k(k+1)/2} && \text{by Lemma 6 (i)}\\ &=x^{-k}y^{-(k+j)}z^{k+l}u^{m+kj+k(k+1)/2} && \text{by Lemma 4 (iii)}\end{aligned}$$

where j, k, l, m are as in the statement of the lemma. Now

$$-k=(-1)^{i+1}f_i,$$

$$-(k+j)=(-1)^{i+1}(f_{i-1}+f_i)=(-1)^{i+1}f_{i+1},$$

$$k+l=(-1)^{i+2}(f_i-f_{i-2})-1=(-1)^{i+2}f_{i-1}-1,$$

and, using Lemma 7 (i) and (ii), we have

$$\begin{aligned} m+kj+k(k+1)/2 &= e_i+f_{i-1}f_i+f_i(f_i+(-1)^i)/2 \\ &\equiv e_i+e_i+e_{i+1}+e_{i+2}+e_{i+2} \pmod 2 \\ &\equiv e_{i+1} \pmod 2. \end{aligned}$$

The result follows from Lemma 5. □

5. Proof of Theorem A (ii)

Assume that n is odd, $n>0$, and let H be the subgroup $\langle x,y,z,u\rangle$ of G. From Lemma 4 (i) and (iii), we have

$$b^nxb^{-n}=x, \quad b^ny^{-1}b^{-n}=y^{-1}, \quad b^n(xyz^{-1})b^{-n}=xyz^{-1}.$$

Using Lemma 8 and $x^{-1}b^nxb^{-n}=1$, we have

$$x^jy^kz^lu^m=1, \tag{1}$$

where

$$j=-f_{n-1}-1, \quad k=-f_n, \quad l=f_{n-2}-1, \quad m=e_n.$$

Since $bxb^{-1}=y^{-1}$ by Lemma 1 (iv), $b^{n+1}xb^{-n-1}=y^{-1}$, and Lemma 8 yields

$$x^py^qz^ru^s=1, \tag{2}$$

where

$$p=f_n, \quad q=f_{n+1}+1, \quad r=-f_{n-1}-1, \quad s=e_{n+1}.$$

With this notation, we get the following presentation for H:

$$\langle x,y,z,u:[x,z]=[y,z]=1, z^{n+2}=u, u^2=1, [x,y]=u, x^jy^kz^lu^m=x^py^qz^ru^s=1\rangle.$$

Let $K=\langle u\rangle$. Then H/K is an abelian group with order

$$\begin{vmatrix} -j & -k & -l \\ p & q & r \\ 0 & 0 & n+2 \end{vmatrix} = (n+2)\begin{vmatrix} -j & -k \\ p & q \end{vmatrix}$$

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$$=(n+2)(f_{n-1}f_{n+1}+f_{n-1}+f_{n+1}+1-f_n^2)$$
$$=(n+2)(f_{n-1}+f_{n+1}+1+(-1)^n)$$
$$=(n+2)g_n$$

since n is odd. Conjugating relations (1) and (2) by a and using Lemma 1 (i) and (ii) and Lemma 4 (ii) yields

$$x^{-j}y^{-k}z^{-l}u^{-m}=1, \tag{3}$$

$$x^{-p}y^{-q}z^{-r}u^{-s}=1. \tag{4}$$

From (1) and (3) we have

$$x^jy^k=y^kx^j, \tag{5}$$

and, from (2) and (4),

$$x^py^q=y^qx^p. \tag{6}$$

Using Lemma 6 (i), (5) becomes $x^jy^k=x^jy^ku^{jk}$, and (6) becomes $x^py^q=x^py^qu^{pq}$, so that $u^{jk}=u^{pq}=1$. If $n\equiv 1 \pmod 3$, then f_{n-1} is even and f_n odd, so that $jk=(f_{n-1}+1)f_n\equiv 1 \pmod 2$. On the other hand, if $n\equiv 2 \pmod 3$, then $pq=f_n(f_{n-1}+1)\equiv 1 \pmod 2$. So, if $(n,3)=1$, then we have $|K|=1$.

Consider now the case $(n,3)=3$. By Lemma 5, $|K|\leqq 2$, so we only need to show that K is non-trivial here. Let $c=aba$, $N=\langle b,c\rangle$, so that N is normal in G of index 2 with presentation

$$\langle b,c: c^2b^{-2}c^{-1}b=b^n, b^2c^{-2}b^{-1}c=c^n\rangle.$$

Since H/K is abelian of order $(n+2)g_n$, and since $n+2$ is odd and $g_n\equiv 4 \pmod 8$ for $n\equiv 3 \pmod 6$, H/K is a direct product $O\times S$, where O has odd order and S is elementary abelian of order 4. Also, N/K is an extension of H/K by C_n. We can form a new group N_1 by replacing S with a quaternion group T of order 8, so that N_1 is an extension of $H_1=O\times T$ by C_n, with the action of N_1/H_1 on H_1 reducing to the action of N/H on H/K if we factor out the central involution. If $|H|<|H_1|$, i.e. if $|K|=1$, then the Schur multiplier $M(N)$ is non-trivial, a contradiction, as N has deficiency zero (see [**6**]). So K is non-trivial, as required.

If $n<0$, put $h=-n$. Repeating the above arguments with h in place of n, except in the relation $z^{n+2}=u$, yields the result. □

6. Proof of Theorem A (iii)

It is immediate that $[G:G']=2|n|$. The results concerning the three finite groups in the cases $n=2$, 4 or -4 may be obtained using computer implementations of various group theory algorithms. We used a Todd–Coxeter program, to which the second author has added a Reidemeister–Schreier routine based on [3] and the Tietze transformation program described in [4]. Alternatively, algebraic proofs for all these cases may be found in [1], as well as some further details about the groups $G(n)$.

If $n=-2$, we have the presentation

$$\langle a,b: a^2=1, ab^2ab^{-2}ab^{-1}ab=b^{-2}\rangle,$$

so that $bab^2 a(bab^2)^{-1}=b^{-2}a$, giving that $(b^{-2}a)^2=1$, and hence $ab^2a=b^{-2}$. The presentation now reduces to

$$\langle a,b: a^2=1, aba=b^{-1}\rangle,$$

and we have the infinite dihedral group.

If $|n|\geqq 6$, then, arguing as in Section 5 and using the fact that n is even, we obtain

$$[G':G'']=|n+2|\begin{vmatrix} f_{n-1}+1 & f_n \\ f_n. & f_{n+1}+1 \end{vmatrix}=|n+2|(g_n-2).$$

Since we know the G is an infinite group by [2, Theorem 6.1], we must have that G'' is infinite. □

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