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UNIQUENESS THEOREMS FOR DIRICHLET SERIES

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Abstract

We obtain uniqueness theorems for L-functions in the extended Selberg class when the functions share values in a finite set and share values weighted by multiplicities.

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1. Introduction

Let $\mathcal{M}(\mathbb{C})$ be the field of meromorphic functions over the field $\mathbb{C}$ of complex numbers. In this paper, we will study the uniqueness problem for meromorphic functions in the extended Selberg class $\mathcal{S}^\sharp$ of $\mathcal{M}(\mathbb{C})$. The extended Selberg class $\mathcal{S}^\sharp$ is the set of L-functions

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{1.1}$$

in a complex variable $s \in \mathbb{C}$ which satisfy the following axioms (see [9]).

(i) *Ramanujan hypothesis*. $a(n) \ll n^\varepsilon$ for any $\varepsilon > 0$, where the implicit constant may depend on ε.

(ii) *Analytic continuation*. There is a nonnegative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) *Functional equation*. $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j and complex numbers μ_j, ω with $\operatorname{Re}\mu_j \geq 0$ and $|\omega| = 1$.

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Further, an L-function $\mathcal{L}$ in $\mathcal{S}^\sharp$ is in the Selberg class $\mathcal{S}$ if $\mathcal{L}$ also satisfies the following additional axiom (see [9]).

(iv) *Euler product*. $\mathcal{L}(s)$ satisfies

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s),$$

where

$$\mathcal{L}_p(s) = \exp\Big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\Big)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

In the sequel, we mainly consider a subset $\mathcal{S}^\sharp(1)$ of $\mathcal{S}^\sharp$ defined by

$$\mathcal{S}^\sharp(1) = \{\mathcal{L} \in \mathcal{S}^\sharp \mid \mathcal{L} \text{ is expressed by a series of the form (1.1) with } a(1) = 1\}.$$

The classical question in the uniqueness theory of meromorphic functions is as follows.

Question 1.1. For a family $\mathcal{F}$ in $\mathcal{M}(\mathbb{C})$, determine subsets S of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of minimal cardinal such that any two elements f and g of $\mathcal{F}$ are algebraically dependent if $f^{-1}(a) = g^{-1}(a)$ counting multiplicity for each $a \in S$, that is, if f and g share each element of S CM (counting multiplicity).

For the case $\mathcal{F} = \mathcal{M}(\mathbb{C})$, a famous theorem of Nevanlinna claims that any subset $S \subset \overline{\mathbb{C}}$ of four distinct elements is a solution of Question 1.1 and the number 'four' is sharp (see, for example, [4] or [10]). Furthermore, two such elements in $\mathcal{M}(\mathbb{C})$ are related by a fractional linear transformation.

If $\mathcal{F} = \mathcal{S}^\sharp(1)$, a result due to Steuding (see [8] or [9]) shows that any subset S of $\mathbb{C}$ of one element is a solution of Question 1.1. Further, two such L-functions in $\mathcal{S}^\sharp(1)$ must be equal.

In 1976, Gross (see [3]) extended Question 1.1 as follows.

Question 1.2. For a family $\mathcal{F}$ in $\mathcal{M}(\mathbb{C})$, determine subsets $S_1, \ldots, S_q$ of $\overline{\mathbb{C}}$ in which the cardinal of each S_j is as small as possible and minimise the number q such that any two elements f and g of $\mathcal{F}$ are algebraically dependent if $f^{-1}(S_j) = g^{-1}(S_j)$ counting multiplicity for each j, that is, if f and g share each S_j CM (counting multiplicity).

Denote the pre-image of a subset $S \subset \overline{\mathbb{C}}$ under f by

$$E(S, f) = \bigcup_{c \in S} \{s \in \mathbb{C} \mid f(s) - c = 0\},$$

where a zero of $f - c$ with multiplicity m counts m times in $E(S, f)$. If $E(S, f) = E(S, g)$, then f and g share the set S CM.

If $q \geq 4$, Question 1.2 is completely answered by the theorem due to Nevanlinna. But it is still interesting in the cases $q < 4$. For the family $\mathcal{F} = \mathcal{A}(\mathbb{C}) \subset \mathcal{M}(\mathbb{C})$ of entire functions over $\mathbb{C}$, Gross partially solved Question 1.2 by finding three finite

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sets S_j ($j = 1, 2, 3$). Since then, there have been many studies of the uniqueness of meromorphic functions sharing sets (see, for example, [1, 2, 6, 7, 10–13]).

For the family $\mathcal{F} = \mathcal{S}^\sharp(1)$, we answer Question 1.2 completely as follows.

Theorem 1.3. *Fix a positive integer n and take a subset $S = \{c_1, \ldots, c_n\} \subset \mathbb{C} - \{1\}$ of distinct complex numbers satisfying*

$$n + (n-1)\sigma_1(c_1, \ldots, c_n) + \cdots + 2\sigma_{n-2}(c_1, \ldots, c_n) + \sigma_{n-1}(c_1, \ldots, c_n) \neq 0,$$

where σ_j are the elementary symmetric polynomials defined by

$$\sigma_j(c_1, \ldots, c_n) = (-1)^j \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} c_{i_1} c_{i_2} \cdots c_{i_j}, \quad j = 1, 2, \ldots, n-1.$$

If two L-functions $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ in $\mathcal{S}^\sharp(1)$ share S CM, then $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$.

In particular, the result due to Steuding is a special case of Theorem 1.3 corresponding to the case $n = 1$.

Let k denote a nonnegative integer or $+\infty$. For any $c \in \overline{\mathbb{C}}$, we denote by $E_k(c, f)$ the set of all c-points of f, where a c-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. For $S \subseteq \overline{\mathbb{C}}$, we define

$$E_k(S, f) = \bigcup_{c \in S} E_k(c, f).$$

If $E_k(S, f) = E_k(S, g)$, then f and g share the set S weighted by k (or with weight k, or truncated multiplicity $k + 1$). For the notation and basic results from Nevanlinna theory and further details related to $\mathcal{F} = \mathcal{M}(\mathbb{C})$ or $\mathcal{A}(\mathbb{C})$, see [4, 10].

Questions 1.1 and 1.2 are special cases of the following general question.

Question 1.4. For a family $\mathcal{F}$ in $\mathcal{M}(\mathbb{C})$, determine subsets $S_1, \ldots, S_q$ of $\overline{\mathbb{C}}$ in which the cardinal of each S_j is as small as possible and minimise the number q such that any two elements f and g of $\mathcal{F}$ are algebraically dependent if f and g share each S_j weighted by k (or with truncated multiplicity $k + 1$).

For the case $\mathcal{F} = \mathcal{M}(\mathbb{C})$, $q \ge 5$, $k = 0$, Nevanlinna completely settled Question 1.4 by choosing $S_j = \{c_j\}$ for distinct elements c_j of $\overline{\mathbb{C}}$, and proved that two such functions must be equal. However, Question 1.4 is still interesting for the cases $q \le 4$.

If $\mathcal{F}$ is the subfamily $\mathcal{S}_e^\sharp(1)$ of the family $\mathcal{S}^\sharp(1)$ satisfying the same functional equation and with an additional condition, Steuding (see [9] or [8]) partially answered Question 1.4 for the case $k = 0$, $q = 2$, where the $S_j = \{c_j\}$ consist of two distinct elements c_j of $\mathbb{C}$. For the case $\mathcal{F} = \mathcal{S}_e^\sharp(1)$, Li (see [5]) completely solved this case by removing the additional condition in Steuding's result. Moreover, in order to extend Steuding–Li's result from the subfamily $\mathcal{S}_e^\sharp(1)$ to the global family $\mathcal{S}^\sharp(1)$, it would be desirable to remove the assumption that both L-functions satisfy the same functional equation (see [8]). By including weights, we can reach this goal as follows.

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THEOREM 1.5. *Let c_1, c_2 be two distinct complex numbers and take two positive integers k_1, k_2 with $k_1k_2 > 1$. If two L-functions $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ in $\mathcal{S}^\sharp(1)$ share c_1, c_2 weighted k_1, k_2, respectively, then $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$.*

THEOREM 1.6. *Let k_1, k_2 be two positive integers with $k_1k_2 > 1$ and take a complex number c and a nonempty subset $S = \{c_1, \ldots, c_n\} \subset \mathbb{C} - \{1, c\}$ of distinct complex numbers satisfying*

$$n + (n-1)\sigma_1(c_1, \ldots, c_n) + \cdots + 2\sigma_{n-2}(c_1, \ldots, c_n) + \sigma_{n-1}(c_1, \ldots, c_n) \neq 0,$$

where

$$\sigma_j(c_1, \ldots, c_n) = (-1)^j \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} c_{i_1} c_{i_2} \cdots c_{i_j}, \quad j = 1, 2, \ldots, n-1.$$

If two L-functions $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ in $\mathcal{S}^\sharp(1)$ share c, S weighted k_1, k_2, respectively, then $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$.

2. Proofs of the theorems

2.1. Proof of Theorem 1.3. First of all, assume that $\mathcal{L}_1(s)$, $\mathcal{L}_2(s)$ are both entire functions and share the set $S = \{c_1, c_2, \ldots, c_n\}$ CM. We obtain an entire function

$$l(s) = \frac{(\mathcal{L}_1(s) - c_1)(\mathcal{L}_1(s) - c_2) \cdots (\mathcal{L}_1(s) - c_n)}{(\mathcal{L}_2(s) - c_1)(\mathcal{L}_2(s) - c_2) \cdots (\mathcal{L}_2(s) - c_n)}$$

with $l(s) \neq 0, \infty$. By the first fundamental theorem,

$$T\Big(r, \frac{1}{\mathcal{L}_2(s) - c_i}\Big) = T(r, \mathcal{L}_2) + O(1)$$

for $i = 1, 2, \ldots, n$. If we denote the order of a meromorphic function f by $\rho(f)$, then it follows that

$$\rho\Big(\frac{1}{\mathcal{L}_2 - c_i}\Big) = \rho(\mathcal{L}_2) = 1.$$

Moreover,

$$\rho(\mathcal{L}_1 - c_i) = \rho(\mathcal{L}_1) = 1, \quad i = 1, 2, \ldots, n.$$

Since the order of a finite product of functions of finite order is less than or equal to the maximum of the order of these factors (see [10]), we have $\rho(l) \le 1$. This implies that $l(s)$ is of the form

$$l(s) = e^{P(s)},$$

where $P(s)$ is a polynomial of degree at most $\rho(l) \le 1$. Since $\mathcal{L}_j(s) \to 1$ as $s \to +\infty$ for $j = 1, 2$,

$$\lim_{s \to +\infty} l(s) = \frac{(1 - c_1)(1 - c_2) \cdots (1 - c_n)}{(1 - c_1)(1 - c_2) \cdots (1 - c_n)} = 1.$$

This implies that the polynomial $P(s) \equiv 0$, that is, $l(s) \equiv 1$.

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If $\mathcal{L}_1(s)$ or $\mathcal{L}_2(s)$ has a pole at $s = 1$ with multiplicity k_1 (≥ 0) or k_2 (≥ 0), respectively, we may set

$$l(s) = \frac{(s-1)^k(\mathcal{L}_1(s)-c_1)(\mathcal{L}_1(s)-c_2)\cdots(\mathcal{L}_1(s)-c_n)}{(\mathcal{L}_2(s)-c_1)(\mathcal{L}_2(s)-c_2)\cdots(\mathcal{L}_2(s)-c_n)},$$

where $k = n(k_2 - k_1)$ is an integer. Repeating the argument above, we see that $l(s)$ is of the form

$$l(s) = e^{P(s)},$$

where $P(s)$ is a polynomial of degree at most $\lambda(l) \leq 1$. If $P(s)$ is a polynomial of degree one, denote it as $As + B$, where $A(\neq 0), B$ are constants. This leads to a contradiction because

$$\lim_{s\to+\infty}(s-1)^{-k}e^{As+B} = \lim_{s\to+\infty}(s-1)^{-k}l(s) = \lim_{s\to+\infty}\frac{(1-c_1)(1-c_2)\cdots(1-c_n)}{(1-c_1)(1-c_2)\cdots(1-c_n)} = 1. \tag{2.1}$$

Therefore, $P(s)$ is a constant. In view of (2.1), we get $k = 0$. Then it follows that $l(s) \equiv 1$.

If $\mathcal{L}_1(s) \not\equiv \mathcal{L}_2(s)$, on account of

$$l(s) = \frac{(\mathcal{L}_1(s)-c_1)(\mathcal{L}_1(s)-c_2)\cdots(\mathcal{L}_1(s)-c_n)}{(\mathcal{L}_2(s)-c_1)(\mathcal{L}_2(s)-c_2)\cdots(\mathcal{L}_2(s)-c_n)} \equiv 1,$$

we have the following equations:

$$(\mathcal{L}_1-c_1)(\mathcal{L}_1-c_2)\cdots(\mathcal{L}_1-c_n) \equiv (\mathcal{L}_2-c_1)(\mathcal{L}_2-c_2)\cdots(\mathcal{L}_2-c_n),$$
$$\mathcal{L}_1^n + \sigma_1\mathcal{L}_1^{n-1} + \cdots + \sigma_{n-2}\mathcal{L}_1^2 + \sigma_{n-1}\mathcal{L}_1 \equiv \mathcal{L}_2^n + \sigma_1\mathcal{L}_2^{n-1} + \cdots + \sigma_{n-2}\mathcal{L}_2^2 + \sigma_{n-1}\mathcal{L}_2,$$
$$(\mathcal{L}_1^n - \mathcal{L}_2^n) + \sigma_1(\mathcal{L}_1^{n-1} - \mathcal{L}_2^{n-1}) + \cdots + \sigma_{n-2}(\mathcal{L}_1^2 - \mathcal{L}_2^2) + \sigma_{n-1}(\mathcal{L}_1 - \mathcal{L}_2) \equiv 0$$

and

$$\begin{aligned}(\mathcal{L}_1 - \mathcal{L}_2)((\mathcal{L}_1^{n-1} + \mathcal{L}_1^{n-2}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-1}) + \sigma_1(\mathcal{L}_1^{n-2} + \mathcal{L}_1^{n-3}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-2})\\ + \cdots + \sigma_{n-2}(\mathcal{L}_1 + \mathcal{L}_2) + \sigma_{n-1}) \equiv 0,\end{aligned}$$

where

$$\sigma_j = (-1)^j \sum_{1\leq i_1<i_2<\cdots<i_j\leq n} c_{i_1}c_{i_2}\cdots c_{i_j}, \quad j = 1, 2, \ldots, n-1.$$

Set

$$\begin{aligned}h(s) = (\mathcal{L}_1^{n-1} + \mathcal{L}_1^{n-2}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-1}) + \sigma_1(\mathcal{L}_1^{n-2} + \mathcal{L}_1^{n-3}\mathcal{L}_2 + \cdots + \mathcal{L}_2^{n-2})\\ + \cdots + \sigma_{n-2}(\mathcal{L}_1 + \mathcal{L}_2) + \sigma_{n-1}.\end{aligned}$$

Since $\mathcal{L}_j(s)$ tends to 1 as $s \to +\infty$ for $j = 1, 2$, it is easy to deduce that

$$\lim_{s\to+\infty} h(s) = n + (n-1)\sigma_1 + \cdots + 2\sigma_{n-2} + \sigma_{n-1} \neq 0.$$

Thus, we have $\mathcal{L}_1 \equiv \mathcal{L}_2$. This completes the proof of Theorem 1.3.

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2.2. Proof of Theorem 1.5. We first look at the simple case when one of $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$, say $\mathcal{L}_1(s)$, is constant. Then $\mathcal{L}_1(s) \equiv 1$ by the assumption that $a(1) = 1$. Since $\mathcal{L}_2(s) - c_j$ and $\mathcal{L}_1(s) - c_j$ $(j = 1, 2)$ have the same zeros by the assumption, it is easy to see that $\mathcal{L}_2(s) \equiv 1$ when c_1 or c_2 is 1, or $\mathcal{L}_2(s) \neq c_1, c_2$ in $\mathbb{C}$ when $c_1, c_2 \neq 1$. In the latter case, noting that an L-function has at most one pole, $\mathcal{L}_2(s)$ must be constant and thus $\mathcal{L}_2(s) \equiv 1$ since $a(1) = 1$, by the class Picard theorem (see for example [10]) that a nonconstant meromorphic function in $\mathbb{C}$ assumes each value in $\mathbb{C} \cup \{\infty\}$ infinitely many times with at most two exceptions. Therefore, $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$.

We thus assume, in the following, that $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ are nonconstant. We consider the following two auxiliary functions:

$$F_1(s) = \frac{\mathcal{L}_1'(s)}{\mathcal{L}_1(s) - c_1} - \frac{\mathcal{L}_2'(s)}{\mathcal{L}_2(s) - c_1}, \tag{2.2}$$

$$F_2(s) = \frac{\mathcal{L}_1'(s)}{\mathcal{L}_1(s) - c_2} - \frac{\mathcal{L}_2'(s)}{\mathcal{L}_2(s) - c_2}. \tag{2.3}$$

If $F_1(s) \equiv 0$, by integration, we have from (2.2) that

$$\mathcal{L}_1(s) - c_1 \equiv A(\mathcal{L}_2(s) - c_1),$$

where $A \neq 0$ is a constant. This implies that $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ share c_1 CM; thus, $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$. If $F_2(s) \equiv 0$, by repeating the argument above, we also get $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$. Next, we assume that $F_1(s) \not\equiv 0$ and $F_2(s) \not\equiv 0$. Since $\mathcal{L}_1(s)$, $\mathcal{L}_2(s)$ share $(c_1, k_1), (c_2, k_2)$, from (2.2),

$$\begin{aligned} k_2 \overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) &\leq \overline{N}_{(2}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) + (k_2 - 1)\overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) \\ &\leq N\Big(r, \frac{1}{F_1}\Big) \leq T(r, F_1) + O(1) \leq N(r, F_1) + m(r, F_1) + O(1) \\ &\leq \overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) + \overline{N}(r, \mathcal{L}_1) + \overline{N}(r, \mathcal{L}_2) + S(r, \mathcal{L}_1) + S(r, \mathcal{L}_2) \\ &\leq \overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) + O(\log r). \end{aligned} \tag{2.4}$$

Similarly, from (2.3),

$$\begin{aligned} k_1 \overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) &\leq \overline{N}_{(2}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) + (k_1 - 1)\overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) \\ &\leq N\Big(r, \frac{1}{F_2}\Big) \leq T(r, F_2) + O(1) \leq N(r, F_2) + m(r, F_2) + O(1) \\ &\leq \overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) + \overline{N}(r, \mathcal{L}_1) + \overline{N}(r, \mathcal{L}_2) + S(r, \mathcal{L}_1) + S(r, \mathcal{L}_2) \\ &\leq \overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) + O(\log r). \end{aligned} \tag{2.5}$$

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Combining (2.4) and (2.5),

$$\overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) \le \frac{1}{k_1}\overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) + O(\log r)$$
$$\le \frac{1}{k_1 k_2}\overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) + O(\log r).$$

Since $k_1 k_2 > 1$,

$$\overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) = O(\log r). \tag{2.6}$$

Substituting (2.6) into (2.4),

$$\overline{N}_{(k_2+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) = O(\log r). \tag{2.7}$$

Furthermore,

$$\overline{N}_{(2}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) = O(\log r). \tag{2.8}$$

Substituting (2.7) into (2.5),

$$\overline{N}_{(2}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) = O(\log r).$$

In addition, from (2.2) and (2.6),

$$\begin{aligned} T(r, F_1) &= N(r, F_1) + m(r, F_1) \\ &\le \overline{N}_{(k_1+1}\Big(r, \frac{1}{\mathcal{L}_1 - c_1}\Big) + \overline{N}(r, \mathcal{L}_1) + \overline{N}(r, \mathcal{L}_2) + S(r, \mathcal{L}_1) + S(r, \mathcal{L}_2) \\ &= O(\log r); \end{aligned}$$

this implies that $F_1(s)$ is a rational function. Set $F_1(s) = (P(s)/Q(s))$, that is,

$$\frac{\mathcal{L}_1'(s)}{\mathcal{L}_1(s) - c_1} - \frac{\mathcal{L}_2'(s)}{\mathcal{L}_2(s) - c_1} = \frac{P(s)}{Q(s)}; \tag{2.9}$$

integrating both sides of the equality (2.9),

$$\frac{\mathcal{L}_1(s) - c_1}{\mathcal{L}_2(s) - c_1} = e^{\int (P(s)/Q(s))\, ds}. \tag{2.10}$$

Since $\mathcal{L}_j(s) \to 1$ as $s \to +\infty$ for $j = 1, 2$,

$$\lim_{s\to+\infty} \frac{\mathcal{L}_1(s) - c_1}{\mathcal{L}_2(s) - c_1} = 1$$

for $c_1 \neq 1$. If $c_1 = 1$, then we can replace c_1 by c_2. Thus,

$$\lim_{s\to+\infty} \int \frac{P(s)}{Q(s)}\, ds = 0.$$

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It follows that $\deg(P(s)) < \deg(Q(s))$. In addition, by a simple calculation, we see that all poles of $F_1(s)$ are simple. Therefore, we can rewrite $F_1(s)$ as

$$F_1(s) = \frac{P(s)}{Q(s)} = \frac{c\prod_{i=1}^{m}(s-a_i)}{\prod_{j=1}^{n}(s-b_j)} = \sum_{j=1}^{n}\frac{\lambda_j}{s-b_j},$$

where $c \neq 0$ is a constant, m, n are two positive integers satisfying $m < n$ and a_i $(i = 1, 2, \ldots, m)$, b_j $(j = 1, 2, \ldots, n)$ with $b_i \neq b_j$ $(i \neq j)$ being the zeros and poles of $F_1(s)$, respectively. Then

$$\int \frac{P(s)}{Q(s)}\,ds = \sum_{j=1}^{n}\int \frac{\lambda_j}{s-b_j}\,ds = \sum_{j=1}^{n}\lambda_j \ln(s-b_j) + C_1,$$

where C_1 is a constant. Note that the λ_j $(j = 1, 2, \ldots, n)$ are integers because $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ are both meromorphic functions. Using this in (2.10),

$$\mathcal{L}_1(s) - c_1 = A(\mathcal{L}_2(s) - c_1)\prod_{j=1}^{n}(s-b_j)^{\lambda_j} = A(\mathcal{L}_2(s) - c_1)\frac{\prod_{j=1}^{t_1}(s-b_j)^{\lambda_j}}{\prod_{j=t_1+1}^{n}(s-b_j)^{-\lambda_j}}, \quad (2.11)$$

where $A \neq 0$ is a constant, $\lambda_j > 0$ $(j = 1, \ldots, t_1)$ and $\lambda_j < 0$ $(j = t_1 + 1, \ldots, n)$.

If $N_{1)}(r, 1/\mathcal{L}_1 - c_2) \neq S(r, \mathcal{L}_1)$, then, for any s_0 such that $\mathcal{L}_1(s_0) = c_2$, we have $\mathcal{L}_2(s_0) = c_2$. Thus, from (2.11),

$$A\prod_{j=1}^{t_1}(s_0-b_j)^{\lambda_j} = \prod_{j=t_1+1}^{n}(s_0-b_j)^{-\lambda_j}.$$

Set

$$M(s) = A\prod_{j=1}^{t_1}(s-b_j)^{\lambda_j} - \prod_{j=t_1+1}^{n}(s-b_j)^{-\lambda_j}.$$

Then $M(s)$ has at most n zeros, which contradicts $N_{1)}(r, 1/\mathcal{L}_1 - c_2) \neq S(r, \mathcal{L}_1)$. Therefore, we have $N_{1)}(r, 1/\mathcal{L}_1 - c_2) = S(r, \mathcal{L}_1)$. Combining this with (2.8),

$$\overline{N}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) = N_{1)}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{\mathcal{L}_1 - c_2}\Big) = S(r, \mathcal{L}_1) = O(\log r). \quad (2.12)$$

Since $\mathcal{L}_1$, $\mathcal{L}_2$ share c_2 weighted k_2,

$$\overline{N}\Big(r, \frac{1}{\mathcal{L}_2 - c_2}\Big) = O(\log r).$$

In the following, we consider the function

$$H = \frac{\mathcal{L}_1''}{\mathcal{L}_1'} - \frac{2\mathcal{L}_1'}{\mathcal{L}_1' - c_1} - \Big(\frac{\mathcal{L}_2''}{\mathcal{L}_2'} - \frac{2\mathcal{L}_2'}{\mathcal{L}_2' - c_1}\Big).$$

If $H \not\equiv 0$, then it follows that

$$m(r, H) = S(r, \mathcal{L}_1) + S(r, \mathcal{L}_2) = O(\log r)$$

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and

$$\begin{aligned} N(r,H) \le\ & \overline{N}_{(2}(r,\mathcal{L}_1) + \overline{N}_{(2}(r,\mathcal{L}_2) + \overline{N}_{(2}\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{\mathcal{L}_2-c_1}\Big) \\ & + \overline{N}_{(2}\Big(r,\frac{1}{\mathcal{L}_1-c_2}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{\mathcal{L}_2-c_2}\Big) + \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_1'}\Big) + \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_2'}\Big) \\ \le\ & \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_1'}\Big) + \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_2'}\Big) + O(\log r), \end{aligned}$$

where $\overline{N}_{\otimes}(r,1/\mathcal{L}_1')$ denotes the reduced counting function of the zeros of $\mathcal{L}_1'$ which are not the zeros of $(\mathcal{L}_1-c_1)(\mathcal{L}_1-c_2)$. Since $\mathcal{L}_1$, $\mathcal{L}_2$ share c_1 weighted k_1 (≥ 1), by a simple calculation, we can deduce that the simple zeros of $\mathcal{L}_1-c_1$ are the zeros of H. Thus, by the first fundamental theorem,

$$N_{1)}\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) \le N\Big(r,\frac{1}{H}\Big) \le N(r,H)+m(r,H) \le \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_1'}\Big) + \overline{N}_{\otimes}\Big(r,\frac{1}{\mathcal{L}_2'}\Big) + O(\log r).$$

Noting that the zeros of $\mathcal{L}_1-c_1$ with multiplicity $k\ge 2$ are the zeros of $\mathcal{L}_1'$ with multiplicity $k-1$,

$$N\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) = N_{1)}\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) + N_{(2}\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) \le N_0\Big(r,\frac{1}{\mathcal{L}_1'}\Big) + N_0\Big(r,\frac{1}{\mathcal{L}_2'}\Big) + O(\log r),$$

where $N_0(r,1/\mathcal{L}_1')$ denotes the counting function of the zeros of $\mathcal{L}_1'$ which are not the zeros of $\mathcal{L}_1-c_2$. Suppose that

$$\psi = \frac{\mathcal{L}_1'}{\mathcal{L}_1-c_2};$$

then it is easy to see that

$$m(r,\psi) = S(r,\mathcal{L}_1), \quad N(r,\psi) \le \overline{N}(r,\mathcal{L}_1) + \overline{N}\Big(r,\frac{1}{\mathcal{L}_1-c_2}\Big)$$

and

$$N_0\Big(r,\frac{1}{\mathcal{L}_1'}\Big) \le N\Big(r,\frac{1}{\psi}\Big).$$

By the first fundamental theorem and (2.12),

$$N_0\Big(r,\frac{1}{\mathcal{L}_1'}\Big) = S(r,\mathcal{L}_1) = O(\log r).$$

The same argument shows that

$$N_0\Big(r,\frac{1}{\mathcal{L}_2'}\Big) = S(r,\mathcal{L}_2) = O(\log r).$$

Therefore, we have $N(r,1/\mathcal{L}_1-c_1) = O(\log r)$. By the second fundamental second theorem,

$$T(r,\mathcal{L}_1) \le \overline{N}(r,\mathcal{L}_1) + \overline{N}\Big(r,\frac{1}{\mathcal{L}_1-c_1}\Big) + \overline{N}\Big(r,\frac{1}{\mathcal{L}_1-c_2}\Big) + S(r,\mathcal{L}_1) = O(\log r),$$

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which is a contradiction. Thus, $H \equiv 0$. By integration,

$$\frac{1}{\mathcal{L}_1 - c_1} \equiv \frac{A}{\mathcal{L}_2 - c_1} + B,$$

where $A \neq 0, B$ are two constants. It shows that $\mathcal{L}_1$, $\mathcal{L}_2$ share c_1 CM. Thus, we have $\mathcal{L}_1 \equiv \mathcal{L}_2$. This completes the proof of Theorem 1.5.

2.3. Proof of Theorem 1.6. By the same argument as in the proof of Theorem 1.5, we see that if one of $\mathcal{L}_1$ and $\mathcal{L}_2$ is constant, then $\mathcal{L}_1 \equiv \mathcal{L}_2$. In the following, we consider the case that $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ are nonconstant. Define two functions

$$l_1(s) = (\mathcal{L}_1(s) - c_1)(\mathcal{L}_1(s) - c_2) \cdots (\mathcal{L}_1(s) - c_n),$$
$$l_2(s) = (\mathcal{L}_2(s) - c_1)(\mathcal{L}_2(s) - c_2) \cdots (\mathcal{L}_2(s) - c_n).$$

Then $l_1(s)$, $l_2(s)$ share the values $a = (c - c_1)(c - c_2) \cdots (c - c_n) \neq 0$ and 0 with the weights k_1 and k_2, respectively. Next, we consider the following two auxiliary functions:

$$F_1(s) = \frac{l_1'(s)}{l_1(s)} - \frac{l_2'(s)}{l_2(s)}, \tag{2.13}$$

$$F_2(s) = \frac{l_1'(s)}{l_1(s) - a} - \frac{l_2'(s)}{l_2(s) - a}. \tag{2.14}$$

If $F_1(s) \equiv 0$, by integration, then we have $l_1(s) \equiv A l_2(s)$ from (2.13), where $A \neq 0$ is a constant. This implies that $l_1(s)$, $l_2(s)$ share the value 0 CM. From the definition of $l_i(s)$ $(i = 1, 2)$, we deduce that $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ share the set $S = \{c_1, c_2, \ldots, c_n\}$ CM. By Theorem 1.3, $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$. If $F_2(s) \equiv 0$, from (2.14), we have $l_1(s) - a \equiv A(l_2(s) - a)$. Since $l_1(s)$, $l_2(s)$ share the value 0 with weight k_2, we have $A = 1$. Thus, $l_1(s) \equiv l_2(s)$. From the definition of $l_i(s)$ $(i = 1, 2)$, we deduce that $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$ share the set $S = \{c_1, c_2, \ldots, c_n\}$ CM and, by Theorem 1.3, we get $\mathcal{L}_1(s) \equiv \mathcal{L}_2(s)$. If $F_1(s) \not\equiv 0$ and $F_2(s) \not\equiv 0$, then we repeat the argument from the proof of Theorem 1.5 to reach the same conclusion. This completes the proof of Theorem 1.6.

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