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Liouville's Theorem in the Radially Symmetric Case

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Abstract. We present a very short proof of Liouville's theorem for solutions to a non-uniformly elliptic radially symmetric equation. The proof uses the Ricatti equation satisfied by the Dirichlet to Neumann map.

1 Introduction

The classical version of Liouville's theorem asserts that if a harmonic function defined on all of Euclidean space is bounded, it must be constant. This fundamental result has been generalized in many directions, and its study for manifolds is a large field of research. Several years ago, in connection with their work on the De Giorgi conjecture, Ghoussoub and Gui [GG] raised the question of whether Liouville's theorem holds in Euclidean space for solutions of non uniformly elliptic equations of the form

$$\nabla \cdot \sigma^2 \nabla \varphi = 0$$

in place of harmonic functions. Here $\sigma(x)^2$ is bounded and positive, but need not be bounded away from zero. It was known from the work of Berestycki, Caffarelli and Nirenberg [BCN] that in two dimensions this version of Liouville's theorem holds. However Ghoussoub and Gui [GG] showed that that in dimensions 7 and higher it does not. The remaining cases, dimensions 3 through 6 were settled, also in the negative, by Barlow [B].

The situation is quite different if σ is assumed to be radially symmetric. In this case Liouville's theorem does hold in any dimension. This is known from work of Losev [L] (see also [LM], which contains further references). Of course, the radially symmetric case is much simpler, since separation of variables reduces the problem to a question about ODE's. Specifically, any solution φ is a linear combination of solutions of the form $\varphi(x) = u(|x|)Y(x/|x|)$, where Y is an eigenfunction of the spherical Laplacian with eigenvalue μ^2, and u satisfies the ODE (1) below.

This paper is just a small remark about this simple case. We show that the Ricatti equation for the Dirichlet to Neumann map leads to a very short proof. Of course, in our setting the Dirichlet to Neuman map is just a function, namely the function f defined by (3) below. But it is interesting to note that the Dirichlet to Neumann map satisfies an operator version of the same Ricatti equation (4) that f satisfies, even when σ is not radially symmetric. Perhaps this can be used to give a proof of

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Liouville's theorem for perturbations of radially symmetric σ. However, we did not see a simple way of doing this.

2 Growth of Solutions

Theorem 1 *Suppose that u solves the radial Laplace equation*

$$(1) \qquad \big(\sigma^2(r)u'(r)\big)' + \frac{(n-1)\sigma^2(r)}{r}u'(r) - \frac{\mu^2\sigma^2(r)}{r^2}u(r) = 0$$

for $r \in (0, \infty)$. *Assume* $n \geq 3$ *and* $\sigma^2(r) \in C^1\big([0,\infty)\big)$ *with* $0 < \sigma^2(r) \leq 1$. *Define* $\beta(\mu)$ *to be*

$$\beta(\mu) = -\frac{n-2}{2} + \sqrt{\Big(\frac{n-2}{2}\Big)^2 + \mu^2}$$

If u is bounded by

$$(2) \qquad |u(r)| \leq C(1 + r^\beta)$$

then $\beta \geq \beta(\mu)$.

Remark This theorem gives a minimal growth rate, depending on μ, for a solution u to the radial Laplace equation. It implies Liouville's theorem, since a bounded solution ($\beta = 0$) is only possible when $\beta(\mu) = 0$. In this case $\mu = 0$, and $u(r) = C$ is the only solution bounded at the origin.

Remark When $\sigma(r) = 1$ the solution bounded at the origin is $u(r) = r^{\beta(\mu)}$. Thus the value of $\beta(\mu)$ is optimal.

Proof Suppose that u is a solution satisfying (2). We must show that $\beta \geq \beta(\mu)$.
Define

$$(3) \qquad f(r) = \frac{r^{n-1}\sigma^2(r)u'(r)}{u(r)}$$

Then f satisfies the Ricatti type equation

$$(4) \qquad f'(r) = r^{n-3}\mu^2\sigma^2(r) - \frac{1}{r^{n-1}\sigma^2(r)}f^2(r)$$

We begin by showing that f is well defined and positive. This follows from the ODE version of Calderon's identity, namely,

$$(5) \qquad \sigma^2(r)u'(r)u(r)r^{n-1} = \int_0^r \sigma^2(s)\big(u'(s)^2 + \frac{\mu^2}{s^2}u^2(s)\big)\,s^{n-1}\,ds > 0$$

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To prove this we first integrate by parts to obtain

$$(6)\qquad 0 < \int_a^r \sigma^2(s)\big(u'(s)^2 + \frac{\mu^2}{s^2}u^2(s)\big)\, s^{n-1}\, ds$$
$$= \sigma^2(s)u'(s)u(s)s^{n-1}|_a^r - \int_a^r u(s)\Big\{\frac{d}{ds}\big(\sigma^2 u' s^{n-1}\big) - \frac{\mu^2}{s^2}u(s)s^{n-1}\Big\}\, ds$$
$$= \sigma^2(s)u'(s)u(s)s^{n-1}|_a^r$$

Then (5) follows from

$$(7)\qquad \lim_{a\to 0}\sigma^2(a)u'(a)u(a)a^{n-1} = 0$$

To see this, notice that the equation implies that $(\sigma^2 r^{n-1}u')' = \mu^2\sigma^2 r^{n-3}u$, so that the bounds on σ^2 and u imply that near zero,

$$0 \le (\sigma^2 r^{n-1}u')' \le Cr^{n-3}$$

Integrating from 0 to a and letting a tend to zero gives (7) and thus (5). This shows that neither u nor u' can vanish, and that f is well defined, does not blow up, and is positive. Since u cannot change sign, we may as well assume that $u > 0$.

The idea of the proof is to estimate the quantity

$$(8)\qquad Q(r) = \int_1^r u'(x)/u(x) + \epsilon f'(x)/f(x)\, dx$$

from above and below for large r.

We begin with the upper bound. Performing the integral in (8) yields

$$Q(r) = \ln\big(u(r)\big) + \epsilon\ln\big(f(r)\big) + C$$

Here C denotes a generic constant that may depend on μ but not r. Dropping the negative second term in (4) and recalling that $\sigma^2(r) \le 1$ yields

$$f'(r) \le r^{n-3}\mu^2\sigma^2(r) \le r^{n-3}\mu^2$$

Integrating this yields

$$f(r) \le \frac{\mu^2}{n-2}r^{n-2} + C.$$

This estimate together with our growth assumptions on u imply

$$Q(r) \le \beta\ln(r) + \epsilon(n-2)\ln(r) + C$$

Now we turn to the lower bound. Using the expression given by (4) for f' we obtain

$$u'(x)/u(x) + \epsilon f'(x)/f(x) = \frac{f(r)}{r^{n-1}\sigma^2(r)} + \epsilon\Big(\frac{r^{n-3}\mu^2\sigma^2(r)}{f(r)} - \frac{f}{r^{n-1}\sigma^2(r)}\Big)$$
$$\ge (1-\epsilon)\frac{f(r)}{r^{n-1}\sigma^2(r)} + \epsilon\frac{r^{n-3}\mu^2\sigma^2(r)}{f(r)}$$

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Now assume that $0 \leq \epsilon \leq 1$. Since for positive numbers a, b and c we have

$$ac + b/c \geq 2\sqrt{ab}$$

we find

$$u'(x)/u(x) + \epsilon f'(x)/f(x) \geq 2\mu\sqrt{\epsilon(1-\epsilon)}/r.$$

This implies

$$Q(r) \geq 2\mu\sqrt{\epsilon(1-\epsilon)}\ln(r) + C$$

Comparing the upper and lower bounds for $Q(r)$ for large r yields

$$\beta \geq 2\mu\sqrt{\epsilon(1-\epsilon)} - \epsilon(n-2)$$

Since this holds for every $\epsilon \in [0,1]$, the theorem now follows from the computation

$$\max_{\epsilon\in[0,1]} 2\mu\sqrt{\epsilon(1-\epsilon)} - \epsilon(n-2) = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu^2} = \beta(\mu) \qquad \blacksquare$$

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