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SMALL SETS OF k-TH POWERS

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ABSTRACT. Let $k \geq 2$ and $q = g(k) - G(k)$, where $g(k)$ is the smallest possible value of r such that every natural number is the sum of at most r k-th powers and $G(k)$ is the minimal value of r such that every sufficiently large integer is the sum of r k-th powers. For each positive integer $r \geq q$, let $u'_r = g(k) + r - q$. Then for every $\varepsilon > 0$ and $N \geq N(r, \varepsilon)$, we construct a set A of k-th powers such that $|A| \leq (r(2+\varepsilon)^r + 1)N^{1/(k+r)}$ and every nonnegative integer $n \leq N$ is the sum of u'_r k-th powers in A. Some related results are also obtained.

The famous Waring's problem states that for every $k \geq 2$ there exists a number $r \geq 1$ such that every natural number is the sum of at most r k-th powers. Let $g(k)$ be the smallest possible value for r. Analogous to $g(k)$, let $G(k)$ denote the minimal value of r such that every sufficiently large integer is the sum of r k-th powers. Clearly $G(k) \leq g(k)$. In 1770, Lagrange proved that $g(2) = 4$. Since every positive integer of the form $8t + 7$ cannot be written as the sum of three squares, $G(2)$ cannot be 3, and so $G(2) = g(2) = 4$. In 1909, Wieferich [8] proved $g(3) = 9$. Landau [2] and Linnik [3] obtained $G(3) \leq 8$ and $G(3) \leq 7$ in 1909 and 1943 respectively. Though forty-nine years have passed without an improvement to $G(3)$, it is never-the-less conjectured that $G(3) = 4$ (*cf.* [5], p. 240).

Choi, Erdős and Nathanson [1] showed that for every $N > 1$, there is a set A of squares such that $|A| < (4/\log 2)N^{1/3}\log N$ and every $n \leq N$ is a sum of four squares in A; here and below we denote by $|A|$ the cardinality of set A. Nathanson [4] proved the following more general result.

THEOREM A. *Let $k \geq 2$ and $s = g(k) + 1$. For any $\varepsilon > 0$ and all $N \geq N(\varepsilon)$ there exists a finite set A of k-th powers such that*

$$|A| \leq (2+\varepsilon)N^{1/(k+1)}$$

and each nonnegative integer $n \leq N$ is the sum of s elements belonging to A.

Our Theorem 1 is a generalization of Theorem A (Theorem A is the special case $r = 1$).

THEOREM 1. *Let $k \geq 2$ and for any positive integer r let $u_r = g(k) + r$. Then for every $\varepsilon > 0$ and all $N \geq N(r, \varepsilon)$, there exists a finite set A of k-th powers such that*

$$|A| \leq C(r, \varepsilon)N^{1/(k+r)}$$

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and every nonnegative integer $n \leq N$ *is the sum of* u_r *k-th powers in A, where* $C(r, \varepsilon) = r(1+\varepsilon)^r + 1$.

Since in most cases $G(k) < g(k)$, one could naturally think of sharpening Theorem 1 in terms of $G(k)$. Our Theorem 2 achieves this goal.

THEOREM 2. *Let* $k \geq 2$ *and* $q = g(k) - G(k)$. *For each positive integer* $r \geq q$ *let* $u'_r = g(k) + r - q$. *Then for every* $\varepsilon > 0$ *and all* $N \geq N(r, \varepsilon)$, *there exists a finite set A of k-th powers such that*

$$|A| \leq C'(r, \varepsilon) N^{1/(k+r)}$$

and every nonnegative integer $n \leq N$ *is the sum of* u'_r *elements of A, where* $C'(r, \varepsilon) = r(2+\varepsilon)^r + 1$.

We list known values and estimations for some $g(k)$ and $G(k)$ in order to facilitate the comparing of Theorem 1 and 2 (*cf.* [5], Chapter 4, [6], and [7]):

$$g(4) = 19,\ G(4) = 16;\ g(5) = 37,\ 6 \leq G(5) \leq 18;\ g(6) = 73,\ 9 \leq G(6) \leq 28;$$
$$143 \leq g(7) \leq 3806,\ 8 \leq G(7) \leq 41;\ 279 \leq g(8) \leq 36119,\ 32 \leq G(8) \leq 57;$$
$$g(9) \geq 548,\ 13 \leq G(9) \leq 75;\ g(10) \geq 1079,\ 12 \leq G(10) \leq 93.$$

To compare Theorems 1 and 2 let the r of Theorem 1 equal the $r - q$ of Theorem 2. For example, if $k = 6$ let $r = q+1 \geq 46$. Theorem 2 gives $|A| \leq \big(6(2+\varepsilon)^6+1\big)N^{1/52}$ and Theorem 1 gives $|A| \leq \big(6(1+\varepsilon)^6 + 1\big)N^{1/7}$ and in both cases all $n \leq N$ (for sufficiently large N) are the sum of 74 elements of A. It appears that q is large for all $k \geq 3$ (even small k).

We give a corollary which is an application of Theorem 2 to cubes.

COROLLARY. *For every* $\varepsilon > 0$ *and all* $N \geq N(\varepsilon)$, *there exists a finite set A of cubes such that*

$$|A| \leq N^{(1/5)+\varepsilon}$$

and every nonnegative integer $n \leq N$ *is the sum of nine cubes in A.*

Next, Theorem 3 is for squares.

THEOREM 3. *For every* $N > 2$, *there is a set A of squares such that*

$$|A| < 7N^{1/4}$$

and every nonnegative integer $n \leq N$ *is the sum of at most five squares in A.*

Since $g(2) = 4$, $g(2) + 1 = 5$. Taking $k = 2$ in Theorem A, the conclusion is that there exists a finite set of squares such that $|A| \leq (2+\varepsilon)N^{1/3}$ and every nonnegative integer $n \leq N$ is the sum of 5 squares. Hence our Theorem 3 is better, for large N, than the case $k = 2$ in Theorem A. For example, if $N = 10^{12}$, then Theorem A gives $|A| < (2+\varepsilon)N^{1/3} \approx 20{,}000$ while Theorem 3 gives $|A| < 7N^{1/4} = 7{,}000$.

Unfortunately our methods do not readily lead to infinite basic sets A of k-th powers with $|A\cap\{1,2,\ldots,N\}|\le cN^{\alpha}$ for all N where $\alpha<1/k$.

PROOF OF THEOREM 1. Let $\varepsilon>0$ and r and N be positive integers. Define

$$A_0=\{a^k:0\le a\le(1+\varepsilon)^rN^{1/(k+r)}\},$$
$$A_1=\{[s_1^{1/k}N^{k/(k(k+r))}]^k:1\le s_1\le(1+\varepsilon)^{r-1}N^{1/(k+r)}\},$$
$$A_2=\{[s_2^{1/k}N^{(k+1)/(k(k+r))}]^k:1\le s_2\le(1+\varepsilon)^{r-2}N^{1/(k+r)}\},$$
$$\vdots$$
$$A_r=\{s_r^{1/k}N^{(k+r-1)/(k(k+r))}]^k:1\le s_r\le N^{1/(k+r)}\}.$$

Let $A=A_0\cup A_1\cup A_2\cup\cdots\cup A_r$. Then

$$|A|\le\big(1+(1+\varepsilon)+(1+\varepsilon)^2+\cdots+(1+\varepsilon)^r\big)N^{1/(k+r)}\le C(r,\varepsilon)N^{1/(k+r)}.$$

It follows from the definition of $g(k)$ that each integer $n\in[0,(1+\varepsilon)^{rk}N^{k/(k+r)}]$ is a sum of $g(k)$, hence of $u_r=g(k)+r$, elements of $A_0\subseteq A$.

We need two lemmas.

LEMMA 1. *If $N^{k/(k+r)}<n\le(1+\varepsilon)^{r-1}N^{(k+1)/(k+r)}$, then there is an integer $t_1^k\in A_1$ such that $n-t_1^k$ is a sum of $g(k)$ elements of A_0.*

PROOF. Suppose $N^{k/(k+r)}<n\le n(1+\varepsilon)^{r-1}N^{(k+1)/(k+r)}$. Define $s_1=\left[\frac{n}{N^{k/(k+r)}}\right]$ and $t_1=[s_1^{1/k}N^{1/(k+r)}]$. Then $s_1\le(1+\varepsilon)^{r-1}N^{1/(k+r)}$,

$$n-t_1^k\ge s_1N^{k/(k+r)}-s_1N^{k/(k+r)}=0,$$

and

$$\begin{aligned}n-t_1^k&<(s_1+1)N^{k/(k+r)}-(s_1^{1/k}N^{1/(k+r)}-1)^k\\&=(s_1+1)N^{k/(k+r)}-s_1N^{k/(k+r)}-\sum_{j=0}^{k-1}\binom{k}{j}(-1)^{k-j}s_1^{j/k}N^{j/(k+r)}\\&\le N^{k/(k+r)}+2^k(s_1)^{(k-1)/k}N^{(k-1)/(k+r)}\\&\le\big(1+2^k(1+\varepsilon)^{r(k-1)/k}N^{-1/(k(k+1))}\big)N^{k/(k+r)}\\&\le(1+\varepsilon)N^{k/(k+r)},\end{aligned}$$

provided N is sufficiently large. So $n-t_1^k$ is a sum of $g(k)$ elements of $A_0\subseteq A$ and consequently n is a sum of $g(k)+1$ elements of A. This completes the proof of Lemma 1.

LEMMA 2. *Let $N^{(k+i)/(k+r)}<n\le(1+\varepsilon)^{r-i-1}N^{(k+i+1)/(k+r)}$, where $1\le i\le r-1$. Then there exists an integer $t_{i+1}^k\in A_{i+1}$ such that $n-t_{i+1}^k\in[0,(1+\varepsilon)N^{(k+i)/(k+r)}]\subseteq[0,(1+\varepsilon)^{r-i}N^{(k+i)/(k+r)}]$.*

PROOF. Suppose $N^{(k+i)/(k+r)}<n\le(1+\varepsilon)^{r-i-1}N^{(k+i+1)/(k+r)}$, where $1\le i\le r-1$. Define $s_{i+1}=\left[\frac{n}{N^{(k+i)/(k+r)}}\right]$ and $t_{i+1}=[s_{i+1}^{1/k}N^{(k+i)/(k(k+r))}]$. Then $t_{i+1}^k\in A_{i+1}$, $s_{i+1}N^{(k+i)/(k+r)}\le n<(s_{i+1}+1)N^{(k+i)/(k+r)}$, and $s_{i+1}^{1/k}N^{(k+i)/(k(k+r))}-1<t_{i+1}\le s_{i+1}^{1/k}N^{(k+1)/(k(k+r))}$. So

$$n-t_{i+1}^k\ge s_{i+1}N^{(k+i)/(k+r)}-s_{i+1}N^{(k+i)/(k+r)}=0$$

and

$$\begin{aligned}
n - t_{i+1}^k &< (s_{i+1}+1)N^{(k+i)/(k+r)} - \left(s_{i+1}^{1/k}N^{(k+1)/(k(k+r))} - 1\right)^k \\
&= (s_{i+1}+1)N^{(k+i)/(k+r)} - s_{i+1}N^{(k+i)/(k+r)} \\
&\quad - \sum_{j=0}^{k-1}\binom{k}{j}(-1)^{k-j}s_{i+1}^{j/k}N^{j(k+i)/(k(k+r))} \\
&\le N^{(k+i)/(k+r)} + 2^k(s_{i+1})^{(k-1)/k}N^{(k-1)/(k+r)} \\
&\le N^{(k+i)/(k+r)} + 2^k(1+\varepsilon)^{(r-i)(k-1)/k}N^{(k-1)/(k(k+r))+(k-1)/(k+r)} \\
&= \left(1 + 2^k(1+\varepsilon)^{(r-i)(k-1)/k}N^{-(i+1/k)/(k+r)}\right)N^{(k+i)/(k+r)} \\
&\le (1+\varepsilon)N^{(k+i)/(k+r)},
\end{aligned}$$

for sufficiently large N. This completes the proof of Lemma 2.

If $N^{k/(k+r)} < n \le (1+\varepsilon)^{r-1}N^{(k+1)/(k+r)}$, then it follows from Lemma 1 that there exists an integer $t_1^k \in A_1$ such that $n - t_1^k$ is a sum of $g(k)$, hence of $g(k)+r$, elements of $A_0 \subseteq A$.

Suppose $N^{(k+i)/(k+r)} < n \le (1+\varepsilon)^{r-i-1}N^{(k+i+1)/(k+r)}$, $1 \le i \le r-1$. By Lemma 2, there exists an integer $t_{i+1}^k \in A_{i+1}$ such that $n - t_{i+1}^k \in [0, (1+\varepsilon)^{r-i}N^{(k+i)/(k+r)}]$. Write $m = n - t_{i+1}^k$. If $m \in [0, (1+\varepsilon)^rN^{k/(k+r)}]$, then m is sum of $g(k)$ elements of A_0, and so n is a sum of $g(k)+1$ elements of A. If $m \in \left(N^{k/(k+r)}, (1+\varepsilon)^{r-1}N^{(k+1)/(k+r)}\right]$, then Lemma 1 yields that there is an integer $t_1^k \in A_1$ such that $m - t_1^k$ is a sum of $g(k)$ elements of A_0, and so n is a sum of $g(k)+2$ elements of A (note that in this case $r = 2$). If

$$m \in \left(N^{(k+j)/(k+r)}, (1+\varepsilon)^{r-j-1}N^{(k+j+1)/(k+r)}\right]$$

for some j, $1 \le j < i$, then again by Lemma 2, there exists an integer $t_{j+1}^k \in A_{j+1}$ such that $m - t_{j+1}^k \in [0, (1+\varepsilon)^{r-j}N^{(k+j)/(k+r)}]$. Repeatedly using this method, finally we get a sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_v\}$ of positive integers, where $\alpha_1 > \alpha_2 > \cdots > \alpha_v$, $1 \le v \le i$, such that $t_{\alpha_w}^k \in A_{\alpha_w}$ for all $1 \le w \le v$ and

$$n - t_{\alpha_1}^k - t_{\alpha_2}^k - \cdots - t_{\alpha_v}^k \in [0, (1+\varepsilon)^rN^{k/(k+r)}].$$

Therefore $n - t_{\alpha_1}^k - t_{\alpha_2}^k - \cdots - t_{\alpha_v}^k$ is a sum of $g(k)$ elements of A_0, and so n is a sum of $g(k)+v$, hence of $g(k)+r$ for $v \le r$, elements of A, as required.

PROOF OF THEOREM 2. Let $\varepsilon > 0$. Define

$$A_0 = \{a^k : 0 \le a \le (2+\varepsilon)^rN^{1/(k+r)}\},$$

$$A_i = \{[s_i^{1/k}N^{(k+i-1)/(k(k+r))}]^k : 1 \le s_i \le (2+\varepsilon)^{r-i}N^{1/(k+r)}\}, \quad i = 1, \ldots, r.$$

Let $A = A_0 \cup A_1 \cup \cdots \cup A_r$, then

$$\begin{aligned}
|A| &\le \left(1 + (2+\varepsilon) + (2+\varepsilon)^2 + \cdots + (2+\varepsilon)^r\right)N^{1/(k+r)} \\
&\le \left(r(2+\varepsilon)^r + 1\right)N^{1/(k+r)} \\
&= C'(r,\varepsilon)N^{1/(k+r)},
\end{aligned}$$

for sufficiently large N. Now each integer $n \in [0, (2+\varepsilon)^{rk}N^{k/(k+r)}]$ is a sum of $g(k)$ (of course of u_r' ($\ge g(k)$)) elements of A_0. Again we need two lemmas. We omit the proofs which are analogous to those of Lemmas 1 and 2. (Just let s_{i+1} here be one less than the s_{i+1} in Lemmas 1 and 2 ($0 \le i \le r-1$).)

LEMMA 3. *If* $N^{k/(k+r)} < n \le (2+\varepsilon)^{r-1}N^{(k+1)/(k+r)}$, *then there is an integer* $t_1^k \in A_1$ *such that* $n - t_1^k$ *is a sum of* $G(k)$ *elements of* A_0.

LEMMA 4. *Let* $N^{(k+i)/(k+r)} < n \le (2+\varepsilon)^{r-i-1}N^{(k+i+1)/(k+r)}$, *where* $1 \le i \le r-1$. *Then there exists an integer* $t_{i+1} \in A_{i+1}$ *such that* $n - t_{i+1}^k \in [N^{(k+i)/(k+r)}, (2+\varepsilon)N^{(k+i)/(k+r)}] \subseteq [N^{(k+i)/(k+r)}, (2+\varepsilon)^{r-i}N^{(k+i)/(k+r)}]$.

If $N^{k/(k+r)} < n \le (2+\varepsilon)^{r-1}N^{(k+1)/(k+r)}$, then it follows from Lemma 3 that there exists an integer $t_1^k \in A_1$ such that $n - t_1^k$ is a sum of $G(k)$ elements of A_0 and so n is a sum of $G(k)+1$ elements of A.

Suppose $N^{(k+i)/(k+r)} < n \le (2+\varepsilon)^{r-i-1}N^{(k+i+1)/(k+r)}$, $1 \le i \le r-1$. By Lemma 4, there exists an integer $t_{i+1}^k \in A_{i+1}$ such that $n - t_{i+1}^k \in [N^{(k+i)/(k+r)}, (2+\varepsilon)^{r-i}N^{(k+i)/(k+r)}]$. Write $m = n - t_{i+1}^k$. If $m \in [N^{k/(k+r)}, (2+\varepsilon)^r N^{k/(k+r)}]$, then m is a sum of $G(k)$ elements of A_0, and so n is a sum of $G(k)+1$ elements of A. If $m \in \left(N^{k/(k+r)}, (2+\varepsilon)^{r-1}N^{(k+1)/(k+r)}\right]$, then Lemma 3 yields that there is an integer $t_1^k \in A_1$ such that $m - t_1^k$ is a sum of $G(k)$ elements of A_0, and so n is a sum of $G(k)+2$ elements of A (note that in this case $r = 2$). If $m \in \left(N^{(k+j)/(k+r)}, (2+\varepsilon)^{r-j-1}N^{(k+j+1)/(k+r)}\right]$ for some j, $1 \le j < i$, then again by Lemma 4, there exists an integer $t_{j+1}^k \in A_{j+1}$ such that $m - t_{j+1}^k \in [N^{(k+j)/(k+r)}, (2+\varepsilon)^{r-j}N^{(k+j)/(k+r)}]$. Repeatedly using this method, finally we get a sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_v\}$ of positive integers, where $\alpha_1 > \alpha_2 > \cdots > \alpha_v$, $1 \le v \le i$, such that $t_{\alpha_w}^k \in A_{\alpha_w}$ for all $1 \le w \le v$ and

$$n - t_{\alpha_1}^k - t_{\alpha_2}^k - \cdots - t_{\alpha_v}^k \in [N^{k/(k+r)}, (2+\varepsilon)^r N^{k/(k+r)}].$$

Therefore $n - t_{\alpha_1}^k - t_{\alpha_2}^k - \cdots - t_{\alpha_v}^k$ is a sum of $G(k)$ elements of A_0, and so n is a sum of $G(k)+v$, hence of $G(k)+r$ as $v \le r$, elements of A. Since $G(k) = g(k) - q$, we complete the proof of Theorem 2.

PROOF OF COROLLARY. Since $g(3) = 9$ and $G(3) \le 7$ by Linnik's theorem, we can take $r = q \ge 2$ in Theorem 2. Then $u_r' = 9$ and the result follows for sufficiently large N. If $G(3) = 4$, then this corollary is immediately improved to

$$|A| < N^{1/8} + \varepsilon.$$

PROOF OF THEOREM 3. We start with a lemma the simple proof of which may be found in [1].

LEMMA 5. *Let* $a \ge 1$. *Let* $m \ge a^2$ *and* $m \not\equiv 0 \pmod 4$. *Then either* $m - a^2$ *or* $m - (a-1)^2$ *is a sum of three squares.*

Now define $A_1 = \{b^2 : 0 \le b \le 3N^{1/4} \text{ and } b^2 \le N\}$. Let A_2 consist of the squares of all numbers of the form $[k_1^{1/2}N^{1/4}] - i$, where $9 \le k_1 \le N^{1/4}$ and $i \in \{0,1\}$, and let A_3 consist of the squares of all numbers of the form $[k_2^{1/2}N^{3/8}] - j$, where $2 \le k_2 \le N^{1/4}$ and $j \in \{0,1\}$. Then $|A_1| \le 3N^{1/4} + 1$, $|A_2| \le 2N^{1/4} - 16$, and $|A_3| \le 2N^{1/4} - 2$. Let $A = A_1 \cup A_2 \cup A_3$; then $|A| < 7N^{1/4}$.

The set A_1 contains all squares not exceeding $\min(N, 9N^{1/2})$. This implies that if $0 \le n \le \min(N, 9N^{1/2})$ then n is a sum of four squares in $A_1 \subseteq A$.

Now suppose $9N^{1/2} < n \leq N^{3/4}$. Put $k_1 = \left[\frac{n}{N^{1/2}}\right]$, $b = [k_1^{1/2}N^{1/4}]$.

Clearly $9 \leq k_1 \leq N^{1/4}$ and $b^2 \leq n$. If either $c = b$ or $c = b - 1$ then Lagrange's theorem yields that $n - c^2$ is the sum of four squares. Note also $c^2 \in A_2$. Since $k_1N^{1/2} \leq n < (k_1 + 1)N^{1/2}$ and $b \leq k_1^{1/2}N^{1/4} < b + 1$, it follows that

$$
\begin{aligned}
0 \leq n - c^2 &< (k_1 + 1)N^{1/2} - (b - 1)^2 \\
&\leq (k_1 + 1)N^{1/2} - (k_1^{1/2}N^{1/4} - 2)^2 \\
&< N^{1/2} + 4k_1^{1/2}N^{1/4} \\
&< 9N^{1/2}.
\end{aligned}
$$

Thus $n - c^2$ is the sum of four squares in A_1. Hence if $0 \leq n \leq N^{3/4}$ and $n \not\equiv 0 \pmod 4$, then n is a sum of five squares in A.

We now consider the case $N^{3/4} < n \leq N$. Put $k_2 = \left[\frac{n}{N^{3/4}}\right]$, $a = [k_2^{1/2}N^{3/8}]$. If c is either a or $a - 1$, then

$$0 \leq n - c^2 < (k_2 + 1)N^{3/4} - (a - 1)^2 < N^{3/4} + 4N^{1/2}.$$

If $0 \leq n - c^2 \leq 9N^{1/2}$, then $n - c^2$ is a sum of four squares in A_1. Suppose now $9N^{1/2} < n - c^2 \leq N^{3/4} + 4N^{1/2}$. Write $m = n - c^2$ where may choose c so that $m \not\equiv 0 \pmod 4$. Put $k_3 = \left[\frac{m}{N^{1/2}}\right]$ and $b = [k_3^{1/2}N^{1/4}]$. Thus $9 \leq k_3 \leq N^{1/4}+4$, $b^2 \leq k_3N^{1/2} \leq m$. If d is either b or $b - 1$, then d is in A_2 and

$$0 \leq m - d^2 < (k_3 + 1)N^{1/2} - (b - 1)^2 < 9N^{1/2}.$$

Thus, by Lemma 5, we may choose d such that $m - d^2$ is a sum of three squares in A_1. Hence n is the sum of five squares from A. This completes the proof.

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