A NOTE ON CENTRE-BY-FINITE-EXPONENT VARIETIES OF GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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We refer the reader to Hanna Neumann [7] for notation and other undefined terms. Let $\mathfrak{A}(n)$, $\mathfrak{B}(n)$ and $\mathfrak{C}(n)$ denote the varieties of groups defined by the laws $(xy)^n = x^n y^n$, $[x, y]^n = 1$ and $[x, y^n] = 1$ respectively, where n is an integer. $\mathfrak{A}(n)$-groups were termed "n-abelian" by R. Baer [1] and have been a subject matter of investigation by various authors (see [3], [5], [6] and the references therein). Recently Kalužnin [5] has shown that $\mathfrak{A}(n) = \mathfrak{A} \vee \mathfrak{B}_n \vee \mathfrak{B}_{n-1}$ $(n \neq 0, 1)$, thus clarifying the relationship between $\mathfrak{A}(n)$ and the familiar varieties. From the elementary inequalities

$$(1) \quad \mathfrak{A}(n) = \mathfrak{A}(1-n) \leqq [\mathfrak{B}_{n(n-1)}, \mathfrak{C}] = \mathfrak{C}(n(n-1)) \leqq \mathfrak{A}\mathfrak{B}_{n(n-1)} \qquad (n \neq 0, 1)$$

it is easily deduced that

$$(2) \qquad \mathfrak{A}(n) \leqq \mathfrak{B}(n(n-1))$$

(see for instance [5]). If $G = C_m \, Wr \, C_\infty$, then clearly $G \in \mathfrak{B}(m)$ but $G \notin \mathfrak{C}(m^*)$ for any $m^* \neq 0$ and hence $G \notin \mathfrak{A}(m^*)$ for any $m^* \neq 0,1$. Thus $\mathfrak{B}(m) \not\leqq \mathfrak{C}(m^*)$ and $\mathfrak{B}(m) \not\leqq \mathfrak{A}(m^*)$. It is also easy to see that in general $\mathfrak{C}(n(n-1)) \not\leqq \mathfrak{A}(n)$ (see for instance [6] § 5.1) and we are led to ask

Question 1: Does there exist for each positive integer m, an integer $f(m)$ such that $\mathfrak{C}(m) \leqq \mathfrak{A}(f(m))$?

If m is such that $B_{2,m}$ (the unrestricted Burnside group of exponent m on 2 generators) is finite, then for a group $G = \langle x, y \rangle$ in $\mathfrak{C}(m)$ one has $G/Z(G)$ finite and by a well-known theorem of Schur [8] (page 26) G' is finite, say, of exponent m^*. Now for a suitable u in G' we have that $(xy)^{mm^*} = (x^m y^m u)^{m^*} = x^{mm^*} y^{mm^*}$; hence $\mathfrak{C}(m) \leqq \mathfrak{A}(mm^*)$. In particular Question 1 has affirmative answers for $m = 2$, 3, 4 and 6. However not relying on the solution of the Burnside problem we are able to prove

Theorem 1. (i) $\mathfrak{C}(2) \leqq \mathfrak{A}(4)$, (ii) $\mathfrak{C}(3) \leqq \mathfrak{A}(9)$, (iii) $\mathfrak{C}(4) \leqq \mathfrak{A}(32)$.

PROOF: We note that the laws $[x, y^n] = 1$ and $(xy)^n = (yx)^n$ are equivalent.

(i) The law $[x, y^2] = 1$ implies $[x, y, z] = 1$ and so also $[x, y]^2 = 1$. Thus $(xy)^4 = ((xy)^2[x, y])^2 = ((yx)^2[x, y])^2 = (yx^2y)^2 = (y^2x^2)^2 = x^4y^4$.

(ii) $(xy)^9 = ((xy)^3[x, y])^3[y, x]^3 = ((yx)^3[x, y])^3(y, x]^3 = ((yxy)(xxy))^3[y, x]^3 = ((xxy)(yxy))^3[y, x]^3 = (x^3y^3[x, y])^3[y, x]^3 = x^9y^9$.

(iii) $(xy^2)^{16} = ((xy^2)^4[x, y^2])^4[y^2, x]^4 = ((y^2x)^4[x, y^2])^4[y^2, x]^4 = ((y^2x)^3xy^2)^4[y^2, x]^4 = (x^2y^4(xy^2)^2)^4[y^2, x]^4 = (x^4y^8[x, y^2])^4[y^2, x]^4 = x^{16}y^{32}$.

Replace x by x^{-1} and y by xy to get $(yxy)^{16} = x^{-16}(xy)^{32}$. Thus

$$(xy)^{32} = x^{16}(y(xy))^{16} = x^{16}(xy^2)^{16} = x^{32}y^{32}.$$

It follows from (2) that a torsion-free $\mathfrak{A}(n)$-group is abelian (since a torsion-free $\mathfrak{B}(n)$-group is abelian). Here we ask

QUESTION 2: Is every torsion-free $\mathfrak{C}(n)$-group abelian?

This question is not new and in fact there is an outstanding conjecture that this question has an affirmative answer. Obviously Question 2 has positive answer for those integers for which Question 1 has positive answer. Further, since by Schur's Theorem a torsion-free centre-by-finite group is abelian, it follows that a torsion-free locally soluble $\mathfrak{C}(n)$-group is abelian. Without any such assumption we are able to prove

THEOREM 2. *A torsion-free* $\mathfrak{C}(n)$*-group is abelian for* $n = 2^k3^l$ $(k \geqq 0, l = 0, 1)$.

PROOF: Let G be a torsion-free group in $\mathfrak{C}(2^k3^l)$. We prove by reverse induction on $j \in \{k, \cdots, 0\}$ that $G \in \mathfrak{C}(2^j3^l)$. For $j = k$ the result is given. Assume $G \in \mathfrak{C}(2^{i+1}3^l)$ $(0 \leqq i < k)$. We show that $[x, y^{2^i3^l}] = 1$ for all $x, y \in G$. Put $z = y^{2^i3^l}$, so that by induction hypothesis, $[x, z^2] = 1$. Thus $[x, z]^{-1} = [x, z]^z$. But this implies that $[x, z]^{2^{i+1}3^l} = [x, z]^{2^{i+1}3^{l}z} = [x, z]^{-2^{i+1}3^l}$. Hence $[x, z]^{2^{i+2}3^l} = 1$. Since G is torsion-free, $[x, z] = 1$ and $G \in \mathfrak{C}(2^i3^l)$. Thus $G \in \mathfrak{C}(2^j3^l)$ for all $j \in \{k, \cdots, 0\}$, and $G \in \mathfrak{C}(1) = \mathfrak{A}$ or $\mathfrak{C}(3)$ depending on whether $l = 0$ or $l = 1$. In both cases G is abelian by Theorem 1.

REMARK 1. If $G \in \mathfrak{A}(n)$, then for any $x, y \in G$,

$$(x^{-1}y^{-1}xy)^n = (x^{-1}y^{-1})^n(xy)^n = (yx)^{-n}(xy)^n.$$

Thus by Kalužnin's Theorem 3 we have

$$\mathfrak{A}(n) \wedge \mathfrak{B}(n) = \mathfrak{A}(n) \wedge \mathfrak{C}(n) = \mathfrak{A} \vee \mathfrak{B}_n. \tag{3}$$

REMARK 2. It seems worthwhile to remark that if G is a torsion-free Engel group in $\mathfrak{C}(n)$, $(n \neq 0)$ then for any two elements x, y in G, either $[x, y] = 1$ or there exists an integer $r \geqq 1$ such that $[x, ry] \neq 1$ but

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$[x, (r+1)y] = 1$. In the latter case, $1 = [x, (r-1)y, y^n] = [x, ry]^n = [x, ry]$ gives a contradiction. Thus in $\mathfrak{C}(n)$ torsion-free Engel groups are abelian.

REMARK 3. In [3] Durbin considered the problem of characterizing those sequences $\{n_1, \cdots, n_t\}$ of integers for which it is true that $\bigwedge_{k=1}^t \mathfrak{A}(n_k) = \mathfrak{A}$. If $\mathfrak{B}$ denotes the class of all groups of finite exponent, then he proves that $\mathfrak{B} \wedge (\bigwedge_{k=1}^t \mathfrak{A}(n_k)) < \mathfrak{A}$ if and only if $\left(\binom{n_1}{2}, \cdots, \binom{n_t}{2}\right) = 1$, where $\binom{n_k}{2} = \frac{1}{2}n_k(n_k+1)$. He shows further that the hypothesis of finite exponent can be replaced by "periodicity" in the special case $\{n, n+2\}$. We complete the discussion on Durbin's problem by proving,

THEOREM 3. $\bigwedge_{k=1}^{t} \mathfrak{A}(n_k) = \mathfrak{A}$ *if and only if* $\left(\binom{n_1}{2}, \cdots, \binom{n_t}{2}\right) = 1$.

PROOF: The "only if" part of the theorem follows from Durbin's proof. For the rest of the proof we first notice from (1), that

$$\bigwedge_{k=1}^{t} \mathfrak{A}(n_k) \leqq \bigwedge_{k=1}^{t} [\mathfrak{B}_{n_k(n_k-1)}, \mathfrak{E}]$$
$$\leqq [\bigwedge_{k=1}^{t} \mathfrak{B}_{n_k(n_k-1)}, \mathfrak{E}] = [\mathfrak{B}_2, \mathfrak{E}] \leqq \mathfrak{N}_2.$$

But groups in $\mathfrak{N}_2$ satisfy the law $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$ for every integer n. Thus $\bigwedge_{k=1}^t \mathfrak{A}(n_k) = \mathfrak{A}$, as was required.

REMARK 4. In our initial proof of Theorem 3 we made use of the following lemma which seems to be of independent interest (c.f. [7] page 39 and [2].

LEMMA. $\mathfrak{N}_c(\bigwedge_{k=1}^{t} \mathfrak{B}_{m_k}) = \bigwedge_{k=1}^{t} \mathfrak{N}_c \mathfrak{B}_{m_k}$

PROOF: For positive integers c, m, n let $G \in \mathfrak{N}_c\mathfrak{B}_m \wedge \mathfrak{N}_c\mathfrak{B}_n$. Then $[x_1^m, \cdots, x_{c+1}^m] = [x_1^n, \cdots, x_{c+1}^n] = 1$ for all $x_i \in G$. If $d = (m, n)$, both $[x_1^d, \cdots, x_{c+1}^d)^{(m/d)^{c+1}}$ and $[x_1^d, \cdots, x_{c+1}^d]^{(n/d)^{c+1}}$ lie in $(G^d)_{(c+2)}$, where $G^d = \langle x^d;\ x \in G\rangle$. Thus $(G^d)_{(c+1)} = (G^d)_{(c+2)}$. On the other hand $G^d = G^m G^n$ is nilpotent since $G^m, G^n \in \mathfrak{N}_c$. Thus $(G^d)_{(c+1)} = 1$ and $G \in \mathfrak{N}_c\mathfrak{B}_d$. This proves $\mathfrak{N}_c(\mathfrak{B}_m \wedge \mathfrak{B}_n) = \mathfrak{N}_c\mathfrak{B}_m \wedge \mathfrak{N}_c\mathfrak{B}_n$ and the lemma follows.

REMARK 5. In the concluding section of his paper [3], Durbin raised the following number theoretic question: Does there exist, for each positive integer t, a set $\{n_1, \cdots, n_t\}$ of integers satisfying $\left(\binom{n_1}{2}, \cdots, \binom{n_t}{2}\right) = 1$ such that no proper subset satisfies this property? We give an affirmative answer to this question by giving a process of constructing such integers. This construction is due to T. J. Dickson whose co-operation is gratefully acknowledged.

For $t = 2$, the set $\{2, 3\}$ will do. For $t > 2$ we first choose a set p_1, $p_2, \cdots, p_t$ of primes as follows: choose $p_1 = 2$, $p_2 = 3$ and for $3 \leqq i \leqq t$, choose p_i to be of the form $l_i p_1 p_2 \cdots p_{i-1}+1$ for some integer $l_i \geqq 3$. This is possible by Dirichlet's Theorem (see for instance [4] page 13). Thus $p_i \equiv 1\ (p_j)$ for $j = 1, \cdots, i-1$. Let $p'_i = \prod_{j \neq i} p_j$ and define $n_i = 2p'_i+1$. It is now routine to show that the set $\{n_1, \cdots, n_t\}$ has the required properties.

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