Proceedings of the Edinburgh Mathematical Society (2006) **49**, 551–565 ©
DOI:10.1017/S0013091505000301 Printed in the United Kingdom

THE EXISTENCE OF ISOTROPIC MODULI SPACES

MD. SHOWKAT ALI

Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh
(msa417@hotmail.com)

(Received 9 March 2005)

Abstract A theorem on the existence of moduli spaces of compact complex isotropic submanifolds in complex contact manifolds is established.

Keywords: contact manifold; isotropic; Legendre and Kodaira moduli spaces

2000 *Mathematics subject classification:* Primary 53D10; 53D30; 32G10

1. Introduction

In 1962 Kodaira [**3**] proved that if $X \hookrightarrow Y$ is a compact complex submanifold of a complex manifold Y with normal bundle $N_{X|Y}$ such that $H^1(X, N_{X|Y}) = 0$, then there exists a complete analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex submanifolds X_t of Y with the moduli space M. The family is maximal and its moduli space M, called the Kodaira moduli space, is an $h^0(X, N_{X|Y})$-dimensional complex manifold. Kodaira's theorem found many applications in geometry and analysis, especially in twistor theory. Merkulov [**7**] proved that if $X \hookrightarrow Y$ is a compact complex Legendre submanifold of a complex contact manifold Y with contact line bundle L such that $H^1(X, L_X) = 0$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre submanifolds containing X with the moduli space M, which is an $h^0(X, L_X)$-dimensional complex manifold. In this paper we prove that if $X \hookrightarrow Y$ is a compact complex isotropic submanifold of a complex contact manifold Y with contact line bundle L such that $H^1(X, L_X) = H^1(X, S_X) = 0$, where L_X is the restriction of L on Y to X and S_X is a certain canonically defined vector bundle on X which is the kernel of the canonical projection $p : N_{X|Y} \to J^1 L_X$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact isotropic submanifolds containing X with the moduli space M, which is an $h^0(X, L_X) + h^0(X, S_X)$-dimensional smooth complex manifold. There are strong indications in [**8**] that the moduli spaces of such families studied in this paper will play a pivotal role in the twistor theory of G-structures with restricted invariant torsion.

2. Complex contact manifolds

Definition 2.1. A complex contact manifold is a pair (Y, D) consisting of a $(2n+1)$-dimensional complex manifold Y and a rank-$2n$ holomorphic subbundle $D \subset \mathcal{T}Y$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\phi : D \times D \to \mathcal{T}Y/D,$$
$$(v, w) \to [v, w] \mod D$$

is non-degenerate. Define the contact line bundle $L := \mathcal{T}Y/D$ on Y by the exact sequence

$$0 \to D^{2n} \to \mathcal{T}Y^{2n+1} \xrightarrow{\theta} L \to 0,$$

where θ is the tautological projection and $D = \ker\theta$. However, we may also think of θ (in a trivialization of L) as a line bundle-valued 1-form $\theta \in H^0(Y, \Omega^1 Y \otimes L)$, and so attempt to form its exterior derivative $\mathrm{d}\theta$. We can easily verify that the maximal non-degeneracy of the distribution D is equivalent to the fact that the 'twisted' 1-form defined above satisfies the condition

$$\theta \wedge (\mathrm{d}\theta)^n \neq 0.$$

3. Complex isotropic submanifolds

Definition 3.1. A compact complex p-dimensional submanifold $X^p \hookrightarrow Y^{2n+1}$ of a complex contact manifold Y^{2n+1} is called *isotropic* if $\mathcal{T}X \subset D|_X$.

An isotropic submanifold of maximal possible dimension n is called a Legendre submanifold. The normal bundle $N_{X|Y}$ of any Legendre submanifold $X \hookrightarrow Y$ is isomorphic to $J^1 L_X$ [**5**], where $L_X = L|_X$, and, therefore, fits into the exact sequence

$$0 \to \Omega^1 X \otimes L_X \to N_{X|Y} \xrightarrow{\mathrm{pr}} L_X \to 0.$$

Definition 3.2. The bundle S_X is defined to be the kernel of the canonical projection

$$p : N_{X|Y} \to J^1 L_X,$$

i.e. it is defined by the exact sequence

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0.$$

Definition 3.3. Let X be an isotropic submanifold of a complex contact manifold (Y, D). Let

$$TX^{\perp} = \{Z \in D \mid \mathrm{d}\theta(Z, W) = 0,\ \forall W \in TX\}.$$

Then $TX \subseteq TX^{\perp}$ and the bundle S_X is defined by $S_X = TX^{\perp}/TX$.

Theorem 3.4. *Let (Y, D) be a complex contact manifold and $X \subset Y$ be an isotropic submanifold of Y with contact line bundle L. Then there is a short exact sequence*

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0.$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

Proof. Consider a particular 1-form θ that represents the contact structure. Let $p \in X$, $Z \in T_pX$ be a vector in the normal bundle and $Q \in T_pY$. There are then two equations

$$f(p) = \theta(Q), \quad \mathrm{d}\theta(Z, Q) = Z(f)|_p,$$

which uniquely determine the 1-jet on X of a function f at p.

Consider rescaling $\theta \mapsto g\theta$, where g is a function on Y. If we set $\hat{\theta} = g\theta$ and $\hat{f} = gf$, then we have

$$\hat{\theta}(Q) = g\theta(Q) = gf(p) = \hat{f}|_p$$

and

$$\begin{aligned}
\mathrm{d}\hat{\theta}(Z, Q) &= (\mathrm{d}g \wedge \theta)(Z, Q) + g\,\mathrm{d}\theta(Z, Q) \\
&= \mathrm{d}g(Z)\theta(Q) - \mathrm{d}g(Q)\theta(Z) + gZ(f)|_p \\
&= Z(g)f(p) - 0 + gZ(f)|_p \\
&= Z(gf)|_p \\
&= Z(\hat{f})|_p.
\end{aligned}$$

(Since $T_pX \subseteq T_pX^{\perp} \subset D$, we have $Z \in D$, so $\theta(Z) = 0$.) Therefore, this elementary calculation shows that the two conditions above are satisfied by gf and so we can conclude that we have defined a map $N_{X|Y} \to J^1L_X$. Furthermore, it is clear that the kernel is $TX^{\perp}/TX$. Thus, the proof is completed. □

4. Kodaira relative deformation theory

In this section we recall some useful facts about relative deformation theory of compact complex submanifolds of complex manifolds [**6**].

Let Y and M be complex manifolds and let $\pi_1 : Y \times M \to Y$ and $\pi_2 : Y \times M \to M$ be two natural projections. An analytic family of compact submanifolds of the complex manifold Y with moduli space M is a complex submanifold $F \hookrightarrow Y \times M$ such that the restriction of the projection π_2 on F is a proper regular map (regularity means that the rank of the differential of $\nu \equiv \pi_2|_F : F \to M$ is equal to $\dim M$ at every point). Thus, the family F has double fibration structure

$$Y \xleftarrow{\mu} F \xrightarrow{\mu} M,$$

where $\mu = \pi_1|_F$. For each $t \in M$ we say that the compact complex submanifolds $X_t := \mu \circ \nu^{-1}(t) \hookrightarrow Y$ belong to the family F.

5. Existence of Legendre moduli spaces

Let Y be a complex contact manifold. An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds of Y is called an analytic family of compact Legendre submanifolds if, for any point $t \in M$, the corresponding subset $X_t := \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is a Legendre submanifold. The parameter space M is called a Legendre moduli space. In 1995, Merkulov [**7**] proved the following theorem for the existence of complete Legendre moduli spaces.

Theorem 5.1 (Merkulov [7]). *Let X be a compact complex Legendre submanifold of a complex contact manifold Y with contact line bundle L. If $H^1(X, L_X) = 0$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre submanifolds containing X with the moduli space M, which is an $h^0(X, L_X)$-dimensional complex manifold.*

6. Families of complex isotropic submanifolds

Let Y be a complex contact manifold. An analytic family $F \hookrightarrow Y \times M$ of compact submanifolds of the complex manifold Y is called an analytic family of isotropic submanifolds if, for any $t \in M$, the corresponding subset $X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is an isotropic submanifold. We will use the notation $\{X_t \hookrightarrow Y \mid t \in M\}$ to denote an analytic family of isotropic submanifolds.

Let $X = X_{t_0}$ for some $t_0 \in M$. If $X^p \hookrightarrow Y^{2n+1}$ is an isotropic submanifold, then each point in X has a neighbourhood U in Y such that the contact structure in a suitable trivialization of L over U (see [**2**]) is

$$\theta = \mathrm{d}\omega^0 + \sum_{\bar{a}=p+1}^{n} \omega^{\bar{a}}\,\mathrm{d}\omega^{\bar{\bar{a}}} + \sum_{a=1}^{p} \omega^a\,\mathrm{d}z^a$$

and X in U is given by

$$\omega^0 = \omega^a = \omega^{\bar{a}} = \omega^{\bar{\bar{a}}} = 0.$$

There exists an adopted coordinate covering $\{U_i\}$ of a tubular neighbourhood of X inside Y. In view of the above fact one can always choose local coordinate functions $(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, z_i^a)$ in U_i, where $\bar{a}, \bar{\bar{a}} = 1, \dots, n$ and $a = 1, \dots, p$ such that the contact structure in U_i is represented by

$$\theta_i = \mathrm{d}\omega_i^0 + \sum_{\bar{a}=p+1}^{n} \underbrace{\omega_i^{\bar{a}}\,\mathrm{d}\omega_i^{\bar{\bar{a}}}}_{(n-p)\text{-terms}} + \sum_{a=1}^{p} \underbrace{\omega_i^a\,\mathrm{d}z_i^a}_{p\text{-terms}}$$

and $U_i \cap X$ is given by

$$\omega_i^0 = \omega_i^a = \omega_i^{\bar{a}} = \omega_i^{\bar{\bar{a}}} = 0$$

and

$$\theta_i|_{U_i \cap U_j} = A_{ij}\theta_j|_{U_i \cap U_j} \tag{6.1}$$

for some nowhere-vanishing holomorphic functions A_{ij}. They satisfy the condition

$$A_{ik} = A_{ij}A_{jk}$$

on every triple intersection $U_i \cap U_j \cap U_k$. Clearly, $\{A_{ij}\}$ are glueing functions of the contact line bundle L.

On the intersection $U_i \cap U_j$, the coordinates $\omega_i^A := (\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}})$ and z_i^a are holomorphic functions of $\omega_j^B := (\omega_j^0, \omega_j^a, \omega_j^{\bar{a}}, \omega_j^{\bar{\bar{a}}})$ and z_j^b,

$$\left.\begin{aligned} \omega_i^0 &= f_{ij}^0(\omega_j^B, z_j^b) \\ \omega_i^a &= f_{ij}^a(\omega_j^B, z_j^b) \\ \omega_i^{\bar{a}} &= f_{ij}^{\bar{a}}(\omega_j^B, z_j^b) \\ \omega_i^{\bar{\bar{a}}} &= f_{ij}^{\bar{\bar{a}}}(\omega_j^B, z_j^b) \\ z_i^a &= g_{ij}^a(\omega_j^B, z_j^b) \end{aligned}\right\} \iff \left.\begin{aligned} \omega_i^A &= f_{ij}^A(\omega_j^B, z_j^b), \\ z_i^a &= g_{ij}^a(\omega_j^B, z_j^b), \end{aligned}\right\} \tag{6.2}$$

with $f_{ij}^A(0, z_j^b) = 0$. Equation (6.1) puts the following constraints on glueing functions:

$$A_{ij} = \frac{\partial f_{ij}^0}{\partial \omega_j^0} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^0} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^0}, \tag{6.3}$$

$$0 = \frac{\partial f_{ij}^0}{\partial \omega_j^a} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^a} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^a}, \tag{6.4}$$

$$0 = \frac{\partial f_{ij}^0}{\partial \omega_j^{\bar{a}}} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^{\bar{a}}} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^{\bar{a}}}, \tag{6.5}$$

$$A_{ij}\omega_j^{\bar{a}} = \frac{\partial f_{ij}^0}{\partial \omega_j^{\bar{\bar{a}}}} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial \omega_j^{\bar{\bar{a}}}} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial \omega_j^{\bar{\bar{a}}}}, \tag{6.6}$$

$$A_{ij}\omega_j^a = \frac{\partial f_{ij}^0}{\partial z_j^a} + \sum_b f_{ij}^b \frac{\partial g_{ij}^b}{\partial z_j^a} + \sum_{\bar{b}} f_{ij}^{\bar{b}} \frac{\partial f_{ij}^{\bar{\bar{b}}}}{\partial z_j^a}, \tag{6.7}$$

which express the fact that the chosen coordinate charts U_i are glued by the contactomorphisms.

For any point t in a sufficiently small coordinate neighbourhood $M_0 \subset M$ of t_0 with coordinate functions t^α, $\alpha = 1, \dots, m = \dim M$, the associated isotropic submanifold $X_t = \mu \circ \nu^{-1}(t)$ is given in the domain U_i by equations of the form [**2**]

$$\omega_i^A = \phi_i^A(z_i^a, t^\alpha), \quad A = 0, a, \bar{a}, \bar{\bar{a}}.$$

Lemma 6.1. *X_t is isotropic if and only if*

$$\phi_i^a(z_i, t) = -\frac{\partial \phi_i^0(z_i, t)}{\partial z_i^a} - \sum_{\bar{b}=p+1}^{n} \phi_i^{\bar{b}}(z_i, t) \frac{\partial \phi_i^{\bar{\bar{b}}}(z_i, t)}{\partial z_i^a}$$

holds.

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

Proof. Let $X^p \hookrightarrow Y^{2n+1}$ be an isotropic submanifold in a complex contact manifold Y. For an arbitrary X_t, the deformation of X inside Y is given by

$$\left.\begin{aligned} \omega_i^0 &= \phi_i^0(z_i,t) \\ \omega_i^a &= \phi_i^a(z_i,t) \\ \omega_i^{\bar a} &= \phi_i^{\bar a}(z_i,t) \\ \omega_i^{\bar{\bar a}} &= \phi_i^{\bar{\bar a}}(z_i,t) \end{aligned}\right\} \implies \omega_i^A = \phi_i^A(z_i,t).$$

Then, $\{\partial\phi_i^A/\partial t|_0\}$ is a global section of $N_{X|Y}$. X_t is isotropic if and only if

$$\theta_i = \mathrm{d}\omega_i^0 + \omega_i^{\bar a}\,\mathrm{d}\omega_i^{\bar{\bar a}} + \omega_i^a\,\mathrm{d}z_i^a$$

vanishes on X_t. Then

$$\begin{aligned} 0 &= \theta_i|_{X_t} \\ &= \mathrm{d}\phi_i^0(z_i,t) + \phi_i^{\bar a}(z_i,t)\,\mathrm{d}\phi_i^{\bar{\bar a}}(z_i,t) + \phi_i^a(z_i,t)\,\mathrm{d}z_i^a \\ &= \frac{\partial\phi_i^0(z_i,t)}{\partial z_i^a}\,\mathrm{d}z_i^a + \phi_i^{\bar a}(z_i,t)\frac{\partial\phi^{\bar{\bar a}}}{\partial z_i^b}\,\mathrm{d}z_i^b + \phi_i^a(z_i,t)\,\mathrm{d}z_i^a \\ &= \left[\phi_i^a(z_i,t) + \frac{\partial\phi_i^0(z_i,t)}{\partial z_i^a} + \sum_{\bar b=p+1}^{n}\phi_i^{\bar b}(z_i,t)\frac{\partial\phi_i^{\bar{\bar b}}(z_i,t)}{\partial z_i^a}\right]\mathrm{d}z_i^a. \end{aligned}$$

Thus, we obtain

$$\phi_i^a(z_i,t) = -\frac{\partial\phi_i^0(z_i,t)}{\partial z_i^a} - \sum_{\bar b=p+1}^{n}\phi_i^{\bar b}(z_i,t)\frac{\partial\phi_i^{\bar{\bar b}}(z_i,t)}{\partial z_i^a}, \tag{6.8}$$

where $\phi_i^A(z_i,t)$ is a holomorphic function of z_i^a and t, which satisfy the boundary condition $\phi_i^A(z_i,t) = 0$ for $t = t_0$. □

7. Isotropic moduli spaces: completeness and maximality

Let Y be a complex contact manifold and $F \hookrightarrow Y \times M$ be an analytic family of compact complex isotropic submanifolds. The latter is also an analytic family of compact complex submanifolds in the sense of Kodaira and thus, for each $t \in M$, there is a canonical linear map

$$k_t : T_tM \to H^0(X_t, N_{X_t|Y}).$$

The exact sequence

$$0 \to S_{X_t} \to N_{X_t|Y} \to J^1L_{X_t} \to 0$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

can be expanded as follows:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \Omega^1 X_t \otimes L_{X_t} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & S_{X_t} & \longrightarrow & N_{X_t|Y} & \longrightarrow & J^1 L_{X_t} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & L_{X_t} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Hence, there is a canonical map represented by a diagonal arrow,

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & H^0(X_t, \Omega^1 X_t \otimes L_{X_t}) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & H^0(X_t, S_{X_t}) & \longrightarrow & H^0(X_t, N_{X_t|Y}) & \longrightarrow & H^0(X_t, J^1 L_{X_t}) & \longrightarrow & 0 \\
 & & & & & \searrow & \downarrow & & \\
 & & & & & & H^0(X_t, L_{X_t}) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Thus, there is a canonical sequence of linear spaces:

$$0 \to H^0(X_t, S_{X_t}) \to H^0(X_t, N_{X_t|Y}) \to H^0(X_t, L_{X_t}) \to 0,$$

which is not exact, in general.

Definition 7.1. The analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is *complete* at a point $t \in M$ if the Kodaira map k_t makes the induced sequence,

$$0 \to H^0(X_t, S_{X_t}) \to k_t(T_t M) \to H^0(X_t, L_{X_t}) \to 0,$$

exact. The analytic family $F \hookrightarrow Y \times M$ is called complete if it is complete at each point of the moduli space.

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

Lemma 7.2 (Ali [1]). *If an analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is complete at a point $t_0 \in M$, then there is an open neighbourhood $U \subseteq M$ of the point t_0 such that the family $F \hookrightarrow Y \times M$ is complete at all points $t \in U$.*

Definition 7.3. An analytic family $F \hookrightarrow Y \times M$ of compact complex isotropic submanifolds is *maximal* at a point $t_0 \in M$ if, for any other analytic family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ of compact complex isotropic submanifolds such that $\mu \circ \nu^{-1}(t_0) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}_0)$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists a neighbourhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f : \tilde{U} \to M$ such that $f(\tilde{t}_0) = t_0$ and $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}') = \mu \circ \nu^{-1}(f(\tilde{t}'))$ for each $\tilde{t}' \in \tilde{U}$. The family $F \hookrightarrow Y \times M$ is called maximal if it is maximal at each point t in the moduli space M.

Lemma 7.4 (Ali [1]). *If an analytic family of compact complex isotropic submanifolds $F \hookrightarrow Y \times M$ is complete at a point $t_0 \in M$, then it is maximal at the point t_0.*

8. Existence theorem

Theorem 8.1. *If $X \hookrightarrow Y$ is a compact complex isotropic submanifold in a complex contact manifold Y, then its normal bundle $N_{X|Y}$ fits into an extension*

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0.$$

If $H^1(X, L_X) = H^1(X, S_X) = 0$, then there exists a complete and maximal analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of isotropic submanifolds such that

(i) *$X_{t_0} = X$ for some $t_0 \in M$;*

(ii) *the moduli space M is smooth;*

(iii) $\dim M = h^0(X, L_X) + h^0(X, S_X)$;

(iv) *the tangent space $T_t M$, $t \in M$, fits into the extension*

$$0 \to H^0(X_t, S_{X_t}) \to k_t(T_t M) \to H^0(X_t, L_{X_t}) \to 0.$$

Proof. Let $(\omega_i^0, \omega_i^a, \omega_i^{\bar{a}}, \omega_i^{\bar{\bar{a}}}, z_i^a)$ be a coordinate system on Y that is adapted to the isotropic character of the embedding $X \hookrightarrow Y$ as described in § 6. Assume that $\{X_t \hookrightarrow Y \mid t \in M\}$ is a family of compact complex isotropic submanifolds in the complex contact manifold Y. According to § 6, such a family can be described by $\phi_i^0(z_i, t)$, $\phi_i^a(z_i, t)$, $\phi_i^{\bar{a}}(z_i, t)$, $\phi_i^{\bar{\bar{a}}}(z_i, t)$, which solve the equations in $U_i \cap U_j$:

$$\begin{aligned}
\phi_i^0(z_i, t) &= f_{ij}^0(\phi_j^0(z_j, t), \phi_j^a(z_j, t), \phi_j^{\bar{a}}(z_j, t), \phi_j^{\bar{\bar{a}}}(z_j, t), z_j),\\
\phi_i^a(z_i, t) &= f_{ij}^a(\phi_j^0(z_j, t), \phi_j^a(z_j, t), \phi_j^{\bar{a}}(z_j, t), \phi_j^{\bar{\bar{a}}}(z_j, t), z_j),\\
\phi_i^{\bar{a}}(z_i, t) &= f_{ij}^{\bar{a}}(\phi_j^0(z_j, t), \phi_j^a(z_j, t), \phi_j^{\bar{a}}(z_j, t), \phi_j^{\bar{\bar{a}}}(z_j, t), z_j),\\
\phi_i^{\bar{\bar{a}}}(z_i, t) &= f_{ij}^{\bar{\bar{a}}}(\phi_j^0(z_j, t), \phi_j^a(z_j, t), \phi_j^{\bar{a}}(z_j, t), \phi_j^{\bar{\bar{a}}}(z_j, t), z_j),\\
z_i^a &= g_{ij}^a(\phi_j^0(z_j, t), \phi_j^a(z_j, t), \phi_j^{\bar{a}}(z_j, t), \phi_j^{\bar{\bar{a}}}(z_j, t), z_j),
\end{aligned}$$

and equation (6.8). We know that $N_{X|Y}$ fits into a diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \Omega^1 X \otimes L_X & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & S_X & \longrightarrow & N_{X|Y} & \xrightarrow{p} & J^1 L_X & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & L_{X_t} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

There exists a canonical morphism of sheaves of abelian groups, $\alpha : L_X \to J^1 L_X$, which, in our local coordinates, is given explicitly by

$$\Big\{\phi_i^0(z_i,t)\Big\} \to \left\{\begin{matrix} \phi_i^0(z_i,t) \\ -\dfrac{\partial \phi_i^0(z_i,t)}{\partial z_i^a} \end{matrix}\right\}.$$

Define a subsheaf of abelian groups in the sheaves $N_{X|Y}$ as $\tilde{N}_{X|Y} := p^{-1}(\alpha(L_X))$, where $p : N_{X|Y} \to J^1 L_X$ is the canonical epimorphism. By construction, $\tilde{N}_{X|Y}$ fits into an exact sequence:

$$0 \to S_X \to \tilde{N}_{X|Y} \to L_X \to 0.$$

The long exact sequence associated with the sequence above gives

$$0 \to H^0(X, S_X) \to H^0(X, \tilde{N}_{X|Y}) \to H^0(X, L_X) \to H^1(X, S_X) \to \cdots.$$

By assumption, $H^1(X, S_X) = 0$. Hence, we have an exact sequence of vector spaces,

$$0 \to H^0(X, S_X) \to H^0(X, \tilde{N}_{X|Y}) \to H^0(X, L_X) \to 0,$$

implying that

$$\dim H^0(X, \tilde{N}_{X|Y}) = \dim H^0(X, S_X) + \dim H^0(X, L_X) := m.$$

Let θ_α, $\alpha = 1, \dots, m$, be a basis of the global sections of $\tilde{N}_{X|Y}$. In our coordinate system, each θ_α can be represented by a 0-cocycle,

$$\theta_\alpha \Longleftrightarrow \left\{\begin{matrix} \theta_{\alpha i}^0 \\ -\dfrac{\partial \theta_{\alpha i}^0}{\partial z_i^a} \\ \theta_{\alpha i}^{\bar{a}} \\ \theta_{\alpha i}^{\bar{\bar{a}}} \end{matrix}\right\} = \Big\{\theta_{\alpha i}^A\Big\}, \quad A = 0, a, \bar{a}, \bar{\bar{a}}.$$

In $U_i \cap U_j$, we have

$$\theta^A_{\alpha i}(z) = F^A_{ijB}(z)\theta^B_{\beta j}(z), \quad z = (0, z_i), \tag{8.1}$$

where the matrix-valued functions are given by

$$F^A_{ijB} = \begin{bmatrix} A_{ij}|_X & 0 & 0 & 0 \\ \dfrac{\partial f^a_{ij}}{\partial \omega^0_j}\Big|_X & \dfrac{\partial f^a_{ij}}{\partial \omega^b_j}\Big|_X & 0 & 0 \\ \dfrac{\partial f^{\bar a}_{ij}}{\partial \omega^0_j}\Big|_X & \dfrac{\partial f^{\bar a}_{ij}}{\partial \omega^b_j}\Big|_X & \dfrac{\partial f^{\bar a}_{ij}}{\partial \omega^{\bar b}_j}|_X & \dfrac{\partial f^{\bar a}_{ij}}{\partial \omega^{\bar{\bar b}}_j}\Big|_X \\ \dfrac{\partial f^{\bar{\bar a}}_{ij}}{\partial \omega^0_j}\Big|_X & \dfrac{\partial f^{\bar{\bar a}}_{ij}}{\partial \omega^b_j}\Big|_X & \dfrac{\partial f^{\bar{\bar b}}_{ij}}{\partial \omega^{\bar b}_j}|_X & \dfrac{\partial f^{\bar{\bar b}}_{ij}}{\partial \omega^{\bar{\bar b}}_j}\Big|_X \end{bmatrix}.$$

Define

$$\phi^A_i(z_i, t) = \begin{bmatrix} \phi^0_i(z_i, t) \\ \phi^a_i(z_i, t) \\ \phi^{\bar a}_i(z_i, t) \\ \phi^{\bar{\bar a}}_i(z_i, t) \end{bmatrix},$$

where equation (6.8) holds. Let ε be a small positive number. In order to prove theorem 8.1, we must find the holomorphic functions $\phi^A_i(z_i, t)$ in $z_i = (z^1_i, \dots, z^n_i)$ and in $t = (t^1, \dots, t^m)$, $|z_i| < 1$, $|t| < \varepsilon$, with $|\phi^A_i(z_i, t)| < 1$ such that

$$\phi^A_i(g^a_{ij}(\phi^B_j(z_j, t), z_j), t) = f^A_{ij}(\phi^B_j(z_j, t), z_j) \tag{8.2}$$

where $A = 0, a, \bar a, \bar{\bar a}$, equation (6.8) and the boundary conditions

$$\phi^A_i(z_i, 0) = 0 \tag{8.3}$$

and

$$\frac{\partial \phi^A_i(z_i, t)}{\partial t^\alpha}\Big|_{t=0} = \theta^A_{\alpha i}(z), \quad z = (0, z_i), \tag{8.4}$$

are satisfied. If we succeed in solving all these equations for the functions $\{\phi^A_i(z_i, t)\}$, which are holomorphic in t in some neighbourhood $U \subset C^q$ of the origin, then the boundary conditions will guarantee that the resulting analytic family $F \hookrightarrow Y \times U$ is complete at $t = 0$ and, hence, by Lemmas 7.2 and 7.4, is complete and maximal in some neighbourhood $M \subseteq U$ of the origin. Therefore, all we need to prove the theorem is to solve equations (8.2)–(8.4). We shall do this in three steps.

Step 1 (simplification of the basic system of equations). Let us first show that it is sufficient to solve only those equations of system (8.2), corresponding to $A = 0, \bar a, \bar{\bar a}$, which the holomorphic functions $\{\phi^A_i(z_i, t)\}$ satisfy, on overlaps $X \cap U_i \cap U_j$. Then, denoting

$$A^a_b := \left[\sum_{A=0}^{n} \frac{\partial g^a_{ij}}{\partial \omega^A_j} \frac{\partial \phi^A_j}{\partial z^b_j} + \frac{\partial g^a_{ij}}{\partial z^b_j}\right]\Big|_{\omega^A_j = \phi^A_j(z_j, t)}$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

and using equations (6.3)–(6.7), we obtain (see [**1**, pp. 65, 66])

$$\sum_{a=1}^{n}\frac{\partial\phi_i^0}{\partial z_i^a}A_b^a = \left[\sum_{a=1}^{n}\frac{\partial\phi_i^0}{\partial z_i^a}\sum_{A=0}^{n}\frac{\partial g_{ij}^a}{\partial\omega_j^A}\frac{\partial\phi_j^A}{\partial z_j^b}+\frac{\partial g_{ij}^a}{\partial z_j^b}\right]\Bigg|_{\omega_j^A=\phi_j^A(z_j,t)}$$
$$= \left[\sum_{A=0}^{n}\frac{\partial f_{ij}^0}{\partial\omega_j^A}\frac{\partial\phi_j^A}{\partial z_j^b}+\frac{\partial f_{ij}^0}{\partial z_j^b}\right]\Bigg|_{\omega_j^A=\phi_j^A(z_j,t)}$$
$$= -\sum_{c} f_{ij}^c A_b^c - \sum_{\bar c} f_{ij}^{\bar c}\frac{\partial f_{ij}^{\bar{\bar c}}}{\partial z_i^a}A_b^a,$$

which implies that

$$\sum_{a=1}^{n}\left(\frac{\partial\phi_i^0}{\partial z_i^a}+\sum_{\bar c} f_{ij}^{\bar c}\frac{\partial f_{ij}^{\bar{\bar c}}}{\partial z_i^a}\right)A_b^a = -\sum_{c=1}^{n} f_{ij}^c A_b^c. \tag{8.5}$$

Since the Jacobian of the coordinate transformation

$$\det\frac{\partial(\omega_i^0,\omega_i^a,\omega_i^{\bar a},\omega_i^{\bar{\bar a}},z_i^a)}{\partial(\omega_j^0,\omega_j^b,\omega_j^{\bar b},\omega_j^{\bar{\bar b}},z_j^b)}\Bigg|_X$$
$$= \frac{\partial f_{ij}^0}{\partial\omega_j^0}\Bigg|_X \det\left(\frac{\partial f_{ij}^a}{\partial\omega_j^b}\right)\Bigg|_X \det\left(\frac{\partial f_{ij}^{\bar a}}{\partial\omega_j^{\bar b}}\right)\Bigg|_X \det\left(\frac{\partial f_{ij}^{\bar{\bar a}}}{\partial\omega_{ij}^{\bar{\bar b}}}\right)\Bigg|_X \det(A_b^a)|_{t=0}$$

is nowhere zero on X, the matrix A_b^a is non-degenerate at $t=0$ and hence is non-degenerate for all t in some small neighbourhood U' of the zero in C^m. Equation (8.5) then implies that

$$\left(-\frac{\partial\phi_i^0}{\partial z_i^a}-\sum_{\bar a} f_{ij}^{\bar a}\frac{\partial f_{ij}^{\bar{\bar a}}}{\partial z_i^a}\right)\Bigg|_{z_i=g_{ij}(\phi_j^B(z_j,t),z_j)} = f_{ij}^a|_{\omega_j^A=\phi_j^A(z_j,t)},$$

i.e. that equation (8.2) with $A=a$ is automatically satisfied. Thus, we must solve equations (8.2) for $A=0,\bar a,\bar{\bar a}$ with boundary conditions (8.3), (8.4).

Step 2 (existence of formal solutions). In what follows we write the power-series expansion of an arbitrary holomorphic function $P(t)$ in $t^1,\dots,t^m$, defined on a neighbourhood of the origin, in the form

$$P(t)=P_0(t)+P_1(t)+\cdots+P_q(t)+\cdots,$$

where each term $P_q(t)$ denotes a homogeneous polynomial of degree q in $t^1,\dots,t^m$, and denote by $P^{[q]}(t)$ the polynomial

$$P^{[q]}(t)=P_0(t)+P_1(t)+\cdots+P_q(t).$$

If $Q(t)$ is another holomorphic function in t, we write $P(t)\overset{q}{\equiv}Q(t)$ if $P^{[q]}(t)=Q^{[q]}(t)$.

Now we expand each component $\phi_i^A(z_i,t)$ of $\phi_i(z_i,t)$ into a power series

$$\phi_i^A(z_i,t)=\phi_{i|1}^A(z_i,t)+\cdots+\phi_{i|q}^A(z_i,t)+\cdots$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

in $t^1, \dots, t^m$, and write

$$\phi^A_{i|q}(z_i, t) = (\phi^1_{i|q}(z_i, t), \dots, \phi^A_{i|q}(z_i, t), \dots, \phi^p_{i|q}(z_i, t)),$$
$$\phi^{A[q]}_i(z_i, t) = \phi^A_{i|1}(z_i, t) + \cdots + \phi^A_{i|q}(z_i, t).$$

The equality (8.2) is then reduced to the following system of congruences:

$$\phi^{A[q]}_i(g^a_{ij}(\phi^{B[q]}_j(z_j, t), z_j), t) \overset{q}{\equiv} f^A_{ij}(\phi^{B[q]}_j(z_j, t), z_j), \quad q = 1, 2, 3, \dots. \tag{8.6}$$

We note that the congruence $(8.6)_1$ is equivalent to

$$\phi^A_{i|1}(z_i, t) = F^A_{ijB}(z) \,.\, \phi^B_{j|1}(z_j, t), \quad z = (0, z_i) = (0, z_j).$$

First, we shall construct the polynomials $\phi^{A[q]}_i(z_i, t)$ by induction on q. In view of the boundary conditions (8.3), (8.4), we define

$$\phi^A_{i|1}(z_i, t) = \sum_\alpha \theta^A_{\alpha i}(z) t^\alpha.$$

It is clear by (8.1) that the linear forms $\phi^A_{i|1}(z_i, t)$, $i \in I$, satisfy $(8.6)_1$.

Assume that the polynomials $\phi^{A[q]}_i(z_i, t)$, $i \in I$, satisfying $(8.6)_q$ are already determined for an integer $q \geqslant 1$. For the sake of simplicity we write

$$\begin{aligned}
\phi^{A[q]}_j(t) &= \phi^{A[q]}_j(z_j, t),\\
f^A_{ij}(\omega^B_j) &= f^A_{ij}(\omega^B_j, z_j),\\
f^A_{kj}(\omega^B_j) &= f^A_{kj}(\omega^B_j, z_j),\\
g^a_{ij}(\omega^B_j) &= g^a_{ij}(\omega^B_j, z_j),\\
&\vdots
\end{aligned}$$

and we set

$$\psi^A_{ij}(z_j, t) \overset{q+1}{=} \phi^{A[q]}_i(z_i, t)\big|_{z^a_i = g^a_{ij}(\phi^{B[q]}_j(z_j, t), z_j)} - f^A_{ij}(\omega^B_j, z_j)\big|_{\omega^B_j = \phi^{B[q]}_j(z_j, t)}. \tag{8.7}$$

Note that $\psi^A_{ij}(z_j, t)$ is a homogeneous polynomial of degree $q + 1$ in $t^1, \dots, t^m$ whose coefficients are vector-valued holomorphic functions of z_j, $|z_j| < 1$, $|g_{ij}(0, z_j)| < 1$, and that

$$\psi^A_{ij}(z_j, t) \overset{q+1}{\equiv} \phi^{A[q]}_i(g_{ij}(\phi^{B[q]}_j(t)), t) - f^A_{ij}(\phi^{B[q]}_j(t)). \tag{8.8}$$

We define

$$\psi^A_{ij}(z, t) = \psi^A_{ij}(z_j, t) \quad \text{for } z = (0, z_j) \in U_i \cap U_j.$$

We have the equality [**1**]

$$\psi^A_{ij}(z, t) = \psi^A_{ik}(z, t) + F^A_{ikB}(z) \,.\, \psi^B_{kj}(z, t) \quad \text{for } z \in U_i \cap U_j \cap U_k. \tag{8.9}$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

We now have to prove that the 1-cocycle $\{\psi^A_{ij}(z_i,t)\}$ takes values in $\tilde{N}_{X|Y}$ rather than in $N_{X|Y}$. By definition, we obtain

$$\psi^0_{ij}(z_j,t) \overset{q+1}{=} \phi^{0[q]}_i(z_i,t)|_{z^a_i=g^a_{ij}(\phi^{B[q]}_j(z_j,t),z_j)} - f^0_{ij}(\omega^B_j,z_j)|_{\omega^B_j=\phi^{B[q]}_j(z_j,t)} \tag{8.10}$$

and

$$\psi^a_{ij}(z_j,t) \overset{q+1}{=} \phi^{a[q]}_i(z_i,t)|_{z^a_i=g^a_{ij}(\phi^{B[q]}_j(z_j,t),z_j)} - f^a_{ij}(\omega^B_j,z_j)|_{\omega^B_j=\phi^{B[q]}_j(z_j,t)}. \tag{8.11}$$

Then $\{\psi^A_{ij}(z_j,t)\}$ represents a cohomology class in $H^1(X,\tilde{N}_{X|Y})$ if and only if

$$\psi^a_{ij}(z_j,t) = -\frac{\partial\psi^0_{ij}(z_j,t)}{\partial z^b_j}(A^{-1})^b_a$$

or

$$\frac{\partial\psi^0_{ij}(z_j,t)}{\partial z^b_j} = -\sum_a \psi^a_{ij}(z_j,t)A^a_b.$$

To prove this, differentiate (8.10) with respect to z^b_j, and using equations (6.3)–(6.7) and (8.11) with Lemma 6.1, we obtain (see [**1**])

$$\begin{aligned}
\frac{\partial\psi^0_{ij}}{\partial z^b_j} &= \frac{\partial\phi^{0[q]}_i(z_i,t)}{\partial z^a_i}\bigg|_{z^a_i=g^a_{ij}(\phi^{B[q]}_j(z_j,t),z_j)}\bigg(\frac{\partial g^a_{ij}}{\partial\omega^A_j}\frac{\partial\phi^A_j}{\partial z^b_j}+\frac{\partial g^a_{ij}}{\partial z^b_j}\bigg)\\
&\quad -\frac{\partial f^0_{ij}}{\partial\omega^B_j}\bigg|_{\omega^B_j=\phi^{B[q]}_j(z_j,t)}\frac{\partial\phi^{B[q]}_j}{\partial z^b_j}-\frac{\partial f^0_{ij}}{\partial z^b_j}\bigg|_{\omega^B_j=\phi^{B[q]}_j(z_j,t)}\\
&= -\phi^{a[q]}_i A^a_b - \sum_{\bar{b}}\phi^{\bar{b}[q]}_i\frac{\partial\phi^{\bar{\bar{b}}[q]}_i}{\partial z^a_i}A^a_b + \sum_a f^a_{ij}\bigg|_{\omega^B_j=\phi^{B[q]}_j(z_j,t)}A^a_b + \sum_{\bar{b}}\phi^{\bar{b}[q]}_i\frac{\partial\phi^{\bar{\bar{b}}[q]}_i}{\partial z^a_i}A^a_b\\
&= -\sum_a \psi^a_{ij}(z_j,t)A^a_b.
\end{aligned}$$

Hence,

$$\frac{\partial\psi^0_{ij}(z_j,t)}{\partial z^b_j} = -\sum_a \psi^a_{ij}(z_j,t)A^a_b.$$

From the exact sequence

$$0\to S_X\to\tilde{N}_{X|Y}\to L_X\to 0,$$

it follows that

$$\cdots\to H^1(X,S_X)\to H^1(X,\tilde{N}_{X|Y})\to H^1(X,L_X)\to\cdots$$

as $H^1(X,S_X)=H^1(X,L_X)=0$, and hence we get $H^1(X,\tilde{N}_{X|Y})=0$. Therefore, there exists a collection $\{\phi^A_{i|q+1}(z,t)\}$ of homogeneous polynomials $\phi^A_{i|q+1}(z,t)$ of degree $q+1$ in $t^1,\dots,t^m$, whose coefficients are holomorphic functions of z defined on U_i if we take values in $\tilde{N}_{X|Y}$ such that

$$\psi^A_{ij}(z,t) = F^A_{ijB}(z)\phi^B_{j|q+1}(z,t) - \phi^A_{i|q+1}(z,t) \quad \text{for } z\in U_i\cap U_j. \tag{8.12}$$

https://doi.org/10.1017/S0013091505000301 Published online by Cambridge University Press

Considering the coefficients of $\phi^A_{i|q+1}(z,t)$ as functions of the local coordinate z_i of z, we write $\phi^A_{i|q+1}(z_i,t)$ for $\phi^A_{i|q+1}(z,t)$. The formula (8.12) can then be written in the form

$$\psi^A_{ij}(z_j,t) = F^A_{ijB}(z)\phi^B_{j|q+1}(z_j,t) - \phi^A_{i|q+1}(g_{ij}(0,z_j),t). \tag{8.13}$$

We now define

$$\phi_i^{A[q+1]}(z_i,t) = \phi_i^{A[q]}(z_i,t) + \phi^A_{i|q+1}(z_i,t), \quad i \in I.$$

On writing $\phi_j^{A[q+1]}(t)$ for $\phi_j^{A[q]}(z_j,t)$, we then have

$$\phi_i^{A[q+1]}(g_{ij}(\phi_j^{B[q+1]}(t)),t) \overset{q+1}{\equiv} \phi_i^{A[q]}(g_{ij}(\phi_j^{B[q]}(t)),t) + \phi^A_{i|q+1}(g_{ij}(0,z_j),t),$$
$$f^A_{ij}(\phi_j^{B[q+1]}(t)) \overset{q+1}{\equiv} f^A_{ij}(\phi_j^{B[q]}(t)) + F^A_{ijB}(z)\phi^B_{j|q+1}(z_j,t).$$

Consequently, from (8.8) and (8.9), we obtain the congruence

$$\phi_i^{A[q+1]}(g_{ij}(\phi_j^{B[q+1]}(t)),t) \overset{q+1}{\equiv} f^A_{ij}(\phi_j^{B[q+1]}(t)).$$

This completes our inductive construction of the polynomials $\phi_i^{A[q]}(z_i,t)$, $i \in I$, satisfying $(8.6)_q$. Thus, setting

$$\phi_i^A(z_i,t) = \phi^A_{i|1}(z_i,t) + \cdots + \phi^A_{i|q}(z_i,t) + \cdots,$$

we obtain a formal power series $\phi_i^A(z_i,t)$, $i \in I$, in $t^1,\dots,t^m$, whose coefficients are vector-valued holomorphic functions of z_i, $|z_i| < 1$, which satisfies equations (8.2)–(8.4).

Step 3 (convergence). There is an arbitrariness involved in the construction of the formal power series $\phi_i^A(z_i,t)$. For each $q \geqslant 1$, the 0-cochain $\{\phi^A_{i|q+1}(z_i,t)\}$, whose image under the coboundary map is the 1-cocycle $\{\psi^A_{ij}(z_j,t)\}$, is defined up to the addition of a global holomorphic section of $\tilde{N}_{X|Y}$ over X. We now want to use this freedom to ensure convergence of the formal constructions. The idea is to estimate each holomorphic function involved in the construction of $\phi_i^A(z_i,t)$ and show that, under appropriate choices of $\{\phi^A_{i|q+1}(z_i,t)\}$, $q = 1,2,\dots$, all the resulting power series $\{\phi_i^A(z_i,t)\}$ are majorities by an obviously convergent series

$$A(t) = \frac{a}{16b}\sum_{n=1}^{\infty}\frac{b^n}{n^2}(t_1+t_2+\cdots+t_m)^n,$$

where a and b are some positive constants. Fortunately, what really counts at this stage is the compactness of X and the analyticity of all functions involved in the construction. Therefore, all the estimates obtained by Kodaira [**4**] carry over verbatim to our case. We conclude that polynomials $\phi^A_{i|q+1}(z_i,t)$ can be chosen in such a way that the power series $\phi_i^A(z_i,t)$ converges for $|t| < \varepsilon$, where ε is some positive number. This completes the proof of Theorem 8.1. □

Example 8.2. Let Y be a five-dimensional complex projective space $\mathcal{CP}^5$ with contact structure coming from some non-degenerate skew symmetric product ω on $\mathcal{C}^6$. The contact line bundle L of such a structure is $\mathcal{O}(2)$. Let $X = \mathcal{CP}^1$ be an isotropic complex projective line in Y such that $L_X = \mathcal{O}_X(2)$. The normal bundle of $X \hookrightarrow Y$ is $N_{X|Y} = \mathcal{C}^4 \otimes \mathcal{O}_X(1)$. Since $J^1 L_X = \mathcal{C}^2 \otimes \mathcal{O}_X(1)$, the exact sequence,

$$0 \to S_X \to N_{X|Y} \to J^1 L_X \to 0,$$

implies that $S_X \simeq \mathcal{C}^2 \otimes \mathcal{O}_X(1)$. As $H^1(X, L_X) = H^1(X, S_X) = 0$, Theorem 8.1 then ensures that there is a $(3 + 4) = 7$-dimensional moduli space M of deformations of X in the class of isotropic submanifolds.

In fact, X is a complex projective line, linearly embedded in $\mathcal{CP}^5$ in the usual way. Non-projectively, this corresponds to a 2-plane in $\mathcal{C}^6$, and the condition that $\mathcal{CP}^1$ is isotropic with respect to the contact structure translates into the condition that the 2-plane is isotropic with respect to the symplectic form ω.

Let us consider first the linear deformations of X. These correspond to a subset of the Grassmannian of all 2-planes in $\mathcal{C}^6$ which has dimension $2(6 - 2) = 8$. We may embed this Grassmannian in $\mathcal{P}(\wedge^2 \mathcal{C}^6) = \mathcal{CP}^{14}$ by the Plücker embedding. The isotropic 2-planes then correspond to a hyperplane section of the image of this Grassmannian, since the symplectic form ω is a linear functional on $\wedge^2 \mathcal{C}^6$. The space of isotropic 2-planes therefore has complex dimension 7. Therefore, we can identify the moduli space M of deformations of X with the isotropic Grassmannian of 2-planes in $\mathcal{C}^6$.

References

1. M. S. Ali, Moduli spaces of compact complex isotropic submanifolds in complex contact manifold, PhD thesis, Department of Mathematics, University of Glasgow, UK (2003).
2. V. I. Arnol'd, *Mathematical methods of classical mechanics* (Springer, 1978).
3. K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann. Math.* **75** (1962), 146–162.
4. K. Kodaira, *Complex manifolds and deformation of complex structures* (Springer, 1986).
5. C. LeBrun, Thickenings and conformal gravity, *Commun. Math. Phys.* **139** (1991), 1–43.
6. S. A. Merkulov, Geometry of Kodaira moduli spaces, *Proc. Am. Math. Soc.* **124** (1996), 1499–1506.
7. S. A. Merkulov, Existence and geometry of Legendre moduli spaces, *Math. Z.* **226** (1997), 211–265.
8. S. A. Merkulov, Affine connections on involutive G-structures, in *Geometry and physics*, Lecture Notes in Pure and Applied Mathematics, Volume 184, pp. 261–274 (Dekker, New York, 1997).