Bull. Austral. Math. Soc. 20D60
Vol. 74 (2006) [121–132]

B.H. NEUMANN'S QUESTION ON ENSURING COMMUTATIVITY OF FINITE GROUPS

A. Abdollahi, A. Azad, A. Mohammadi Hassanabadi and M. Zarrin

Dedicated to the memory of Bernhard H. Neumann

This paper is an attempt to provide a partial answer to the following question put forward by Bernhard H. Neumann in 2000: "Let G be a finite group of order g and assume that however a set M of m elements and a set N of n elements of the group is chosen, at least one element of M commutes with at least one element of N. What relations between g, m, n guarantee that G is Abelian?" We find an exponential function $f(m,n)$ such that every such group G is Abelian whenever $|G| > f(m,n)$ and this function can be taken to be polynomial if G is not soluble. We give an upper bound in terms of m and n for the solubility length of G, if G is soluble.

1. Introduction and results

This paper is an attempt to provide a partial answer to the following question put forward by Bernhard H. Neumann in [10]: "Let G be a finite group of order g and assume that however a set M of m elements and a set N of n elements of the group is chosen, at least one element of M commutes with at least one element of N (call this condition $Comm(m,n)$). What relations between g, m, n guarantee that G is Abelian?"

Following Neumann, for given positive integers m and n we say that a group G satisfies the condition $Comm(m,n)$ if and only if for every two subsets M and N of cardinalities m and n respectively, there are elements $x \in M$ and $y \in N$ such that $xy = yx$.

We note that an infinite group G satisfying the condition $Comm(m,n)$ for some m and n is Abelian. This is because every infinite subset of such group contains two commuting elements. Thus by a famous Theorem of Neumann [9], it is centre-by-finite. Therefore $Z(G)$, the centre of G is infinite. Now let M and N be two subsets of $Z(G)$, of sizes m and n respectively. Then for any two elements x and y of G, there are elements $z_1 \in M$ and $z_2 \in N$ such that $xz_1yz_2 = yz_2xz_1$, so that $xy = yx$; namely G is

Received 21st March, 2006
This work was in part supported by the Center of Excellence for Mathematics, University of Isfahan.

Abelian. Therefore in considering non-Abelian groups satisfying $Comm(m, n)$ we need only consider finite cases.

We use the usual notations: for example $C_G(a)$ is the centraliser of an element a in a group G, S_n is the symmetric group on n letters, A_n is the alternating group on n letters, D_{2n} is the dihedral group of order $2n$, Q_8 is the quaternion group of order 8 and T will stand for the group $\langle x, y \mid x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1}\rangle$. If G satisfies the condition $Comm(m, n)$, then we say G is a $C(m, n)$-group, or $G \in C(m, n)$.

Throughout G will denote a finite non-Abelian group unless otherwise is stated. We shall show that a $C(m, n)$-group has order bounded by a function of m and n which may not always be chosen to be a polynomial function in terms of m and n. Our main results are

Theorem 1.1. *Let G be a $C(m, n)$-group. Then $|G|$ is bounded by a function of m and n.*

The solubility length of a soluble $C(m, n)$-group is bounded above in terms of m and n. In fact we prove the following.

Theorem 1.2. *Let $G \in C(m, n)$ be a soluble group of solubility length d. Then*

$$d \leqslant \max\{\lceil \log_2 m \rceil, \lceil \log_2 n \rceil\}$$

We also obtain a solubility criterion for $C(m, n)$-groups in terms of m and n, namely

Theorem 1.3. *Let G be a $C(m, n)$-group and $m + n \leqslant 58$. Then G is a soluble group.*

We give a complete characterisation of $C(m, n)$-groups, where $m + n \leqslant 10$, in the next theorem.

Theorem 1.4. *Let G be a $C(m, n)$-group, where $m + n \leqslant 10$. Then G is isomorphic to one of the following: S_3, D_{2n} for $n \in \{3, 4, 5, 6\}$, Q_8, T or a non-Abelian group of order 16 whose centre is of order 4.*

2. A partial answer to Neumann's question

A subset of a non-Abelian group G no two of whose distinct elements commute is called non-commuting. A non-commuting subset of maximal size is called a maximal non-commuting set and this maximal size will be denoted by $\omega(G)$. In this section we give a partial answer to Neumann's question by proving that a $C(m, n)$-group has the order bounded by a function of m and n.

Proof of Theorem 1.1: Let $Z(G) = \{z_1, z_2, \ldots, z_t\}$, where $t \geqslant \max\{m, n\}$. Choose any two elements a and b in G, and put

$$M = \{az_1, az_2, \ldots, az_m\} \text{ and } N = \{bz_1, bz_2, \ldots, bz_n\}$$

Since G is a $C(m,n)$-group, there exist $az_i \in M$ and $bz_j \in N$, where $1 \leqslant i \leqslant m$ and $1 \leqslant i \leqslant n$, such that $az_i bz_j = bz_j az_i$. This implies that $ab = ba$, and so G is an Abelian group, which is a contradiction. Thus $|Z(G)| < \max\{m,n\}$. Suppose, for a contradiction, that $\omega = \omega(G) \geqslant m + n$. Then there are ω pairwise non-commuting elements $a_1, \ldots, a_{m+n} \in G$. Put

$$M = \{a_1, \ldots, a_m\} \text{ and } N = \{a_{m+1}, \ldots, a_{m+n}\}$$

Since G is a $C(m,n)$-group, there exist $a_i \in M$ and $a_j \in N$ such that $a_i a_j = a_j a_i$, which is a contradiction. Thus $\omega < m + n$. Now the main result of [**11**] implies that $|G : Z(G)| \leqslant c^{\omega}$, where c is a constant. Therefore

$$|G| \leqslant c^{\omega}|Z(G)| \leqslant c^{m+n}\max\{m,n\},$$

which completes the proof. $\square$

REMARK 2.1. Since an extra-special 2-group of order 2^{2k+1}, has maximal non-commuting sets of size $2k+1$ (see [**4**] or [**11**]), if $f(m,n)$ is the least integer such that $|G| \leqslant f(m,n)$ for all $C(m,n)$-groups, then $f(m,n)$ cannot be chosen to be a polynomial in terms of m and n.

The following is a key lemma to some of our results.

LEMMA 2.2. *Let G be a $C(m,n)$-group and let N be a normal subgroup of G such that G/N is non-Abelian. Then $|N| < \max\{m,n\}$.*

PROOF: Suppose on the contrary that $N = \{a_1, a_2, \ldots, a_t\}$ and $t \geqslant \max\{m,n\}$. Choose any two elements x and y in $G \setminus N$, and put

$$X = \{xa_1, xa_2, \ldots, xa_m\} \text{ and}$$
$$Y = \{ya_1, ya_2, \ldots, ya_n\}.$$

Since G is a $C(m,n)$-group, there exist xa_i in X and ya_j in Y such that $[xa_i, ya_j] = 1$. Thus $[x,y] \in N$ and G/N is Abelian, which is a contradiction. $\square$

COROLLARY 2.3. *Let G be an insoluble $C(m,n)$-group. Then*

$$|G| \leqslant 4^4(m+n)^8 \cdot \max\{m,n\}.$$

PROOF: Let S be the largest soluble normal subgroup of G. Then G/S has no non-trivial normal Abelian subgroup and by [**12**, Theorem 1.3], $|G/S| < \big(n(G)\big)^4$, where $n(G)$ is the size of the largest conjugacy class in G. Now by [**11**] we have $n(G) \leqslant 4\omega(G)^2$. Then by the proof of Theorem 1.1, $\omega(G) < m+n$ and by Lemma 2.2, $|S| < \max\{m,n\}$, which completes the proof. $\square$

3. Soluble groups satisfying the condition $Comm(m, n)$

In this section we prove Theorems 1.2 and 1.3. First we need some preliminary lemmas.

Lemma 3.1. *Let G be a $C(m,n)$-group. If $a_1, a_2, \ldots, a_n$ are n distinct elements of G, then $\left|G \setminus \bigcup_{i=1}^{n} C_G(a_i)\right| < m$.*

Proof: Suppose, for a contradiction, that there exist m distinct elements $b_1, b_2, \ldots, b_m$ in $G \setminus \bigcup_{i=1}^{n} C_G(a_i)$. Since G is a $C(m,n)$-group, there exist elements a_i, b_j such that $a_i b_j = b_j a_i$ and so $b_j \in C_G(a_i)$, which is a contradiction. ∎

Lemma 3.2. *If G is a $C(m,n)$-group, then $m + n \geqslant 6$.*

Proof: Suppose, for a contradiction, that $m + n < 6$. We distinguish two cases:
Case 1: $n = 1$. Then $|G| \leqslant 6$ and so $G \cong S_3$, since G is non-Abelian. If $a \in S_3$ is of order 3, then Lemma 3.1 gives $3 = |G \setminus C_G(a)| < m$. It follows that $m = 4$. But S_3 is not a $C(1,4)$-group.

Case 2: $n = 2$. Since G is non-Abelian, there exists an element a in $G \setminus Z(G)$ such that $a^2 \neq 1$; for let $g^2 = 1$ for all $g \in G \setminus Z(G)$. Then $(gz)^2 = 1$ for all $z \in Z(G)$ and $g \in G \setminus Z(G)$. It follows that $1 = g^2 z^2 = z^2$ and so we have $z^2 = 1$ for all $z \in Z(G)$. Hence $g^2 = 1$ for all $g \in G$ which implies that G is Abelian, a contradiction.

Now since $a \neq a^{-1}$, it follows from Lemma 3.1 that

$$\left|G \setminus \left(C_G(a) \cup C_G(a^{-1})\right)\right| \leqslant m - 1 \leqslant 2.$$

Since $C_G(a) = C_G(a^{-1})$, we have that $|G| \leqslant |C_G(a)| + 2$. As $a \in G \setminus Z(G)$, it follows that $|C_G(a)| \leqslant |G|/2$ and so $|G| \leqslant |G|/2 + 2$. Hence $|G| \leqslant 4$, so G is Abelian. This contradiction completes the proof. ∎

Lemma 3.3. *Let G be a $C(m,n)$-group and let N be a non-trivial normal subgroup of G. Then G/N is a $C(m-r, n-t)$-group, for all positive integers r, t such that $2r \leqslant m$ and $2t \leqslant n$.*

Proof: Suppose, for a contradiction, that G/N is not a $C(m-r, n-t)$-group. Thus there exist two subsets

$$X = \{x_1 N, \ldots, x_{m-r} N\} \text{ and } Y = \{y_1 N, \ldots, y_{n-t} N\}$$

such that $[x_i, y_j] \notin N$ for all i, j. Let a be a non-trivial element of N and consider

$$X_1 = \{a x_1, \ldots, a x_{m-r}, x_1, \ldots, x_r\} \text{ and}$$
$$Y_1 = \{a y_1, \ldots, a y_{n-t}, y_1, \ldots, y_t\}.$$

It is clear that $|X| = m$ and $|Y| = n$ and no element of X_1 commutes with no element of Y_1, which completes the proof. ∎

PROOF OF THEOREM 1.2: We argue by induction on d. By hypothesis G is non-Abelian, thus it follows from Lemma 3.2 that either $m \geqslant 3$ or $n \geqslant 3$. Thus for $d = 2$, the result holds, since $\lceil \log_2 3 \rceil = 2$. So assume that $d \geqslant 3$ and the result holds for $d-1$. Now $G/G^{(d-1)}$ has solubility length $d-1$. Let k and ℓ be positive integers such that $2^k < m \leqslant 2^{k+1}$ and $2^\ell < n \leqslant 2^{\ell+1}$. Thus by Lemma 3.3, $G/G^{(d-1)}$ satisfies $Comm(2^k, 2^\ell)$. Thus by the induction hypothesis $d-1 \leqslant \max\{k, \ell\}$ and so $d \leqslant \max\{\lceil \log_2 m \rceil, \lceil \log_2 n \rceil\}$, as required. ∎

To prove Theorem 1.3 we need the following lemma.

If G is a finite group, then for each prime divisor p of $|G|$, we denote by $\nu_p(G)$ the number of Sylow p-subgroups of G.

LEMMA 3.4. *Let G be a $C(m,n)$-group and p be a prime number dividing $|G|$ such that every two distinct Sylow p-subgroups of G have trivial intersection. Then $\nu_p(G) \leqslant m + n - 1$.*

PROOF: It follows from the proof of Theorem 1.1, that $\omega(G) < m + n$. Now [**7**, Lemma 3] completes the proof. ∎

PROOF OF THEOREM 1.3: Suppose, on the contrary, that there exists a non-soluble finite group $G \in C(m,n)$ of the least possible order, where $m + n \leqslant 58$. If there exists a non-trivial proper normal subgroup N of G, then both N and G/N are in $C(m,n)$ and so they are soluble. It follows that G is soluble, which is a contradiction. Therefore G is a minimal simple $C(m,n)$-group. By Thompson's classification of minimal simple groups [**13**], G is isomorphic to one of the following simple groups:

A_5 the alternating group of degree 5,
$PSL(2, 2^p)$, where p is an odd prime,
$PSL(2, 3^p)$, where p is an odd prime,
$PSL(2, p)$, where $5 < p$ is prime and $p \equiv 2 \pmod 5$,
$PSL(3, 3)$, and
$Sz(2^p)$, p an odd prime.

We first prove that A_5 is not a $C(m,n)$-group, where $m + n \leqslant 58$. Let $P_1, \ldots, P_5$, $Q_1, \ldots, Q_{10}$, $R_1, \ldots, R_6$ be Sylow p-subgroups of A_5, for $p = 2, 3, 5$, respectively. It is easy to see that A_5 is the union of these Sylow subgroups and no two distinct non-trivial elements of coprime orders in A_5 commute (see [**3**]). Since every non-trivial element in $\bigcup_{i=1}^{6} R_i \cup Q_1 \cup Q_2$ does not commute with one in

$$\left(\bigcup_{i=1}^{5} P_i \cup \bigcup_{i=3}^{10} Q_i\right) \setminus \{a\}$$

(where a is an arbitrary non-trivial element of Q_{10}), A_5 is not a $C(28,30)$-group and since every non-trivial element in

$$\left(\bigcup_{i=1}^{6} R_i \cup Q_1 \cup Q_2 \cup Q_3\right)\Big\backslash\{b\}$$

(where b is an arbitrary non-trivial element of Q_1) does not commute with one in

$$\bigcup_{i=1}^{5} P_i \cup \bigcup_{i=4}^{10} Q_i,$$

A_5 is not a $C(29,29)$-group. Now suppose that $n \leqslant 27$. Then $n = 4k+\ell$ for some integers k and ℓ, where $0 \leqslant k \leqslant 6$ and $0 \leqslant \ell \leqslant 3$. Let a be an arbitrary non-trivial element of Q_{10} and define

$$\mathcal{A}_n = \begin{cases} \bigcup_{i=1}^{k} R_i & \text{if } \ell = 0 \\ \left(\bigcup_{i=1}^{k} R_i \cup Q_{10}\right)\Big\backslash\{a\} & \text{if } \ell = 1 \\ \bigcup_{i=1}^{k} R_i \cup Q_1 & \text{if } \ell = 2 \\ \bigcup_{i=1}^{k} R_i \cup P_1 & \text{if } \ell = 3 \end{cases}$$

and

$$\mathcal{B}_n = \begin{cases} \left(\bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{10} Q_i\right)\Big\backslash\{a\} & \text{if } \ell = 0 \\ \bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=1}^{5} P_i \cup \bigcup_{i=1}^{9} Q_i & \text{if } \ell = 1 \\ \left(\bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=2}^{5} P_i \cup \bigcup_{i=2}^{10} Q_i\right)\Big\backslash\{a\} & \text{if } \ell = 2 \\ \left(\bigcup_{i=k+1}^{6} R_i \cup \bigcup_{i=2}^{5} P_i \cup \bigcup_{i=1}^{10} Q_i\right)\Big\backslash\{a\} & \text{if } \ell = 3 \end{cases}.$$

Then no non-trivial element of $\mathcal{A}_n$ commutes with one of $\mathcal{B}_n$. It then follows that A_5 is not a $C(n,m)$-group, where $n+m \leqslant 58$.

If G is isomorphic to $PSL(2,2^p)$ or $PSL(2,3^p)$, where p is an odd prime, then by [**1**, Lemma 4.4], $\omega(G) > 64$, which is a contradiction. If $G \cong PSL(3,3)$, then $|G| = 2^4 \times 3^3 \times 13$ so that $\nu_{13}(H) = 144 > 57$, which is not possible by Lemma 3.4. If $G \cong PSL(2,p)$ and $p > 7$ (p is a prime number), then [**1**, Lemma 4.4] implies that $\omega(G) \geqslant 133$, a contradiction. If $G \cong PSL(2,7)$, then by [**1**, Proposition 3.21] and a similar argument as for A_5 we conclude that G is not a $C(m,n)$-group. If $G \cong Sz(2^p)$, then $|G| = 2^{2p} \times (2^p - 1) \times (2^{2p}+1)$ and $\nu_2(G) = 2^{2p}+1 \geqslant 65$ (see Theorem 3.10 (and its proof) of [**8**, Chapter XI]). ∎

We note that the bound 58 in Theorem 1.3 is the best possible. In fact we have

Theorem 3.5. *The alternating group A_5 is the only non-Abelian finite simple $C(m,n)$-group, for some positive integers m and n such that $m+n=59$.*

Proof: First we note that, since every centraliser of A_5 has order at least 3, A_5 is a $C(1,58)$-group. For uniqueness, suppose, on the contrary, that there exists a non-Abelian finite simple group not isomorphic to A_5 and of least possible order which is a $C(m,n)$-group, for some positive integers m and n with $m+n=59$. Then by [**5**, Proposition 3], G is isomorphic to one of the following groups:

$PSL(2,2^p)$, $p=4$ or a prime;
$PSL(2,3^p)$, $PSL(2,5^p)$, p a prime;
$PSL(2,p)$, p a prime and $7 \leqslant p$;
$PSL(3,3)$;
$PSL(2,5)$;
$PSU(3,4)$(the projective special unitary group of degree 3 over the finite field of order 4^2) or
$Sz(2^p)$, p an odd prime.

Now an argument similar to the one in the proof of Theorem 1.3 gives a contradiction in each case. ∎

4. Groups satisfying the condition $Comm(m,n)$ for some small positive integers m and n

In this section we characterise $C(m,n)$-groups for some particular m and n and hence prove Theorem 1.4. First we need some preliminary lemmas.

Lemma 4.1. *Let G be a $C(m,n)$-group. Let x be a non-central element of finite order such that $\varphi(|x|) \geqslant n$, where φ is the Euler φ-function. Then $|G \setminus C_G(x)| < m$.*

Proof: Suppose that

$$\Big\{k \in \mathbb{N} \,:\, 1 \leqslant k \leqslant |x| \text{ and } \gcd(k,|x|)=1\Big\} = \{d_1, d_2, \ldots, d_{\varphi(|x|)}\}.$$

Since $x^{d_i} \neq x^{d_j}$ for all $i \neq j$, by Lemma 3.1

$$\left| G \Big\backslash \bigcup_{i=1}^{d_{\varphi(|x|)}} C_G(x^{d_i}) \right| < m.$$

Also we have $C_G(x) = C_G(x^{d_i})$ for all $1 \leqslant i \leqslant d_{\varphi(|x|)}$. Hence $|G \setminus C_G(x)| < m$. ∎

Lemma 4.2. *Let G be a finite nilpotent $C(m,n)$-group. Then $\prod_{p \mid |G|} p < \max\{m,n\}$.*

https://doi.org/10.1017/S000497270004750X Published online by Cambridge University Press

PROOF: The group G is the direct product of its Sylow subgroups. So $G = \prod_{p \| |G|} P$, where P is the Sylow p-subgroup. Then $Z(G) = \prod_{p \| |G|} Z(P)$ and $\max\{m,n\} \geqslant |Z(G)| \geqslant \prod_{p \| |G|} p$, by the proof of Theorem 1.1. ∎

LEMMA 4.3. *If G is a $C(m,n)$-group, then for any prime divisor p of $|G|$, $p \leqslant \max\{m,n\}$.*

PROOF: Suppose that p is a prime divisor of $|G|$. Let a be an element of order p in G. For any x in G put $X = \{xa, xa^2, \dots, xa^m\}$ and $Y = \{a, a^2, \dots, a^n\}$. Then, by the hypothesis, there exist $xa^i \in X$ and $a^j \in Y$ such that $xa^ia^j = a^jxa^i$. Since $gcd(j,p) = 1$, we have $[x,a] = 1$. Thus $a \in Z(G)$, so that $p \mid |Z(G)|$ and by the proof of Theorem 1.1, $p \leqslant |Z(G)| < \max\{m,n\}$. ∎

LEMMA 4.4. *Let G be a non-Abelian finite group such that $|G/Z(G)| = 4$. Then G is not a $C(z,2z)$-group, where $z = |Z(G)|$.*

PROOF: Since G is non-Abelian, $G/Z(G) \cong C_2 \times C_2$. Thus there exist elements $a, b \in G$ such that

$$G = Z(G) \cup abZ(G) \cup aZ(G) \cup bZ(G).$$

Therefore $\langle aZ(G), bZ(G) \rangle$ is an elementary Abelian 2-group of order 4. Thus $G = \langle a,b \rangle Z(G)$ and so $ab \neq ba$, since G is not Abelian. Now consider the subsets $M = aZ(G) \cup bZ(G)$ and $N = abZ(G)$. We have $xy \neq yx$ for all $x \in M$ and $y \in N$, since $ab \neq ba$. This shows that G is not a $C(z,2z)$-group. ∎

REMARK 4.5.

(1) Let G be a $C(m,n)$-group. Then it is easy to see that G is not a $C\big(t, |G \setminus C_G(a)|\big)$-group, where a is any element of G with $t \leqslant |C_G(a) \setminus Z(G)|$.

(2) If G is a $C(m,n)$-group, then for any two natural numbers m' and n' such that $m \leqslant m'$ and $n \leqslant n'$, G is also a $C(m',n')$-group.

COROLLARY 4.6. *Let G be a $C(1,n)$-group, where $5 \leqslant n \leqslant 9$. Then $G \cong S_3$, D_8, Q_8, D_{10}, T, D_{12} or a non-Abelian group of order 16 whose centre is of order 4.*

PROOF: By Remark 4.5(2), it is enough to consider only the case $n = 9$. Suppose that a is any non-central element of G. By Lemma 3.1 we have $|G \setminus C_G(a)| \leqslant 8$ and so $|G| \leqslant 16$. If $|G| = 12$, then $G \cong A_4$, D_{12} or T. The alternating group A_4 has an element whose centraliser has order 3. Thus by Remark 4.5(1), A_4 is not a $C(1,9)$-group. If $G \cong D_{12}$ or $G \cong T$, then the order of the centraliser of any element in G is at least 4. Thus G is a $C(1,9)$-group. If $|G| = 14$, then $G \cong D_{14}$ and there exists $x \in D_{14}$ such that $|C_G(x)| = 2$. By Remark 4.5(1), D_{14} is not a $C(1,9)$-group. Finally if $|G| = 16$, then $|Z(G)| = 2$ or 4. If $|Z(G)| = 4$, then for all $a \in G$, $|C_G(a)| \geqslant 8$. Thus G is a $C(1,9)$-group. If $|Z(G)| = 2$, then there exists an element a in G such that $|C_G(a)| = 4$, so that by Remark 4.5(1), G is not a $C(1,9)$-group. ∎

Corollary 4.7. *Let G be a $C(2,n)$-group, where $4 \leqslant n \leqslant 8$. Then $G \cong S_3$, Q_8, D_8 or D_{10}.*

Proof: By Remark 4.5(2), it is enough to consider only the case $n = 8$. Since G is non-Abelian, there exists an element a in $G \setminus Z(G)$ such that $a^2 \neq 1$. By Lemma 3.1, $|G \setminus C_G(a)| \leqslant 7$, from which it follows that $|G| \leqslant 14$. If $|G| = 12$, then G contains centraliser of order 4. Thus by Remark 4.5(1), G is not a $C(2,8)$-group. If $|G| = 14$, then $G \cong D_{14}$, and it is not a $C(2,8)$-group since D_{14} contains centralisers of order 2. ∎

Lemma 4.8. *Let G be a $C(3,n)$-group, where $3 \leqslant n \leqslant 7$. Then $G \cong S_3$, D_8, Q_8 or D_{10}.*

Proof: By Remark 4.5(2), it is enough to consider only the case $n = 7$. Since G is non-Abelian, there exists non-central element a in G such that $a^2 \neq 1$. Let $b \in G \setminus Z(G)$ be such that $b \neq a, a^{-1}$. Then by Lemma 3.1,

$$|G \setminus C_G(a) \cup C_G(a^{-1}) \cup C_G(b)| \leqslant 6$$

Hence $|G \setminus C_G(a) \cup C_G(b)| \leqslant 6$. Clearly $|G| \in \{8, 10, 12, 14, 16, 20\}$. If $|G| = 12$, then $G \cong A_4$, D_{12} or T. As before A_4 is not a $C(3,7)$-group. For D_{12}, the subsets $M = \{b, ba^3, ba^5\}$ and $N = \{a, a^2, a^4, a^5, ba, ba^2, ba^4\}$ show that D_{12} is not a $C(3,7)$-group. For T, the subsets $M = \{y, yx^3, x^2y\}$ and $N = \{x, x^2, x^4, x^5, yx, xy, yx^2\}$ show that T is not a $C(3,7)$-group. For $|G| = 14$, $G \cong D_{14}$ and there exists an element $x \in D_{14}$ such that $|C_{D_{14}}(x)| = 7$, showing that D_{14} is not a $C(3,7)$-group. If $|G| = 16$, then G has centralisers of order 8. By Remark 4.5(1), G is not a $C(3,7)$-group. Every non-Abelian group of order 20, has centralisers of order 4, and by Remark 4.5(1), is not a $C(3,7)$-group. ∎

Lemma 4.9. *If G is a $C(4,6)$-group and $Z(G) \neq 1$, then $G \cong Q_8$ or D_8.*

Proof: By Lemma 3.3, $G/(Z(G))$ is an Abelian group and by Lemma 4.2, $\prod_{p||G|} p \leqslant 5$. Thus G is a p-group for $p \in \{2,3,5\}$. If G is a 5-group, then there exists an element a in $G \setminus Z(G)$ whose order is 5. Thus

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(a^4)| \leqslant 5.$$

Hence $|G \setminus C_G(a^2)| \leqslant 5$ and therefore $|G| \leqslant 10$, which is a contradiction. If G is a 3-group, then by the proof of Theorem 1.1, $Z(G) = \langle z \rangle$, and there exists an element a in $G \setminus Z(G)$ such that

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(az) \cup C_G(az^2)| \leqslant 5$$

Hence $|G \setminus C_G(a)| \leqslant 5$ and so $|G| \leqslant 10$, which is not possible. Therefore G is a 2-group and by the proof of Theorem 1.1, $|Z(G)| = 2$ or 4. Let $|Z(G)| = 2$ and $Z(G) = \langle z \rangle$. Then there exists an element a in $G \setminus Z(G)$ of order 4. Now we distinguish two cases:

CASE 1: $a^2 \notin Z(G)$. In this case

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(az)| \leqslant 5.$$

Hence $|G \setminus C_G(a^2)| \leqslant 5$, so that $|G| \leqslant 10$, which cannot happen.

CASE 2: $a^2 \in Z(G)$. In this case there exists an element b in $G \setminus \langle a \rangle$ such that

$$|G \setminus C_G(a) \cup C_G(a^{-1}) \cup C_G(b) \cup C_G(ba^2)| \leqslant 5 \quad \text{and}$$
$$|G \setminus C_G(a) \cup C_G(b)| \leqslant 5.$$

Clearly $|G| = 8$. Now suppose that $|Z(G)| = 4$. Say, $Z(G) = \{1, z_1, z_2, z_3\}$. There exists an element a in $G \setminus Z(G)$ of order 4 such that $a^2 \neq z_1$, and

$$|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(az_1)| \leqslant 5.$$

Therefore $|G| \leqslant 10$, which is not possible again. ∎

LEMMA 4.10. *Let G be a $C(4, n)$-group, where $4 \leqslant n \leqslant 6$. Then $G \cong S_3$, Q_8, D_8, or D_{10}.*

PROOF: By Remark 4.5(2), it is enough to consider only the case $n = 6$. Let $a \in G \setminus Z(G)$. By Lemma 4.1, $|a| \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. Let $Z(G) = 1$. We Distinguish three cases:

CASE 1. $|a| \geqslant 5$. In this case $|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(a^4)| \leqslant 5$ and $|G| \leqslant 10$.

CASE 2. $|a| = 4$. For b in $G \setminus \langle a \rangle$, we have $|G \setminus C_G(a) \cup C_G(a^2) \cup C_G(a^3) \cup C_G(b)| \leqslant 5$, from which it follows that $|G| \in \{8, 10, 12\}$. For $|G| = 12$, then $G \cong A_4$, but the subsets

$$M = \{(12)(34), (13)(24), (14)(23), (123)\} \quad \text{and}$$
$$N = \{(124), (142), (134), (143), (234), (243)\}$$

show that A_4 is not a $C(4, 6)$-group.

CASE 3. $|a| \in \{2, 3\}$. In this case there exist elements a and b in G of order 2 and 3, respectively.

CASE 3(i). Suppose that there exists an element c in $G \setminus \langle b \rangle$ of order 3. Then

$$|G \setminus C_G(b) \cup C_G(b^{-1}) \cup C_G(c) \cup C_G(c^{-1})| \leqslant 5,$$

from which it follows that $|G| = 10$, 12 or 14 so that $G \cong D_{10}$, A_4 or D_{14}. The group D_{14} has centraliser of order 7, and by Remark 4.5(1), it is not a $C(4, 6)$-group.

CASE 3(ii). Every c in $G \setminus \langle b \rangle$ has order two. Let a_1, a_2, a_3, a_4, a_5 and a_6 be elements of order two. Then

$$G = C_G(a_1) \cup C_G(a_2) \cup C_G(a_3) \cup C_G(a_4) \cup C_G(a_5) \cup C_G(a_6) \cup C_G(b).$$

Now by [**2**, Theorem B], $|G| \leqslant 81$. But $|G| = 2^k \cdot 3$ and hence $|G| \in \{6, 12, 24, 48\}$. Since A_4 and S_4 are the only centreless groups of order 12 and 24 respectively which are not $C(4,6)$-groups, $|G| \neq 12$ or 24.

Finally any centreless group of order 48, has more than two elements of order 3, so that $|G| \neq 48$. Now if $Z(G) \neq 1$, then by Lemma 4.9, $G \cong Q_8$ or D_8, and the proof is complete. ∎

LEMMA 4.11. *If G is a $C(5,5)$-group, then $G \cong S_3$, Q_8, D_8 or D_{10}.*

PROOF: A similar proof to that of Lemma 4.9, gives the result. ∎

PROOF OF THEOREM 1.4: It follows easily from Lemmas 4.6–4.11. ∎

REFERENCES

[1] A. Abdollahi, S. Akbari and H.R. Maimani, 'Non-commuting graph of a group', *J. Algebra* **298** (2006), 468–492.

[2] A. Abdollahi and S.M. Jafarian Amiri, 'On groups with an irredundant 7-cover', *J. Pure Appl. Algebra* (to appear).

[3] A. Abdollahi and A. Mohammadi Hassanabadi, 'Finite groups with certain number of elements pairwise generating a non-nilpotent subgroup', *Bull. Iranian Math. Soc.* **30** (2004), 1–20.

[4] E.A. Bertram, 'Some applications of graph theory to finite groups', *Discrete Math.* **44** (1983), 31–43.

[5] R.D. Blyth and D.J.S. Robinson, 'Insoluble groups with the rewriting property P_8', *J. Pure Appl. Algebra* **72** (1991), 251–263.

[6] R.A. Bryce, V. Fedri and L. Serena, 'Covering groups with subgroups', *Bull. Austral. Math. Soc.* **55** (1997), 469–476.

[7] G. Endimioni, 'Groupes finis satisfaisant la condition $(\mathcal{N}, n)$', *C. R. Acad. Sci. Paris Ser. I Math.* **319** (1994), 1245–1247.

[8] B. Huppert and N. Blackburn, *Finite groups, III* (Springer-Verlag, New York, 1982).

[9] B.H. Neumann, 'A problem of Paul Erdös on groups', *J. Austral. Math. Soc. Ser. A* **21** (1976), 467–472.

[10] B.H. Neumann, 'Ensuring commutativity of finite groups', *J. Aust. Math. Soc.* **71** (2001), 233–234.

[11] L. Pyber, 'The number of pairwise non-commuting elements and the index of the centre in a finite group', *J. London Math. Soc. (2)* **35** (1987), 287–295.

[12] D. Segal and A. Shalev, 'On groups with bounded conjugacy classes', *Quart. J. Math. Oxford Ser. (2)* **50** (1999), 505–516.

[13] J.G. Thompson, 'Nonsolvable finite groups all of whose local subgroups are soluble', *Bull. Amer. Math. Soc.* **74** (1968), 383–437.

Department of Mathematics
University of Isfahan
Isfahan 81746-73441
Iran
e-mail: a.abdollahi@math.ui.ac.ir
a-azad@sci.ui.ac.ir
aamohaha@sci.ui.ac.ir
m.zarrin@math.ui.ac.ir