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Hecke Operators on Jacobi-like Forms

Min Ho Lee and Hyo Chul Myung

Abstract. Jacobi-like forms for a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ are formal power series with coefficients in the space of functions on the Poincaré upper half plane satisfying a certain functional equation, and they correspond to sequences of certain modular forms. We introduce Hecke operators acting on the space of Jacobi-like forms and obtain an explicit formula for such an action in terms of modular forms. We also prove that those Hecke operator actions on Jacobi-like forms are compatible with the usual Hecke operator actions on modular forms.

1 Introduction

Jacobi-like forms for a discrete subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ are formal power series with coefficients in the space of functions on the Poincaré upper half plane satisfying a certain functional equation, and they were introduced by Zagier in [7] in connection with Rankin-Cohen brackets for modular forms. A Rankin-Cohen bracket is a bilinear operation which assigns to each pair of modular forms f and g of weights k and l, respectively, a modular form $[f, g]_n$ of weight $k+l+2n$ (*cf.* [1], [6]). The modularity of $[f, g]_n$ follows naturally from the invariance of the product of two Jacobi-like forms. Jacobi-like forms generalize the usual Jacobi forms developed by Eichler and Zagier [4] in some sense, and they are also related to vertex operator algebras and the conformal field theory as is suggested in [3] and [7].

Soliton equations or integrable partial differential equations have been studied extensively for the last few decades, and they include the Korteweg-de Vries (KdV) equation and the Kadomtsev-Petviashvilli (KP) equation which describe waves in shallow water. Pseudodifferential operators are Laurent series in the formal inverse ∂^{-1} of the differential operator $\partial = d/dz$, and they play an important role in the theory of soliton equations. Soliton equations are linked to various areas of mathematics, and one of such areas is the theory of modular forms. For instance, some solutions of the KP equation can be expressed in terms of certain theta functions which are modular forms. In [2] Cohen, Manin and Zagier obtained close connections among pseudodifferential operators, modular forms, and Jacobi-like forms. Among other things, they established a correspondence between Jacobi-like forms and sequences of certain modular forms. One of the applications of this correspondence is the construction of the Jacobi-like form associated to a modular form of even weight, called the Cohen-Kuznetsov lifting of the modular form.

In this paper we introduce Hecke operators acting on the space of Jacobi-like forms and obtain an explicit formula for such an action in terms of modular forms. We also prove that those Hecke operator actions on Jacobi-like forms are compatible

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with the usual Hecke operator actions on modular forms via the Cohen-Kuznetsov lifting map and the correspondence between Jacobi-like forms and modular forms.

2 Jacobi-like forms, modular forms, and Hecke operators

In this section we review Jacobi-like forms, modular forms and Hecke operators acting on the space of modular forms and discuss some of their properties.

Let $\mathrm{GL}^+(2,\mathbb{R})$ (resp. $\mathrm{SL}(2,\mathbb{R})$) be the multiplicative group of 2×2 real matrices of positive determinant (resp. determinant one), which acts on the Poincaré upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

by linear fractional transformations. Thus, if $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $\mathrm{GL}^+(2,\mathbb{R})$ and if $z \in \mathcal{H}$, we have

$$\alpha z = \frac{az+b}{cz+d}.$$

Given such $\alpha \in \mathrm{GL}^+(2,\mathbb{R})$, $z \in \mathcal{H}$ and a function $f\colon \mathcal{H} \to \mathbb{C}$, we set $j(\alpha, z) = cz + d$ and

$$(f \mid_k \alpha)(z) = \det(\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha z) \tag{2.1}$$

for $k \in \mathbb{Z}$. Then it can be shown by induction that

$$\frac{d^r}{dz^r}(f \mid_k \gamma)(z) = \sum_{\ell=0}^{r} \frac{r!}{\ell!} \binom{k+r-1}{r-\ell} \frac{(\det \gamma)^{k/2+\ell}(-c)^{r-\ell}}{j(\alpha, z)^{k+r+\ell}} f^{(\ell)}(\gamma z) \tag{2.2}$$

for each positive integer r, where $f^{(\ell)}$ denotes the derivative of f of order ℓ and $f \mid_k \gamma$ is as in (2.1) (see [2] where the formula for $\gamma \in \mathrm{SL}(2,\mathbb{R})$ is given).

Definition 2.1 Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and let R be the space of complex-valued functions on $\mathcal{H}$. A formal power series $\Phi(z, X) \in R[[X]]$ in X with coefficients in R is a *Jacobi-like form for* Γ if

$$\Phi\big(\gamma z, (cz+d)^{-2} X\big) = e^{cX/(cz+d)} \Phi(z, X) \tag{2.3}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. We shall denote by $\mathcal{J}(\Gamma)$ the space of all Jacobi-like forms for Γ.

Consider a Jacobi-like form $\Phi(z, X) \in \mathcal{J}(\Gamma)$ given by

$$\Phi(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k. \tag{2.4}$$

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Then by (2.3) we have

$$\sum_{k=1}^{\infty} \phi_k(\gamma z)(cz+d)^{-2k} X^k = \Big(\sum_{k=0}^{\infty} \frac{1}{k!} \Big(\frac{c}{cz+d} \Big)^k X^k \Big) \Big(\sum_{k=1}^{\infty} \phi_k(z) X^k \Big).$$

Comparing the coefficients of X^k, we see that a formal power series $\Phi(z, X) \in R[[X]]$ given by (2.4) is a Jacobi-like form for Γ if and only if

$$(\phi_k \mid_{2k} \gamma)(z) = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \Big(\frac{c}{cz+d} \Big)^{\ell} \phi_{k-\ell}(z) \tag{2.5}$$

for all $k \geq 1$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Definition 2.2 Let Γ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, and let k be an integer. A *modular form of weight* k *for* Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$(f \mid_k \gamma)(z) = f(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. We shall denote by $\mathcal{M}_k(\Gamma)$ the space of modular forms of weight k for Γ.

Remark 2.3 The above definition is more general than the usual definition of modular forms, which requires Γ to be of finite covolume and f to be holomorphic at the cusps.

Proposition 2.4 *A formal power series* $\Phi(z, X) \in R[[X]]$ *given by*

$$\Phi(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k \tag{2.6}$$

is an element of $\mathcal{J}(\Gamma)$ *if and only if there is a sequence* $\{f_\ell\}_{\ell=1}^{\infty}$ *of modular forms with* $f_\ell \in \mathcal{M}_{2\ell}(\Gamma)$ *for each* ℓ *such that*

$$\phi_k(z) = \sum_{r=0}^{k-1} \frac{1}{r!\,(2k-r-1)!} f_{k-r}^{(r)}(z)$$

for all $k \geq 1$ *and* $z \in \mathcal{H}$.

Proof This follows from Proposition 2 in [2]. ∎

Two subgroups Γ_1 and Γ_2 of $\mathrm{GL}^+(2, \mathbb{R})$ are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2, in which case we write $\Gamma_1 \sim \Gamma_2$. Given a subgroup Δ of $\mathrm{GL}^+(2, \mathbb{R})$, we denote by $\widetilde{\Delta} \subset \mathrm{GL}^+(2, \mathbb{R})$ the commensurator of Δ, that is,

$$\widetilde{\Delta} = \{ g \in \mathrm{GL}^+(2, \mathbb{R}) \mid g \Delta g^{-1} \sim \Delta \}.$$

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Given a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ and an element $\alpha \in \widetilde{\Gamma}$, it is known that the double coset $\Gamma\alpha\Gamma$ has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i \tag{2.7}$$

for some $\alpha_i \in \mathrm{GL}^+(2,\mathbb{R})$, $i = 1, \dots, s$ (see *e.g.* [5]).

Definition 2.5 The *Hecke operator on* $\mathcal{M}_k(\Gamma)$ for $k \in \mathbb{Z}$ associated to the double coset $\Gamma\alpha\Gamma$ is the linear map $T_k^\alpha \colon \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma)$ defined by

$$T_k^\alpha f = \sum_{i=1}^{s} (f \mid_k \alpha_i) \tag{2.8}$$

for all $f \in \mathcal{M}_k(\Gamma)$.

3 Hecke Operators on Jacobi-like Forms

In this section we introduce Hecke operators associated to double cosets of discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ which act on the space of Jacobi-like forms. We obtain an explicit formula for such an action in terms of the associated modular forms and show that it is compatible with the usual Hecke operator actions on modular forms.

Given an element $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}^+(2,\mathbb{R})$ and a formal power series $\Phi(z,X) \in R[[X]]$, we set

$$(\Phi \mid \delta)(z,X) = e^{-cX/j(\delta,z)} \Phi\bigl(\delta z, (\det \delta) j(\delta,z)^{-2} X\bigr), \tag{3.1}$$

where $j(\delta,z) = cz + d$ for all $z \in \mathcal{H}$. Thus, if Γ is a subgroup of $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}^+(2,\mathbb{R})$, the formal power series $\Phi(z,X)$ is a Jacobi-like form for Γ if and only if $(\Phi \mid \gamma)(z,X) = \Phi(z,X)$ for all $\gamma \in \Gamma$.

Lemma 3.1 *Given a formal power series $\Phi(z,X) \in R[[X]]$, we have*

$$(\Phi \mid \delta) \mid \delta' = (\Phi \mid \delta\delta')$$

for all $\delta, \delta' \in \mathrm{GL}^+(2,\mathbb{R})$.

Proof This can be obtained by a straightforward calculation. ■

Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and let α be an element of its commensurator $\widetilde{\Gamma} \subset \mathrm{GL}^+(2,\mathbb{R})$ such that the associated double coset has a decomposition of the form

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i$$

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with $\alpha_i \in \mathrm{GL}^+(2,\mathbb{R})$ for $i = 1,\dots,s$. Given a Jacobi-like form $\Phi(z,X) \in \mathcal{J}(\Gamma)$, we set

$$(T^\alpha_{\mathcal{J}}\Phi)(z,X) = \sum_{i=1}^{s}(\Phi \mid \alpha_i)(z,X) \tag{3.2}$$

for all $z \in \mathcal{H}$.

Proposition 3.2 *For each $\alpha \in \widetilde{\Gamma}$ the power series $T^\alpha_{\mathcal{J}}\Phi$ given by* (3.2) *is independent of the choice of the coset representatives $\alpha_1,\dots,\alpha_s$, and the map $\Phi \mapsto T^\alpha_{\mathcal{J}}\Phi$ determines a linear endomorphism*

$$T^\alpha_{\mathcal{J}} \colon \mathcal{J}(\Gamma) \to \mathcal{J}(\Gamma)$$

on the space $\mathcal{J}(\Gamma)$ of Jacobi-like forms for Γ.

Proof Suppose that $\{\beta_1,\dots,\beta_s\}$ is another set of coset representatives with $\beta_i = \gamma_i\alpha_i$ for $1 \le i \le s$. Using Lemma 3.1 and the fact that Φ is a Jacobi-like form for Γ, we have

$$\sum_{i=1}^{s}(\Phi \mid \beta_i)(z,X) = \sum_{i=1}^{s}\big((\Phi \mid \gamma_i) \mid \alpha_i\big)(z,X) = \sum_{i=1}^{s}(\Phi \mid \alpha_i)(z,X),$$

and hence $T^\alpha_{\mathcal{J}}\Phi$ is independent of the choice of the coset representatives. Since the linearity of the map $\Phi \mapsto T^\alpha_{\mathcal{J}}\Phi$ is clear, it suffices to show that $T^\alpha_{\mathcal{J}}\big(\mathcal{J}(\Gamma)\big) \subset \mathcal{J}(\Gamma)$. Let $\Phi(z,X) \in \mathcal{J}(\Gamma)$ and $\gamma \in \Gamma$. By Lemma 3.1 we see that

$$(T^\alpha_{\mathcal{J}}\Phi) \mid \gamma = \sum_{i=1}^{s}(\Phi \mid \alpha_i) \mid \gamma = \sum_{i=1}^{s}\Phi \mid (\alpha_i\gamma).$$

However, the set $\{\alpha_1\gamma,\dots,\alpha_s\gamma\}$ is another complete set of coset representatives, and hence we have

$$\sum_{i=1}^{s}\Phi \mid (\alpha_i\gamma) = \sum_{i=1}^{s}\Phi \mid \alpha_i.$$

Thus we see that $(T^\alpha_{\mathcal{J}}\Phi) \mid \gamma = (T^\alpha_{\mathcal{J}}\Phi)$, and therefore the proposition follows. ■

Definition 3.3 A linear operator of the form $T^\alpha_{\mathcal{J}} \colon \mathcal{J}(\Gamma) \to \mathcal{J}(\Gamma)$ for $\alpha \in \widetilde{\Gamma}$ given by (3.2) is called a *Hecke operator on* $\mathcal{J}(\Gamma)$.

We now consider the expression of the image of a Jacobi-like form under the Hecke operator in terms of the associated modular forms described in Proposition 2.4.

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Theorem 3.4 *Let* $\Phi(z,X) = \sum_{k=1}^{\infty} \phi_k(z)X^k \in \mathcal{J}(\Gamma)$ *be a Jacobi-like form for a discrete subgroup* Γ *of* $\mathrm{SL}(2,\mathbb{R})$*, and let* $\{f_\nu\}_{\nu=1}^{\infty}$ *be the sequence of modular forms with*

$$\phi_k = \sum_{r=0}^{k-1} \frac{1}{r!\,(2k-r-1)!} f_{k-r}^{(r)} \tag{3.3}$$

and $f_\nu \in \mathcal{M}_{2\nu}(\Gamma)$ *for all* $k, \nu \geq 1$*. If* α *is an element of* $\widetilde{\Gamma} \subset \mathrm{GL}^+(2,\mathbb{R})$ *such that*

$$\Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i$$

with $\alpha_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \mathrm{GL}^+(2,\mathbb{R})$ *for* $i = 1,\dots,s$*, then we have*

(3.4)
$$T_{\mathcal{J}}^{\alpha}\Phi(z,X) = \sum_{k=1}^{\infty}\sum_{i=1}^{s}\sum_{\mu=0}^{k-1}\sum_{r=0}^{\mu} \frac{(\det\alpha_i)^{k-\mu+r}(-c_i)^{\mu-r}}{(\mu-r)!\,r!\,(2k-2\mu+r-1)!} \frac{f_{k-\mu}^{(r)}(\alpha_i z)}{(c_i z+d_i)^{2k-\mu+r}} X^k$$

for all $z \in \mathcal{H}$.

Proof By (3.2) we have $(T_{\mathcal{J}}^{\alpha}\Phi)(z,X) = \sum_{i=1}^{s} \Phi_i(z,X)$, where

$$\Phi_i(z,X) = (\Phi \mid \alpha_i)(z,X) \in R[[X]]$$

for each i. We set $\Phi_i(z,X) = \sum_{k=1}^{\infty} \psi_{i,k}(z)X^k$ for $1 \leq i \leq s$. Then by (3.1) we see that

$$\begin{aligned}
\sum_{k=1}^{\infty} \psi_{i,k}(z)X^k &= (\Phi \mid \alpha_i)(z,X) \\
&= \exp[-c_i X/(c_i z+d_i)]\Phi\big(\alpha_i z, (\det\alpha_i)(c_i z+d_i)^{-2}X\big) \\
&= \left(\sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{-c_i}{c_i z+d_i}\right)^r X^r\right) \\
&\qquad \times \left(\sum_{k=0}^{\infty} \phi_k(\alpha_i z)(\det\alpha_i)^k (c_i z+d_i)^{-2k} X^k\right).
\end{aligned}$$

Comparing the coefficients of X^k, we obtain

$$\begin{aligned}
\psi_{i,k}(z) &= \sum_{\ell=0}^{k-1} \frac{(\det\alpha_i)^{k-\ell}}{\ell!}\left(\frac{-c_i}{c_i z+d_i}\right)^{\ell} \frac{\phi_{k-\ell}(\alpha_i z)}{(c_i z+d_i)^{2k-2\ell}} \\
&= \sum_{\ell=0}^{k-1} \frac{(\det\alpha_i)^{k-\ell}(-c_i)^{\ell}}{\ell!} \frac{\phi_{k-\ell}(\alpha_i z)}{(c_i z+d_i)^{2k-\ell}}.
\end{aligned}$$

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By (3.3) we have

$$\phi_{k-\ell}(\alpha_i z) = \sum_{r=0}^{k-1-\ell} \frac{1}{r!\,(2k-2\ell-r-1)!} f_{k-\ell-r}^{(r)}(\alpha_i z),$$

and hence it follows that

$$T_{\mathcal{J}}^{\alpha}\Phi(z,X) = \sum_{k=1}^{\infty}\sum_{i=1}^{s}\sum_{\ell=0}^{k-1}\sum_{r=0}^{k-1-\ell} \frac{(\det\alpha_i)^{k-\ell}(-c_i)^{\ell}}{\ell!\,r!\,(2k-2\ell-r-1)!} \frac{f_{k-\ell-r}^{(r)}(\alpha_i z)}{(c_i z+d_i)^{2k-\ell}} X^k.$$

Using the index $\mu = \ell + r$, we see that

$$T_{\mathcal{J}}^{\alpha}\Phi(z,X) = \sum_{k=1}^{\infty}\sum_{i=1}^{s}\sum_{r=0}^{k-1}\sum_{\mu=r}^{k-1} \frac{(\det\alpha_i)^{k-\mu+r}(-c_i)^{\mu-r}}{(\mu-r)!\,r!\,(2k-2\mu+r-1)!} \frac{f_{k-\mu}^{(r)}(\alpha_i z)}{(c_i z+d_i)^{2k-\mu+r}} X^k.$$

Replacing $\sum_{r=0}^{k-1}\sum_{\mu=r}^{k-1}$ by $\sum_{\mu=0}^{k-1}\sum_{r=0}^{\mu}$ in this equation we obtain (3.4), and therefore the proof of the theorem is complete. ∎

The Hecke operator on Jacobi-like forms can also be expressed in terms of the Hecke operators on the associated modular forms as in the next theorem.

Theorem 3.5 *Let $\Phi(z,X) \in \mathcal{J}(\Gamma)$, $\{f_\nu\}_{\nu=1}^{\infty}$ and $\alpha \in \widetilde{\Gamma}$ be as in Theorem 3.4, and let $T_{\mathcal{J}}^{\alpha}$ be the Hecke operator on $\mathcal{J}(\Gamma)$ determined by α. Then we have*

$$T_{\mathcal{J}}^{\alpha}\Phi(z,X) = \sum_{k=1}^{\infty}\sum_{\mu=0}^{k-1} \frac{1}{\mu!\,(2k-\mu-1)!} \frac{d^{\mu}}{dz^{\mu}}\big(T_{k-\mu}^{\alpha} f_{k-\mu}(z)\big) X^k, \tag{3.5}$$

where $T_{k-\mu}^{\alpha}$ is the Hecke operator on the space $\mathcal{M}_{2k-2\mu}(\Gamma)$ of modular forms of weight $2k-2\mu$ for Γ associated to α given by (2.8).

Proof Using (2.8) and (2.2), we have

$$\begin{aligned}\frac{d^{\mu}}{dz^{\mu}}\big(T_{k-\mu}^{\alpha} f_{k-\mu}(z)\big) &= \frac{d^{\mu}}{dz^{\mu}} \sum_{i=1}^{s} (f_{k-\mu}\mid_{2k-2\mu} \alpha_i)(z) \\ &= \sum_{i=1}^{s}\sum_{r=0}^{\mu} \frac{\mu!}{r!}\binom{2k-\mu-1}{\mu-r} \frac{(\det\alpha_i)^{k-\mu+r}(-c_i)^{\mu-r}}{(c_i z+d_i)^{2k-\mu+r}} f_{k-\mu}^{(r)}(\alpha_i z).\end{aligned}$$

Thus (3.5) is equivalent to

$$\begin{aligned}&T_{\mathcal{J}}^{\alpha}\Phi(z,X)\\ &\quad= \sum_{k=1}^{\infty}\sum_{i=1}^{s}\sum_{\mu=0}^{k-1}\sum_{r=0}^{\mu} \frac{(\det\alpha_i)^{k-\mu+r}(-c_i)^{\mu-r}}{r!\,(2k-\mu-1)!}\binom{2k-\mu-1}{\mu-r} \frac{f_{k-\mu}^{(r)}(\alpha_i z)}{(c_i z+d_i)^{2k-\mu+r}} X^k\\ &\quad= \sum_{k=1}^{\infty}\sum_{i=1}^{s}\sum_{\mu=0}^{k-1}\sum_{r=0}^{\mu} \frac{(\det\alpha_i)^{k-\mu+r}(-c_i)^{\mu-r}}{(\mu-r)!\,r!\,(2k-2\mu+r-1)!} \frac{f_{k-\mu}^{(r)}(\alpha_i z)}{(c_i z+d_i)^{2k-\mu+r}} X^k,\end{aligned}$$

and hence it follows that (3.5) is equivalent to (3.4). ∎

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By Proposition 2.4 to each Jacobi-like form $\Phi(z,X) \in \mathcal{J}(\Gamma)$ we can associate a sequence $\{f_k\}_{k=1}^{\infty}$ of modular forms with $f_k \in \mathcal{M}_{2k}(\Gamma)$ for all $k \geq 1$. According to Theorem 3.5, the sequence $\{T_k^{\alpha} f_k\}_{k=1}^{\infty}$ of the images of the f_k under the Hecke operators T_k^{α} is a sequence of modular forms associated to $T_k^{\mathcal{J}}\Phi(z,X)$.

4 Cohen-Kuznetsov Liftings

As is described in Section 2, a Jacobi-like form determines a sequence of modular forms. On the other hand, to each modular form we can associate a Jacobi-like form called its Cohen-Kuznetsov lifting. In this section we show that such liftings are compatible with Hecke operator actions.

For each positive integer m we set

$$\mathcal{J}(\Gamma)_m = \mathcal{J}(\Gamma) \cap X^m R[[X]],$$

where $\mathcal{J}(\Gamma) \subset R[[X]]$ is the space of Jacobi-like forms for a discrete group $\Gamma \subset SL(2,\mathbb{R})$ as in Section 2. Given an element $\Phi(z,X)$ in $\mathcal{J}(\Gamma)_m$, using (2.5), we see that its leading coefficient $\phi_m(z)$ satisfies $\phi_m \mid_{2m} \gamma = \phi_m$ for all $\gamma \in \Gamma$. Thus the map $\Phi(z,X) \mapsto \phi_m(z)$ determines a linear map $\mathcal{J}(\Gamma)_m \to \mathcal{M}_{2m}(\Gamma)$ whose kernel is $\mathcal{J}(\Gamma)_{m+1}$. On the other hand, given a modular form $f(z) \in \mathcal{M}_{2m}(\Gamma)$, let $\{f_k\}_{k=1}^{\infty}$ be the sequence of modular forms with $f_m = f$ and $f_k = 0$ for $k \neq m$. Then by Proposition 2.4 this sequence determines an element

$$\Phi_f(z,X) = \sum_{k=1}^{\infty} \psi_k(z) X^k \in \mathcal{J}(\Gamma).$$

Using (2.6), we see that $\psi_k = 0$ for $k < m$ and

$$\psi_k(z) = \frac{1}{(k-m)!\,(k+m-1)!} f^{(k-m)}(z)$$

for $k \geq m$; hence we have

$$\Phi_f(z,X) = \sum_{k=m}^{\infty} \frac{1}{(k-m)!\,(k+m-1)!} f^{(k-m)}(z) X_k \in \mathcal{J}(\Gamma)_m,$$

which is called the Cohen-Kuznetsov lifting of $f(z)$ (*cf.* [2], [7]). We shall denote by $L_m \colon \mathcal{M}_{2m}(\Gamma) \to \mathcal{J}(\Gamma)_m$ the corresponding lifting map $f(z) \mapsto \Phi_f(z,X)$. Since the leading coefficient of $\Phi_f(z,X)$ is $\psi_m(z) = f(z)/(2m-1)!$, if we define the map $K_m \colon \mathcal{J}(\Gamma)_m \to \mathcal{M}_{2m}(\Gamma)$ by

$$K_m\Big(\sum_{k=1}^{\infty} \phi_k(z) X^k\Big) = (2m-1)!\,\phi_m(z), \tag{4.1}$$

then we have $K_m \circ L_m = \mathrm{id}$, and therefore we obtain a split exact sequence

$$0 \to \mathcal{J}(\Gamma)_{m+1} \to \mathcal{J}(\Gamma)_m \xrightarrow{K_m} \mathcal{M}_{2m}(\Gamma) \to 0$$

for each positive integer m.

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Theorem 4.1 *Let* $T^{\alpha}_{\mathcal{J}} \colon \mathcal{J}(\Gamma) \to \mathcal{J}(\Gamma)$ *and* $T^{\alpha}_{m} \colon \mathcal{M}_{2m}(\Gamma) \to \mathcal{M}_{2m}(\Gamma)$ *with* $m \geq 1$ *be the Hecke operators associated to an element* $\alpha \in \widetilde{\Gamma} \subset \mathrm{GL}^{+}(2, \mathbb{R})$*. Then we have*

$$T^{\alpha}_{m} \circ K_m = K_m \circ T^{\alpha}_{\mathcal{J}}, \quad T^{\alpha}_{\mathcal{J}} \circ L_m = L_m \circ T^{\alpha}_{m}$$

for all $m \geq 1$.

Proof Given a positive integer m, let $\Phi(z, X) = \sum_{k=m}^{\infty} \psi_k(z) X^k \in \mathcal{J}(\Gamma)_m$. Then there is a sequence $\{f_k\}_{k=1}^{\infty}$ such that

$$\phi_{\ell} = \sum_{r=0}^{\ell-1} \frac{1}{r!\,(2\ell - r - 1)!} f^{(r)}_{\ell - r}, \quad f_k = \sum_{r=0}^{k-1} \frac{(2k-2-r)!}{r!\,(2k-1)} \phi^{(r)}_{k-r}$$

for all $k \geq 1$ and $\ell \geq m$. Thus we see that $f_k = 0$ for $k < m$ and $\phi_m = f_m$. Using (3.5) and (4.1), we obtain

$$\begin{aligned}
(K_m \circ T^{\alpha}_{\mathcal{J}}) \Phi(z, X) &= \sum_{\mu=0}^{m-1} \frac{1}{\mu!\,(2m - \mu - 1)!} \frac{d^{\mu}}{dz^{\mu}} \big(T^{\alpha}_{m-\mu} f_{m-\mu}(z) \big) \\
&= \frac{1}{(2m-1)!} T^{\alpha}_{m} f_m(z) \\
&= \frac{1}{(2m-1)!} T^{\alpha}_{m} \phi_m(z) = (T^{\alpha}_{m} \circ K_m) \Phi(z, X).
\end{aligned}$$

On the other hand, given an element $f \in \mathcal{M}_{2m}(\Gamma)$, we have

$$\begin{aligned}
(T^{\alpha}_{\mathcal{J}} \circ L_m) f &= T^{\alpha}_{\mathcal{J}} \Big(\sum_{k=m}^{\infty} \frac{1}{(k-m)!\,(k+m-1)!} f^{(k-m)}(z) X^k \Big) \\
&= \sum_{k=m}^{\infty} \sum_{\mu=1}^{k-1} \frac{1}{\mu!\,(2k - \mu - 1)!} \Big(\frac{d^{\mu}}{dz^{\mu}} T^{\alpha}_{k-\mu} f_m(z) \Big) X^k \\
&= \sum_{k=m}^{\infty} \frac{1}{(k-m)!\,(k+m-1)!} \frac{d^{k-m}}{dz^{k-m}} T^{\alpha}_{k-\mu} f_m(z) \\
&= (L_m \circ T^{\alpha}_{m}) f.
\end{aligned}$$

Thus the theorem follows. ■

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