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ON A PROBLEM OF M. P. SCHÜTZENBERGER

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A class of finite semigroups is called a *genus* if it is closed under homomorphic images, subsemigroups and finite direct products. During a talk at the Symposium on Semigroups held at the University of St Andrews, in 1976, M. P. Schützenberger posed the problem of characterising the smallest genus $\mathscr{G}$ which contains finite groups and finite semigroups, all of whose subgroups are trivial.

If $D \in \mathscr{G}$ then, as pointed out by Schützenberger, the subsemigroup $IG(S)$, generated by the idempotents of S, has only trivial subgroups. In the first section of this note we deduce a property of the members of $\mathscr{G}$ which may be used to show that the converse is false. This property also shows that, if a finite regular semigroup S belongs to $\mathscr{G}$, then $\mathscr{H}$ is a congruence on S. In the second section we show that, on the other hand if S is orthodox and $\mathscr{H}$ is a congruence, then $S \in \mathscr{G}$. A corollary is that a finite semigroup which is a union of groups belongs to $\mathscr{G}$ if and only if it is an orthodox band of groups.

1. A necessary condition

A class of finite semigroups is called a *genus* if it is closed under homomorphic images, subsemigroups and finite direct products. For example, the class of all finite groups is a genus, as is the class of all finite bands. Another example of a genus is given by the class $\mathscr{A}$ of all finite semigroups A in which each subgroup is trivial. (Following Eilenberg (**1**), we shall say that a semigroup A is *aperiodic* if each subgroup of A of trivial). It is easy to see that $\mathscr{A}$ is closed under subsemigroups and finite direct products. That it is also closed under homomorphic images, is a consequence of the following lemma (c.f. (**6**)) which is also used later in the paper.

Lemma 1.1. *Let T be a finite semigroup and let θ be a homomorphism of T onto a semigroup S. Then, for each subgroup H of S there is a subgroup K of T with $K\theta = H$. Thus, for each idempotent $e \in S$ there is an idempotent $f \in T$ with $f\theta = e$.*

Corollary 1.2. *Let $\mathscr{A}$ be the class of finite aperiodic semigroups. Then $\mathscr{A}$ is a genus.*

Proof. Let $T \in \mathscr{A}$ and let θ be a homorphism of T onto a semigroup S; suppose that H is a subgroup of S. Then, by Lemma 1.1, there is a subgroup K of T with $K\theta = H$. Since T is *aperiodic* K must be trivial. Hence so is H and therefore $S \in \mathscr{A}$.

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We have already remarked that $\mathscr{A}$ is closed under subsemigroups and finite direct products. Hence, since it is also closed under homomorphic images, $\mathscr{A}$ must be a genus.

The next corollary will also be useful in what follows.

Corollary 1.3. *Let T be a finite semigroup and let θ be a homorphism of T onto a semigroup S. Let $IG(T)$ denote the subsemigroup generated by the idempotents of T and, likewise, let $IG(S)$ denote the subsemigroup generated by the idempotents of S. Then $IG(T)\theta = IG(S)$.*

Proof. Clearly θ maps $IG(T)$ into $IG(S)$. On the other hand, suppose that $x = e_1e_2 \dots e_n \in IG(S)$ where $e_1, \dots, e_n$ are idempotents. Then, by Lemma 1.1, there are idempotents $f_1, \dots, f_n \in T$ with $f_i\theta = e_i$, $1 \leqq i \leqq n$. But then $(f_1f_2 \dots f_n)\theta = e_1e_2 \dots e_n = x$.

If each of $\mathscr{B}$, $\mathscr{C}$ is a genus of finite semigroups then, cf (**1**), page 110, a finite semigroup S belongs to the smallest genus which contains $\mathscr{B}$ and $\mathscr{C}$ if and only if S is a homomorphic image of a subsemigroup of $B \times C$ for $B \in \mathscr{B}$, $C \in \mathscr{C}$. In particular, S belongs to the genus $\mathscr{G}$ generated by finite groups and finite aperiodic semigroups if and only if S is a homomorphic image of subsemigroup of $G \times A$ for some finite group G and finite aperiodic semigroup A; that is, $S \in \mathscr{G}$ if and only if S *divides* $G \times A$.

Lemma 1.4. (Schützenberger). *Let $S \in \mathscr{G}$, then the subsemigroup $IG(S)$ of S, generated by the idempotents of S, is aperiodic.*

Proof. Since $S \in \mathscr{G}$, S divides $G \times A$ for some finite group G and finite aperiodic semigroup A. Thus there is a subsemigroup T of $G \times A$ and a homomorphism θ of T onto S.

Each idempotent of T has the form $(1, u)$, where 1 denotes the identity of G and u is an idempotent of A. Hence $IG(T) \subseteq \{(1, x) \colon x \in IG(A)\}$. Since A is aperiodic, so is $IG(A)$; thus so is $\{(1, x) \colon x \in IG(A)\}$. Hence $IG(T)$ is aperiodic and hence, by Lemma 1.1, so is $IG(S) = IG(T)\theta$.

Lemma 1.5. *Let $S \in \mathscr{G}$ and let H be a subgroup of S, with identity e. Then, for each $h \in H$, $x \in IG(S)$,*

$$hxh^{-1} = exe.$$

Proof. Let T be a subsemigroup of $G \times A$ for some finite group G and finite aperiodic semigroup A and let θ be a homomorphism of T onto S. By Lemma 1.1, there is a subgroup K of T, with identity u, such that $K\theta = H$. Similarly, by Corollary 1.3, there exists $y \in IG(T)$ such that $y\theta = x$.

Now, since A is aperiodic, $K = K_1 \times \{u_1\}$ where K_1 is a subgroup of G and u_1 is idempotent. Further, since G is a group, $y = (1, y_1)$ where 1 denotes the identity of G and $y_1 \in IG(A)$. Let $k = (k_1, u_1) \in K$ be such that $k\theta = h$. Then

$$\begin{aligned} kyk^{-1} &= (k_1, u_1)(1, y_1)(k_1^{-1}, u_1) \\ &= (1, u_1y_1u_1) = uyu \end{aligned}$$

so that, on applying θ, one gets $hxh^{-1} = exe$.

Let S be a regular semigroup and let e, f be idempotents of S. Then we denote by $S(e, f)$ the set

$$S(e,f)=\{u^2=u\in S:\quad fu=u=ue \quad \text{and} \quad euf=ef\}.$$

Let $a, b \in S$ with inverse a', b' respectively and set $e=a'a, f=bb'$. Then Nambooripad (**5**) shows that $S(e,f)\neq\square$ and that $b'ua'$ is an inverse of ab for each $u\in S(e,f)$.

We shall use these ideas in the proof of the following proposition.

Proposition 1.6. *Let $S\in\mathscr{G}$ be a finite regular semigroup. Then $\mathscr{H}$ is a congruence on S.*

Proof. Let $a, b\in S$ with $a\mathscr{H}b$; then there exists inverses a' of a and b' of b such that $aa'=bb'$, $a'a=b'b$. Set $h=ab'$, $h^{-1}=ba'$; then $hh^{-1}=ab'ba'=aa'aa'=aa'=bb'=ba'ab'=h^{-1}h$. Thus h belongs to a subgroup of S, with identity $e=aa'$, and has h^{-1} as inverse there.

Let $x\in S$ and let x' be an inverse for x. Pick $u\in S(a'a, xx')$; then $x'ua'$ is an inverse for ax. Now

$$\begin{aligned} ax.x'ua' &= axx'ua' \\ &= aua' && \text{since } u\in S(a'a, xx') \\ &= aa'aua'aa' \\ &= ab'bub'ba' \\ &= hyh^{-1} && \text{where } y=bub'. \end{aligned}$$

Now $bub'.bub'=bua'aub=bu^2b'=bub'$ since $u\in S(a'a, xx')$. Thus, by Lemma 1.2,

$$hyh^{-1}=eye=aa'bub'aa'=bub'=bx.x'ub'.$$

Hence $ax\mathscr{R}bx$ and, clearly $ax\mathscr{L}bx$; thus $ax\mathscr{H}bx$. Similarly $xa\mathscr{H}xb$ so that $\mathscr{H}$ is a congruence.

Corollary 1.7. *For any $n\geqq 2$, the symmetric inverse semigroup $\mathscr{I}_n$ on n letters does not belong to $\mathscr{G}$.*

Proof. $\mathscr{I}_n$ is fundamental inverse semigroup (**4**) but $\mathscr{H}$ is not a congruence on I_n. Hence I_n does not belong to $\mathscr{G}$.

The idempotents in an inverse semigroup S commute so they form a subsemigroup of S. Hence $IG(S)$ consists entirely of idempotents and so is aperiodic. Since I_n, $n\geqq 2$ does not belong to $\mathscr{G}$ it follows the converse of Lemma 1.4 is false.

2. Orthodox semigroups

In this section we prove that, if S is a finite orthodox semigroup on which $\mathscr{H}$ is a congruence, then $S\in\mathscr{G}$.

A regular semigroup S is called *E-unitary* if the idempotents form a unitary subset of S. Equivalently, S is E-unitary if the idempotents form a class of some group congruence (the minimum group congruence) on S.

Lemma 2.1. *Let S be an orthodox semigroup. Then there is an E-unitary semigroup T and an idempotent separating homomorphism of T onto S. If S is finite, T can also be chosen to be finite.*

Proof. Hall (2) has shown that S is a subdirest product of S/μ and $S/\mathscr{Y}$ where μ is the maximum idempotent separating congruence on S and $\mathscr{Y}$ is the minimum inverse semigroup congruence on S. By (3), there is an E-unitary inverse semigroup V and an idempotent separating homomorphism ϕ of V onto $S/\mathscr{Y}$; further if $S/\mathscr{Y}$ is finite, V also can be chosen finite.

Let $T=\{(A, v) \in S/\mu\times V : A = s\mu, v\phi = s\mathscr{Y}$ for some $s \in S\}$. Then T is easily seen to be a regular subsemigroup of $S/\mu\times V$. Let σ denote the minimum group congruence on V and define $\psi : T\rightarrow V/\sigma$ by $(A, v)\psi = v\sigma$. Then ψ is a homomorphism of T onto a group. Suppose $(A, v)\psi = 1$ where 1 denoted the identity of V/σ; then since V is E-unitary, $v^2 = v$ so that $s\mathscr{Y}$ is idempotent. By (1), this implies $s^2 = s$ so that (A, v) is idempotent. Hence T is E-unitary.

Now, for $(A, v) \in T$, set $(A, v)\theta = s$ if $A = s\mu$, $v\phi = s\mathscr{Y}$. Then, since $\mathscr{Y}\cap\mu = \Delta$, θ is well defined and is a homomorphism of T onto S. Suppose $(A, v)\theta = e = e^2$, then $A = e\mu$ and so, because μ is idempotent separating, θ is idempotent separating.

Lemma 2.2. *Let S be an E-unitary regular semigroup. Then $\mathscr{H}\cap\sigma = \Delta$ on S, where σ is the minimum group congruence on S.*

Proof. Let $(a, b) \in \mathscr{H}\cap\sigma$ and let a', b' be inverses for a, b respectively such that $aa' = bb'$, $a'a = b'b$. Then $ab'\mathscr{H}aa'$ and $(ab', bb') \in \sigma$. Hence, since S is E-unitary, $ab' = aa'$. Similarly $b'a = b'b$. Thus

$$a = aa'a = ab'a = ab'b = aa'b = bb'b = b.$$

Theorem 2.3. *Let S be a finite orthodox semigroup. Then $S \in \mathscr{G}$ if and only if $\mathscr{H}$ is a congruence on S.*

Proof. Suppose that $\mathscr{H}$ is a congruence on S. By Lemma 2.1, there is a finite E-unitary regular semigroup T and an idempotent separating homorphism θ of T onto S. By Lemma 2.2, T can be embedded in $T/\mu\times T/\sigma$ where μ is the maximum idempotent separating congruence on T and σ is the minimum group congruence. But, since θ is idempotent separating, $T/\mu \approx S/\mu = S/\mathscr{H}$ since $\mathscr{H}$ is a congruence. That is, S divides $S/\mathscr{H}\times T/\sigma$ so that $S \in \mathscr{G}$.

The converse is immediate from Proposition 1.6.

Corollary 2.4. *Let S be a finite semigroup which is a union of groups. Then $S \in \mathscr{G}$ if and only if S is an orthodox band of groups.*

Proof. Suppose S $\in \mathscr{G}$ and let e, f be idempotents. Then ef belongs to a subgroup of S. Since $IG(S)$ is aperiodic, this implies ef is idempotent. Hence S is orthodox and by Proposition 1.6, $\mathscr{H}$ is a congruence on S. That is, S is an orthodox band of groups.

The converse is immediate from Theorem 2.3.

Corollary 2.5. *The genus of finite semigroups generated by finite bands and finite groups is the genus of finite orthodox bands of groups.*

Proof. If S divides the direct product of a finite band and a finite group, then S is an orthodox band of groups. Conversely, if S is a finite orthodox band of groups then, by the proof of Theorem 2.3, S divides $G \times S/\mu$ for some finite group G. But, since $\mathscr{H}$ is a congruence on S, S/μ is a band.

Remark 2.6. The strategy involved in the proof of Theorem 2.3 is the following. Given a finite regular semigroup S, find a finite regular semigroup T, on which $\mathscr{H} \cap \sigma = \Delta$ and an idempotent separating homomorphism θ of T onto S.

Suppose now that such T, θ exists and let $e_1, \ldots, e_r$ be idempotents in T; let $w = e_1 \ldots e_r$. Then, since T is finite, w^n, w^{n+1} belong to a subgroup of T for some $n \geqq 1$. Further, since w is a product of idempotents $(w^n, w^{n+1}) \in \sigma$. Thus, since $w^n \mathscr{H} w^{n+1}$, it follows from $\mathscr{H} \cap \sigma = \Delta$ that $w^n = w^{n+1}$. Hence ((**1**), Theorem III, 7.6) $IG(T)$ is aperiodic and consequently $IG(S)$ is aperiodic. We therefore pose the problem: if S is a finite regular semigroup in which $IG(S)$ is aperiodic, does there exist a finite regular semigroup T, with $\sigma \cap \mathscr{H} = \Delta$ on T, and an idempotent separating homomorphism of T onto S?

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