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CONNECTIVITY OF FUNCTION SPACES

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Given two spaces X and Y with Y either an AE (metrizable) or ANE (metrizable), little is known with regard to when the function space (Y^x, τ), for some topology τ, is an AE (metrizable) or ANE (metrizable) except when very strong separation properties are imposed on X and Y (see [**5**, pp. 186–189]). One of our tasks will be to eliminate most of these separation property requirements, therefore complementing or extending some of the results of [**5**]. We also attach an appendix, which contains some needful information, as well as the complete local analogue of [**2**, Theorem 5.1].

Throughout, we use either the terminology and notation of [**2**] or of the appendix. We also let co stand for the *compact-open* topology and pc stand for the *pointwise convergence* topology of any function space.

1. Statement of main results.

THEOREM 2.1. *If X is any space and Y is continuously hyperconnected (respectively continuously m-hyperconnected, continuously ∞-hyperconnected) then (Y^X, co) is continuously hyperconnected (respectively continuously m-hyperconnected, continuously ∞-hyperconnected).*

THEOREM 2.2. *If X is any space and Y is hyperconnected (respectively m-hyperconnected, ∞-hyperconnected), then (Y^X, pc) is hyperconnected (respectively m-hyperconnected, ∞-hyperconnected).*

It is easily seen that Theorem 2.2 remains valid if all connectivity concepts in it are replaced by their "continuously" counterparts. However, it seems that Theorem 2.1 *does not remain valid* if "continuously" is everywhere removed from it. To a large extent, this is the reason for the appendix.

THEOREM 2.3. *If X is any space and Y is equiconnected then (Y^X, co) and (Y^X, pc) are equiconnected.*

THEOREM 2.4. *If X is any space and Y is a metrizable AE (stratifiable) space then (Y^X, co) and (Y^X, pc) are AE (stratifiable).*

2. Proofs of main results.

Proof of Theorem 2.1. We prove only the harder case of hyperconnectedness. Let $\{h_n\}$ be a sequence of functions ($h_n : Y^n \times P_{n-1} \to Y$) which satisfy conditions (a), (b) and (c) of [**2**, Definition 2.2]. Define

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$$H_n : (Y^X)^n \times P_{n-1} \to Y^X,$$

by letting

$$H_n((f_1, \ldots, f_n), t)(x) = h_n((f_1(x), \ldots, f_n(x)), t)$$

for each $x \in X$. We will now show that the functions $H_1, H_2, \ldots$ satisfy conditions (a), (b) and (c) of [**2**, Definition 2.2].

Condition (a). Let $F = (f_1, \ldots, f_n) \in (Y^X)^n$ and

$$t = (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \in P_{n-1}.$$

Then

$$\begin{aligned} H_n(F, t)(x) &= h_n((f_1(x), \ldots, f_n(x)), t) \\ &= h_{n-1}((f_1(x), \ldots, f_{i-1}(x), f_{i+1}(x), \ldots, f_n(x)), \delta_i t) \\ &= H_{n-1}(\delta_i F, \delta_i t)(x). \end{aligned}$$

Hence, $H_n(F,t) = H_{n-1}(\delta_i F, \delta_i t)$, as required.

We will now show that the functions H_n are continuous (this automatically shows that they satisfy condition (b) of [**2**, Definition 2.2]). Let

$$H_n(F, t) \in W(K, V)$$

for some $F = (f_1, \ldots, f_n) \in (Y^X)^n$ and $w \in K$. By continuity of h_n there exist neighbourhoods N_{jw} of $f_j(w)$ for $j = 1, 2, \ldots, n$ and N_{tw} of t such that

$$h_n((\prod_{j=1}{}^n N_{jw}) \times N_{tw}) \subset V.$$

Then, by continuity of $f_1, \ldots, f_n$, there exists a neighbourhood K_w of w in K such that

$$f_j(K_w) \subset N_{jw} \quad \text{for } j = 1, 2, \ldots, n.$$

Finitely many K_w cover K; say $K_{w1}, \ldots, K_{wm}$. Let

$$U = \prod_{j=1}{}^n [\bigcap_{i=1}{}^m W(K_{w_i}, N_{jw_i})] \times \bigcap_{i=1}{}^m N_{tw_i}.$$

Clearly, U is a neighbourhood of (F, t) in $(Y^X)^n \times P_{n-1}$; furthermore, $H_n(U) \subset W(K, V)$. (Let $((g_1, \ldots, g_n), s) \in U$ and pick $b \in K$. Then b is in some K_{w_r} with $1 \leqq r \leqq m$, and hence $g_j(b) \in N_{jw_r}$ for $j = 1, 2, \ldots, n$. Then $((g_1(b), \ldots, g_n(b)), s) \in [\prod_{j=1}{}^n N_{jw_r}] \times N_{tw_r}$ and hence

$$h_n((g_1(b), \ldots, g_n(b)), s) \in V.$$

This shows that $H_n(U) \subset W(K, V)$.)

Condition (c). Pick $f \in Y^X$ and sub-base neighbourhood $W(K, V)$ of f. We will show there exists a neighbourhood U of f such that

$$\bigcup_{n=1}^{\infty} H_n(U^n \times P_{n-1}) \subset W(K, V).$$

For each $x \in K$, pick an open neighbourhood $V_x \subset V$ of $f(x)$ such that $\bigcup_{n=1}^{\infty} h_n((V_x)^n \times P_{n-1}) \subset V$ (this can be done because we are assuming that Y

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is hyperconnected). Then pick a compact neighbourhood K_x of x in K such that $f(K_x) \subset V_x$. Since K is compact, there exist finitely many K_x, say $K_{x1}, \ldots, K_{xn}$, covering K. Then let

$$U = \bigcap_{i=1}^{n} W(K_{xi}, V_{xi}).$$

Clearly, $U \subset W(K, V)$ and U is a neighbourhood of f. Furthermore, $\bigcup_{n=1}^{\infty} H_n(U^n \times P_{n-1}) \subset W(K, V)$. Let $t = (t_1, \ldots, t_n) \in P_{n-1}$ and

$$F = (f_1, \ldots, f_n) \in U^n$$

and pick any $x \in K$. Then $x \in K_{xi}$ for some $1 \leqq i \leqq n$, which implies that $\{f_1(x), \ldots, f_n(x)\} \subset V_{xi}$, which implies that

$$((f_1(x), \ldots, f_n(x)), t) \in (V_{xi})^n \times P_{n-1},$$

which implies that $h_n((f_1(x), \ldots, f_n(x)), t) \in V$, which implies that $H_n(F, t)(x) \in V$ for each $x \in K$, which shows that

$$\bigcup_{n=1}^{\infty} H_n(U^n \times P_{n-1}) \subset W(K, V).$$

It is now clear that condition (c) is valid for any neighbourhood of f, which completes the proof.

Proof of Theorem 2.2. This is essentially the same as the proof of Theorem 2.1 with minor and obvious modifications in order to establish that the functions H_n satisfy condition (b) of [**2**, Definition 2.2].

Proof of Theorem 2.3. Let $F : Y \times Y \times I \to Y$ be an equiconnecting function. Define

$$F^* : Y^X \times Y^X \times I \to Y^X$$

by $F^*(f, g, t)(x) = F(f(x), g(x), t)$ for each $x \in X$. Clearly, $F^*(f, g, 0) = f$, $F^*(f, g, 1) = g$ and $F^*(f, f, t) = f$ for each $f, g, \in Y^X$ and $t \in I$.

Now we show that F^* is continuous whenever Y^X has the co topology (the case when Y^X has the pc topology is similar and much easier). This is essentially the same as the proof of *condition* (c) for Theorem 2.1, with $n = 2$.

Proof of Theorem 2.4. This is immediate from Theorems 2.1 and 2.2, and [**2**, Theorem 5.1].

3. Local connectivity. Unfortunately, local connectivity properties of a space Y are not easily inherited by function-spaces Y^X, as the following example shows.

Example 3.1. Let C be a unit circle in the plane and let P be the discrete space of positive integers. Then $(C^P, \text{co}) \equiv (C^P, \text{pc})$ is not an *ANE* (metrizable).

Proof. Clearly, $(C^P, \text{co}) \equiv (\prod_{i=1}^{\infty} C, \text{product topology})$ and it is well-known that $\Pi_{i=1}^{\infty} C$ is not an *ANE* (metrizable). (For a proof of this, see [**7**, Example 1].)

In view of the preceding example, the following results appear to be quite satisfactory.

THEOREM 3.2. *Let X be a compact metrizable space and Y an ANE (stratifiable). Then (Y^X, τ) is an ANE (stratifiable) for any admissible*[1] *topology τ on Y^X.*

Proof. This is essentially the same as the proof of sufficiency for [**5**, Theorem 2.4, p. 186].

THEOREM 3.3. *Let X be a compact Hausdorff space and Y a completely metrizable ANE (stratifiable) space. Then (Y^X, τ) is an ANE (stratifiable) for any admissible topology τ on Y^X.*

Proof. This is essentially the same as the proof of Theorem 3.2 because of [**6**, Theorem 3.1(b)] (note that the product of a compact Hausdorff space and a stratifiable space is a paracompact space; see [**3**]).

Note that Theorem 3.3 could have been stated in terms of hyperlocal-connectedness, for any admissible *metrizable* topology τ, because of our Theorem 4.4 in the appendix.

4. Appendix. In order to obtain useful information on the connectivity and local connectivity of function spaces in a general setting, we need to improve some of the concepts and work of [**2**].

Definition 4.1. A space L will be called respectively *continuously hyperconnected*, *continuously m-hyperconnected*, *continuously ∞-hyperconnected* if the sequence of functions $h_i : L^i \times P_{i-1} \rightarrow L$ which satisfy the appropriate conditions of [**2**, Definition 2.2] can be chosen to be *continuous*.

THEOREM 4.2. *A metrizable hyperconnected space is continuously hyperconnected.*

Proof. This is essentially contained in the proof of [**2**, Theorem 5.1].

Definition 4.3. A space X is said to be *hyper-locally-connected* (respectively *m-hyper-locally-connected*) if there exists an open cover $\mathscr{N}$ of X and functions $h_n \equiv h_{N_n} : N^n P_{n-1} \rightarrow X$ for each $N \in \mathscr{N}$ and $n = 1, 2, \ldots$, which satisfy conditions (a), (b), (c) (respectively conditions (a), (b), (d)).

(a) $t \in P_{n-1}$ and $t_i = 0$ implies that $h_n(x, t) = h_{n-1}(\delta_i, x, \delta_i t)$ for each $x \in N^n$ and $n = 2, 3, \ldots$.

(b) For each $x \in N^n$, the map $t \rightarrow h_n(x, t)$, from P_{n-1} to X, is continuous.

(c) For each $x \in N$ and neighbourhood U of x, there exists a neighbourhood

1. A topology τ for Y^X is *admissible* if the map ω: $Y^X \times X \rightarrow Y$, defined by $\omega(f, x) = f(x)$, is continuous. If X is locally compact Hausdorff then the cotopology is admissible (see [**1**]).

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$V \subset U$ of x such that

$$\bigcup_{i=1}^{\infty} h_i(V^i \times P_{i-1}) \subset U.$$

(d) For each $x \in N$ and neighbourhood U of x, there exists a neighbourhood $V \subset U$ of x such that

$$\bigcup_{i=1}^{m} h_i(V^i \times P_{i-1}) \subset U.$$

The space X will be said to be ∞*-hyper-locally-connected* provided that X is m-hyper-locally-connected for $m = 1, 2, \ldots$.

THEOREM 4.4. *A metrizable space M is hyper-locally-connected if and only if M is an ANE (stratifiable).*

Proof. First we prove the "only if" part. By [**4**, Theorem 28.3] we only need show that if $f : A \to N$ is a continuous function from a closed subset A of a stratifiable space Z to some $N \in \mathcal{N}$ of Definition 4.3, then f has a continuous extension $g : Z \to M$. But the proof of this is exactly the same as the proof of [**2**, Theorem 4.1].

Now we prove the "if" part. This is essentially the same as the proof of the "only if" part of [**2**, Theorem 5.1], except that we consider the retraction $r : U \to M$, where U is a neighbourhood of M in the Banach space B, and the functions $h_{N_n} : N^n \times P_{n-1} \to M$ where $N = J \cap M$ with J an open convex subset of U.

THEOREM 4.5. *If X is a paracompact hyper-locally-connected space then X is an ANE (stratifiable).*

Proof. This is the same as the proof of the "only if" part of Theorem 4.4.

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