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APPROXIMATION OF BANACH SPACE VALUED NON-ABSOLUTELY INTEGRABLE FUNCTIONS BY STEP FUNCTIONS

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Abstract. The approximation of Banach space valued non-absolutely integrable functions by step functions is studied. It is proved that a Henstock integrable function can be approximated by a sequence of step functions in the Alexiewicz norm, while a Henstock–Kurzweil–Pettis and a Denjoy–Khintchine–Pettis integrable function can be only scalarly approximated in the Alexiewicz norm by a sequence of step functions. In case of Henstock–Kurzweil–Pettis and Denjoy–Khintchine–Pettis integrals the full approximation can be done if and only if the range of the integral is norm relatively compact.

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1. Introduction. It is known that each Pettis integrable function can be 'scalarly approximated' by a sequence of simple functions if and only if the range of its indefinite Pettis integral is a separable subset of the Banach space containing the range of the function (see [**14**], [**15**, Theorem 10.1], [**16**, Theorem 6.8] and [**19**]). The approximation can be done in the Pettis norm if and only if the range of the indefinite Pettis integral is norm relatively compact (see [**13**], [**15**, Theorem 9.1] and [**16**, Theorem 6.2]).

It is the objective of this paper to investigate how Henstock–Kurzweil–Pettis, Denjoy–Khintchine-Pettis and Henstock integrable functions behave when we take into account approximation by a sequence of step functions and what can be said about the ranges of the integrals. We prove that in the general case we can always find a sequence of step functions approximating the integrable function scalarly in the Alexiewicz norm (see Theorems 2 and 3 below).

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We show that a Henstock integrable function can always be approximated by a sequence of step functions in the Alexiewicz norm (see Proposition 1), that corresponds to the Pettis norm in the case of Pettis integrable functions. In case of Henstock–Kurzweil–Pettis and Denjoy–Khintchine–Pettis integrals such an approximation may be done only when the range of the integral is norm relatively compact (see Theorem 4). Thus the situation here is similar to that of the Pettis integral. The methods however are completely different.

Now let us consider the ranges of the non-absolute integrals which are under consideration. It is well known that the range of the Pettis integral of a strongly measurable Banach space valued function is always norm relatively compact (cf. [**7**]). In case of Pettis integrable functions with values in a non-separable Banach space the range of the integral may not be norm relatively compact (see [**9**]). However, if the measure is perfect (what is true in case of the Lebesgue measure), then Stegall proved that the range of the Pettis integral is always norm relatively compact ([**9**]). When the Henstock–Kurzweil–Pettis or Denjoy–Khintchine–Pettis integral is examined, the situation becomes complicated. Even in case of separable Banach spaces (and consequently for strongly measurable functions) the range of the Henstock–Kurzweil–Pettis integral may not be norm relatively compact (see Example 2).

This paper can be considered as a continuation of our study of non-absolute vector valued integrals, started in [**3–6**].

2. Notations and preliminaries. Let $[0, 1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0, 1]$ and by $\mathcal{I}$ the family of all non-trivial closed subintervals of $[0, 1]$.

If $E \in \mathcal{L}$, then we denote by $|E|$ its Lebesgue measure. Throughout this paper X is a Banach space with dual X^*. The closed unit ball of X^* is denoted by $B(X^*)$.

A *partition in* $[0, 1]$ is a finite collection $\pi = \{I_1, \dots, I_p\}$ of non-overlapping subintervals $I_1, \dots, I_p$ of $[0, 1]$. If $\cup_{i=1}^{p} I_i = [0, 1]$ we say that π is a *partition of* $[0, 1]$. The *mesh* of a partition π is the number $\mathrm{mesh}(\pi) =: \sup\{|I| : I \in \pi\}$.

A *Perron partition of* $[0, 1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$, where $I_1, \dots, I_p$ are non-overlapping subintervals of $[0, 1]$, $\cup_{i=1}^{p} I_i = [0, 1]$ and $t_i \in I_i$, $i = 1, \dots, p$.

A *gauge* on $[0, 1]$ is a positive function on $[0, 1]$. For a given gauge δ, we say that a Perron partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is δ*-fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \dots, p$.

We recall that a function $f : [0, 1] \to X$ is said to be a *step function* if there is a partition π of $[0, 1]$ such that f is constant on the interior of each $I \in \mathcal{I}$.

Definition 1. A function f: $[0, 1] \to X$ is said to be *Henstock integrable*, or simply *H-integrable*, on $[0, 1]$ if there exists $w \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\left\| \sum_{i=1}^{p} f(t_i)|I_i| - w \right\| < \varepsilon,$$

for each δ-fine Perron partition $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$.

We set $w := (H) \int_0^1 f \, d\lambda$.

It is known that if $f\colon[0,1]\to X$ is H-integrable on $[0,1]$ and $I\in\mathcal{I}$, then $f\chi_I$ is also H-integrable on $[0,1]$. We say in such a case that f is H-integrable on I. We call the additive interval function $F(I):=(H)\int_I f\,d\lambda$ the *H-primitive* of f.

By $H([0,1],X)$ we denote the set of all H-integrable functions $f\colon[0,1]\to X$. In case of $X=\mathbb{R}$ we will rather use the name of Henstock–Kurzweil instead of Henstock only and we will denote by $HK[0,1]$ the space of all HK-integrable functions $f\colon[0,1]\to\mathbb{R}$.

DEFINITION 2. A function $f\colon[0,1]\to X$ is said to be *scalarly* measurable (resp. *scalarly* integrable) if for every $x^*\in X^*$ the function x^*f is Lebesgue measurable (resp. integrable). A scalarly integrable function $f\colon[0,1]\to X$ is said to be *Pettis integrable* if for each set $A\in\mathcal{L}$ there exists a vector $w_A\in X$ such that

$$\langle x^*, w_A\rangle=\int_A x^*f\,d\lambda\,,\quad\text{for every}\quad x^*\in X^*.$$

We call w_A the *Pettis integral of f* over A and we write $w_A:=(P)\int_A f\,d\lambda$.

DEFINITION 3. A function $f\colon[0,1]\to X$ is said to be *scalarly Henstock–Kurzweil integrable* if for each $x^*\in X^*$ the function x^*f is Henstock–Kurzweil integrable. A scalarly Henstock–Kurzweil integrable function f is said to be *Henstock–Kurzweil–Pettis integrable* (or simply *HKP-integrable*) if for each $I\in\mathcal{I}$ there exists $w_I\in X$ such that

$$\langle x^*, w_I\rangle=(HK)\int_I x^*f\,d\lambda\,,\quad\text{for every}\quad x^*\in X^*.$$

We call w_I the *Henstock–Kurzweil–Pettis integral* of f over I and we write $w_I\coloneqq(HKP)\int_I f\,d\lambda$. If $I=[a,b]$, then we write $(HKP)\int_a^b f\,d\lambda$ instead of $(HKP)\int_{[a,b]}f\,d\lambda$. We call the additive interval function $F(I):=(HKP)\int_I f\,d\lambda$ the *HKP-primitive* of f.

We denote by $HKP([0,1],X)$ the set of all X-valued Henstock–Kurzweil–Pettis integrable functions on $[0,1]$ (functions that are scalarly equivalent are identified).

It is known that the HK-primitive (resp. HKP-primitive) F of a function f is continuous (resp. weakly continuous, i.e. x^*F is continuous for every $x^*\in X^*$).

In the following, given a function $F:[0,1]\to X$ we identify F with the additive function $F:\mathcal{I}\to X$ defined by $F(I)=F(b)-F(a)$, if $I=[a,b]$. And conversely, any additive $F:\mathcal{I}\to X$ is identified with $F:[0,1]\to X$, defined by $F(t)=F[0,t]$.

DEFINITION 4. A function $f\colon[0,1]\to\mathbb{R}$ is said to be *Denjoy–Khintchine integrable* on $[0,1]$ (or simply *DK-integrable*) if there exists an ACG function $F\colon[0,1]\to\mathbb{R}$ such that its approximate derivative F'_{ap} coincides with f almost everywhere. A function f is said to be DK-integrable on $I\in\mathcal{I}$ if $f\chi_I$ is DK-integrable on $[0,1]$. It is known that each DK-integrable function on $[0,1]$ is DK-integrable on every $I\in\mathcal{I}$.

We say that a function $f\colon[0,1]\to X$ is *Denjoy–Khintchine–Pettis integrable* (or simply *DKP-integrable*) on $[0,1]$ if for every $x^*\in X^*$ the function x^*f is DK-integrable and for each $I\in\mathcal{I}$ there exists a vector $w_I\in X$ such that $\langle x^*,w_I\rangle=(DK)\int_I x^*f\,d\lambda$ (where (DK) stands for Denjoy–Khintchine), for every $x^*\in X^*$. The vector w_I is called the *Denjoy–Khintchine–Pettis integral of f* over I and we set $w_I:=(DKP)\int_I f\,d\lambda$. The interval function $F(I):=w_I$ is called the *Denjoy–Khintchine–Pettis primitive* of f. Gordon [**11**] and Gamez and Mendoza [**10**] used the name of Denjoy–Pettis in that context.

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Denote by $DK[0, 1]$ the linear space of all real valued Denjoy–Khintchine integrable functions on $[0, 1]$ (we identify functions that are equal a.e.) and endow it with the Alexiewicz norm

$$\|f\|_A = \sup_{0<t\leq 1} \left|(DK)\int_0^t f\,d\lambda\right| .$$

The space $HK[0, 1]$ is dense in $DK[0, 1]$ in the Alexiewicz norm, the completion of $DK[0, 1]$ coincides with the completion of $HK[0, 1]$, and the conjugate spaces $DK^*[0, 1]$ and $HK^*[0, 1]$ are linearly isometric to the space $BV[0, 1]$ of functions of bounded variation (cf. [**1**]). The weak topology of $HK[0, 1]$ and $DK[0, 1]$ will be denoted by $\sigma(HK, BV)$ and $\sigma(DK, BV)$, respectively. We will denote by τ_m the topology of convergence in measure in the space of Lebesgue measurable functions.

A function $f : [0, 1] \to \mathbb{R}$ is said to be *Denjoy–Perron integrable* on $[0, 1]$ if there exists an ACG^* function $F : [0, 1] \to \mathbb{R}$ such that $F' = f$ a.e.

We refer to [**17**] for the definitions of ACG and ACG^* functions and for the definition of Denjoy–Khintchine integral.

It is known that a function $f : [0, 1] \to \mathbb{R}$ is Denjoy–Perron integrable on $[0, 1]$ if and only if it is HK-integrable on $[0, 1]$ (cf. [**12**]).

By $DKP([0, 1], X)$ we will denote the space of all Denjoy–Khintchine–Pettis integrable X-valued functions (functions that are scalarly equivalent are identified). We equip the space $DKP([0, 1], X)$ with the Alexiewicz norm

$$\|f\|_A = \sup\nolimits_{0<t\leq 1} \left\|(DKP)\int_0^t f\,d\lambda\right\|.$$

As

$$H([0, 1], X) \subset HKP([0, 1], X) \subset DKP([0, 1], X),$$

we endow also the spaces $H([0, 1], X)$ and $HKP([0, 1], X)$ with the Alexiewicz norm.

In the following, when no confusion may be generated, we will use the simple symbol $\int f\,d\lambda$ instead of $(H)\int f\,d\lambda$, $(DK)\int f\,d\lambda$, $(HKP)\int f\,d\lambda$, $(DKP)\int f\,d\lambda$.

3. Approximation. Given an additive function $F : \mathcal{I} \to X$ and a partition π of $[0, 1]$, we set

$$F_\pi = \sum_{I\in\pi} \frac{F(I)}{|I|}\chi_I.$$

PROPOSITION 1. *Let* Π *be a directed set of partitions such that* $\lim_\pi \mathrm{mesh}(\pi) = 0$.

If $F : \mathcal{I} \to X$ *is a continuous additive function, then* $\langle F_\pi\rangle_{\pi\in\Pi}$ *is Cauchy in the Alexiewicz norm of* $H([0, 1], X)$.

If F *is weakly continuous, then* $\langle x^*F_\pi\rangle_{\pi\in\Pi}$ *is Cauchy in the Alexiewicz norm of* $HK[0, 1]$, *for every* $x^* \in X^*$.

If $f : [0, 1] \to X$ *is Henstock integrable or* $f : [0, 1] \to \mathbb{R}$ *is Denjoy–Khintchine integrable and* F *is its primitive, then* $\lim_\pi \|F_\pi - f\|_A = 0$. *In particular* $F(\mathcal{I})$ *is norm relatively compact and there exists a sequence of step functions* $f_n : [0, 1] \to X$ *such that* $\lim_n \|f - f_n\|_A = 0$.

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If $f : [0, 1] \to X$ *is Henstock–Kurzweil–Pettis or Denjoy–Khintchine–Pettis integrable with primitive* F*, then* $\lim_\pi \|x^*F_\pi - x^*f\|_A = 0$*, for every* $x^* \in X^*$.

Proof. Let $\varepsilon > 0$. Since F is continuous there exists $\delta > 0$ such that $\|F(I)\| < \varepsilon/3$ for each interval $I \in \mathcal{I}$ with $|I| < \delta$. Now let $\pi_1 < \pi_2$ be two partitions with mesh less than δ. For any $t \in [0, 1]$, let $[a_1, b_1] \in \pi_1$ and $[a_2, b_2] \in \pi_2$ be the unique intervals such that $a_1 \le a_2 \le t \le b_2 \le b_1$. Then

$$\left\| \int_0^t \sum_{I\in\pi_1} \frac{F(I)}{|I|}\chi_I - \int_0^t \sum_{I\in\pi_2} \frac{F(I)}{|I|}\chi_I \right\| \le \left\| \frac{F([a_1, b_1])}{b_1 - a_1}(t - a_1) - F([a_1, a_2]) - \frac{F([a_2, b_2])}{b_2 - a_2}(t - a_2) \right\| \le \varepsilon.$$

Consequently,

$$\|F_{\pi_1} - F_{\pi_2}\|_A < \varepsilon.$$

In case F is only weakly continuous we apply the above property to each set function $x^*F : \mathcal{I} \to \mathbb{R}$ separately.

Let $f : [0, 1] \to X$ be Henstock integrable or let $f : [0, 1] \to \mathbb{R}$ be Denjoy–Khintchine integrable, and let F denote its primitive. Then we have for each $\pi \in \Pi$

$$\left\| \int_0^t (f - F_\pi) d\lambda \right\| = \left\| F([a_1, t]) - \frac{F([a_1, b_1])}{b_1 - a_1}(t - a_1) \right\| \le \|F([a_1, t])\| + \|F([a_1, b_1])\|,$$

where $[a_1, b_1] \in \pi_1$ contains the point $t \in [0, 1]$ and $a_1 < t$.

Since the primitive of an H-integrable or a DK-integrable function is continuous, we get

$$\lim_\pi \|f - F_\pi\|_A = 0.$$

Now it is enough to choose any directed sequence $\langle \pi_n \rangle_n$ of partitions with $\lim_n \text{mesh}(\pi_n) = 0$ to obtain $\lim_n \|f - f_n\|_A = 0$, where $f_n = F_{\pi_n}$.

In case f is only HKP-integrable (resp. DKP-integrable), we apply the above property to each set function $x^*F : \mathcal{I} \to \mathbb{R}$ separately. □

THEOREM 1. *Let* $f : [0, 1] \to X$ *be a Denjoy–Khintchine–Pettis integrable function and let* F *be its primitive. Then*

(i) $F(\mathcal{I})$ *is a separable and relatively weakly compact subset of* X*;*

(ii) *there exists a sequence of step functions* $f_n : [0, 1] \to X$ *such that* $\lim_n \|x^*f - x^*f_n\|_A = 0$*, for every* $x^* \in X^*$.
If moreover, f *is HKP-integrable, then we have also* $\lim_n x^*f_n = x^*f$ *a.e., for every* $x^* \in X^*$*;*

(iii) *the topology of convergence in measure and the weak topology of* $DK[0, 1]$ *(resp. of* $HK[0, 1]$ *if* f *is HKP-integrable) coincide on the set* $\{x^*f : \|x^*\| \le 1\}$*;*

(iv) $\{x^*f : \|x^*\| \le 1\}$ *is a weakly compact subset of* $DK[0, 1]$ *(resp. of* $HK[0, 1]$ *if* f *is HKP-integrable) and it is compact in the topology of convergence in measure.*

Proof. Since f is Denjoy–Khintchine–Pettis integrable, its primitive F is weakly continuous. The collection $\mathcal{J}$ of all subintervals of $[0, 1]$ with rational end points is countable and each element $F(I)$ is the weak limit of a sequence $(F(J_n))_n$, with all $J_n \in \mathcal{J}$. This proves the separability of $F(\mathcal{I})$. The weak relative compactness of $F(\mathcal{I})$ follows from the weak continuity of F and the inclusion $F(\mathcal{I}) \subset \{F(t) - F(s) : \ s, t \in [0, 1]\}$.

To prove *(ii)* let π_n be the partition of $[0, 1]$ generated by $\{I_{nk} := [k/2^n, (k+1)/2^n] : k = 0, 1 \dots, 2^n - 1\}$ and let

$$f_n = \sum_{k=0}^{2^n-1} \frac{F(I_{nk})}{|I_{nk}|} \chi_{I_{nk}}. \tag{1}$$

Applying Proposition 1 to $\Pi = \bigcup_n \pi_n$, we see that

$$\lim_n \|x^*f - x^*f_n\|_A = 0, \quad \text{for every} \quad x^* \in X^*.$$

Moreover, if $f : [0, 1] \to X$ is KHP-integrable then $(x^*F)' = x^*f$ a.e., for each $x^* \in X^*$. So if t_0 is a point such that $(x^*F)'(t_0) = x^*f(t_0)$ and t_0 is not an end point for the intervals of the partitions π_n, then we have

$$\lim_n x^*f_n(t_0) = \lim_n \frac{x^*F(I_{n_0})}{|I_{n_0}|} = x^*f(t_0),$$

where by I_{n_0} we denote the unique interval of π_n containing t_0. Therefore $x^*f_n \to x^*f$ a.e.

The coincidence of the two topologies on the set $\{x^*f : \|x^*\| \le 1\}$ follows from [**5**, Proposition 2 and Remark 4] (resp. [**5**, Proposition 2]).

The weak compactness of $\{x^*f : \|x^*\| \le 1\}$ follows from [**5**, Theorem 2 and Remark 4] (resp. [**5**, Theorem 2]) where it is proved that for each DKP-integrable (resp. HKP-integrable) function $f : [0, 1] \to X$ the operator $X^* \ni x^* \to x^*f \in DK[0, 1]$ (resp. $X^* \ni x^* \to x^*f \in HK[0, 1]$) is weak*-weakly continuous. □

With the help of the above theorem we obtain the following characterizations of HKP-integrable and DKP-integrable functions:

THEOREM 2. *A scalarly HK-integrable function $f : [0, 1] \to X$ is HKP-integrable if and only if the topology of convergence in measure and the weak topology of $HK[0, 1]$ coincide on the set $\{x^*f : \|x^*\| \le 1\}$ and there exists a sequence of step functions $f_n : [0, 1] \to X$ such that*

(i) $\lim_n \|x^*f - x^*f_n\|_A = 0$, *for every $x^* \in X^*$.*
The condition (i) may be replaced by the condition
(ii) $x^*f_n \to x^*f$ *weakly in $HK[0, 1]$, for every $x^* \in X^*$,*
or by the condition
(iii) $\lim_n x^*f_n = x^*f$, *a.e. on $[0, 1]$, for every $x^* \in X^*$.*

Proof. The 'only if' part follows by Theorem 1 and by [**5**, Theorem 3]. According to [**5**, Theorem 3], concerning the 'if' part we only need to show that f is determined by a separable space $Y \subset X$. So let $Y \subseteq X$ be the closed linear span of the set $\{f_n(t) : t \in [0, 1], \ n \in \mathbb{N}\}$. If $x^* \in Y^\perp$, then $x^*f_n(t) = 0$, for every $t \in [0, 1]$ and $n \in \mathbb{N}$. Hence, if (iii) is fulfilled, it follows at once that $x^*f = 0$, a.e. In case (i) or (ii) are fulfilled then $(HK) \int_I x^*f = 0$, for each $I \in \mathcal{I}$. So by applying [**5**, Lemma 1], we obtain again $x^*f = 0$, a.e.. Thus f is determined by a separable space. □

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We remark that even if all the three conditions (i), (ii) and (iii) in the claim of the above theorem are satisfied by a function $f : [0, 1] \to X$, it may not be HKP-integrable, as the following example shows.

EXAMPLE 1. Let $f : [0, 1] \to c_0$ be defined by

$$f(t) = \left(\chi_{(0,1]}(t), 2\chi_{(0,\frac{1}{2}]}(t), \ldots, n\chi_{(0,\frac{1}{n}]}(t), \ldots\right)$$

for $t \in [0, 1]$ (see [**7**, Example 3]).

The function f is scalarly integrable and hence also scalarly HK-integrable. Now let us consider the sequence of step functions

$$f_n(t) = \left(\chi_{(0,1]}(t), \ldots, n\chi_{(0,\frac{1}{n}]}(t), 0, 0, \ldots\right).$$

By definition the sequence $(f_n)_n$ converges pointwise in norm to f. Since $c_0^* = l_1$, for $x^* = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in l_1$ we have $x^*f = \sum_{n=1}^{\infty} \alpha_n n\chi_{(0,\frac{1}{n}]}$, $x^*f_n = \sum_{k=1}^{n} \alpha_k k\chi_{(0,\frac{1}{k}]}$ and, for each $t \in [0, 1]$,

$$\left|\int_0^t x^*[f(s) - f_n(s)]d\lambda\right| = \left|\int_0^t \left(\sum_{k=n+1}^{\infty} \alpha_k k\chi_{(0,\frac{1}{k}]}\right) d\lambda\right| \le \sum_{k=n+1}^{\infty} |\alpha_k|.$$

Consequently $\lim_n \|x^*f - x^*f_n\|_A = 0$. Now we prove that the function f is not HKP-integrable. Indeed, if $(e_n)_n$ is the canonical base in l_1, then $\int_0^1 \langle e_n, f\rangle d\lambda = \int_0^{\frac{1}{n}} n d\lambda = 1$. So $\int_0^1 f d\lambda = (1, 1, \ldots, 1, \ldots) \notin c_0$.

THEOREM 3. *A scalarly DK-integrable function $f : [0, 1] \to X$ is DKP-integrable if and only if the topology of convergence in measure and the weak topology of $DK[0, 1]$ coincide on the set $\{x^*f : \|x^*\| \le 1\}$ and there exists a sequence of step functions $f_n : [0, 1] \to X$ such that*

(i) $\lim_n \|x^*f - x^*f_n\|_A = 0$, *for every* $x^* \in X^*$.
The condition (i) may be replaced by the condition
(ii) $x^*f_n \to x^*f$ *weakly in* $DK[0, 1]$, *for every* $x^* \in X^*$.

Proof. The proof follows as in Theorem 2 but this time we have to apply Theorem 3 and [**5**, Remark 4] and the Denjoy–Khintchine version of Lemma 1 in [**5**]. □

QUESTION 1. Let $f : [0, 1] \to X$ be DKP-integrable. Does there exist a sequence $(f_n)_n$ of step functions such that for each $x^* \in X^*$ the equality $\lim_n x^*f_n = x^*f$ a.e. holds true?

Ene [**8**] proved that the answer is positive in case of real-valued functions, but the provided sequence is not of the form (1).

We recall that a family $\mathcal{A} \subset H([0, 1], X)$ is *Henstock equi-integrable* (*Henstock-Kurzweil* equi-integrable in case of $X = \mathbb{R}$) (or simply *H-equi-integrable*) on [0, 1] if for every $\epsilon > 0$ there exists a gauge δ on [0, 1] such that

$$\sup_{f\in\mathcal{A}} \left\| \sum_{i=1}^{p} f(t_i)|I_i| - (H)\int_0^1 f d\lambda \right\| < \varepsilon ,$$

for each δ-fine Perron partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of [0, 1].

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REMARK 1. In general, as the following example shows, if a Henstock integrable function f is the limit in the norm topology of $H([0, 1], X)$ of a sequence $(f_n)_n$ of step functions, then it may happen that the sequence $(f_n)_n$ does not converge a.e. to f or that the functions of the sequence are not Henstock equi-integrable.

Let $\alpha > 3$ be arbitrary and let C_α be the Cantor set in [0, 1] with its measure equal to $\frac{\alpha-3}{\alpha-2}$. Denote by $\rho_k^n = (a_k^n, b_k^n)$, $k = 1, \ldots, 2^{n-1}$, $n = 1, 2, \ldots$, the contiguous intervals of the set C_α. Then $|\rho_k^n| = \alpha^{-n}$.

Define for every $n \in \mathbb{N}$

$$f_n(t) = \begin{cases} -2^{1-n}\alpha^i, & \text{if } t \in \overline{\rho_s^i},\ i = 1, \ldots, n,\ s = 1, \ldots, 2^{i-1}, \\ 2\left(1 - \frac{1}{\alpha}\sum_{p=0}^{n-1}(\frac{2}{\alpha})^p\right)^{-1}, & \text{if } t \notin \cup_{i=1}^{n} \cup_{s=1}^{2^{i-1}} \overline{\rho_s^i}\ ; \end{cases}$$

$\overline{\rho_s^i}$ being the closure of ρ_s^i. It can be easily seen that $(f_n(t))$ converges to $2(\alpha-2)(\alpha-3)^{-1}$ on C_α and to zero on the complement of C_α.

We denote the limit function by f. In [**2**] (see also [**5**]) it is proved that the sequence (f_n) is weakly convergent to zero in $HK[0, 1]$. Applying the Mazur Theorem we get the existence of a sequence (h_n) of convex combinations of the step functions f_n such that (h_n) converges to zero in the Alexiewicz norm and to the function f a.e. The sequence (h_n) cannot be Henstock equi-integrable. Indeed, if (h_n) were Henstock equi-integrable, then (h_n) would converge to f in the Alexiewicz norm (see [**18**]) and so also weakly in $HK[0, 1]$. □

4. Integrals with norm relatively compact range. We begin with an easy lemma (cf. [**15**, Lemma 9.3.]).

LEMMA 1. *Let Pi be a directed set of partitions such that* $\lim_\pi \operatorname{mesh}(\pi) = 0$. *Moreover, let Y be a normed space and $U_\pi : Y \to Y$ be a bounded linear operator, for each $\pi \in \Pi$. If $\sup_\pi \|U_\pi\| < \infty$ and $\lim_\pi U_\pi(y) = y$ for every $y \in Y$, then the convergence is uniform on each relatively compact subset of Y.*

LEMMA 2. *Let Π be a directed set of partitions such that* $\lim_\pi \operatorname{mesh}(\pi) = 0$ *and let $U_\pi : DK[0, 1] \to DK[0, 1]$ (resp. $U_\pi : HK[0, 1] \to HK[0, 1]$) be given by $U_\pi(g) = \sum_{I\in\pi} \frac{G(I)}{|I|}\chi_I$, where G is the primitive of g. Then $\|U_\pi(g)\|_A \le \|g\|_A$.*

Proof. Since G is continuous, given $\varepsilon > 0$ there is $\delta > 0$ such that $G(I) < \varepsilon$ for $|I| < \delta$. If we take a partition with intervals of length less than δ, then we have, for each $0 < t \le 1$,

$$\begin{aligned} \left|\int_0^t \sum_{I\in\pi} \frac{G(I)}{|I|}\chi_I \, d\lambda\right| &= \left|\sum_{I\subset[0,t]} G(I) + G(J)\frac{|J\cap[0,t]|}{|J|}\right| \\ &\le \left|\sum_{I\subset[0,t]} G(I) + G(J\cap[0,t])\right| + \left|-G(J\cap[0,t]) + G(J)\frac{|J\cap[0,t]|}{|J|}\right| \\ &< \left|\int_0^t g\, d\lambda\right| + 2\varepsilon\,, \end{aligned}$$

where J is the only interval containing t with $J \cap (t, 1] \neq \emptyset$. Consequently, we have $\|U_\pi(g)\|_A \le \|g\|_A$. □

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PROPOSITION 2. *Let $F : [0, 1] \to X$ be a weakly continuous function such that x^*F is, for each x^*, almost everywhere approximately differentiable (resp. almost everywhere differentiable) and $(x^*F)'_{ap} \in DK[0, 1]$ (resp. $(x^*F)' \in HK[0, 1]$). If the set $W_F = \{(x^*F)'_{ap} : \|x^*\| \le 1\}$ (resp. $W_F = \{(x^*F)' : \|x^*\| \le 1\}$) is a norm relatively compact subset of $DK[0, 1]$ (resp. of $HK[0, 1]$), then for each $\varepsilon > 0$ there exists an X-valued step function $h : [0, 1] \to X$ such that*

$$\sup_{I\in\mathcal{I}} \left\| F(I) - \int_I h\, d\lambda \right\| \le \varepsilon.$$

Proof. We will present the proof only in case of Denjoy–Khintchine integrability. The HK case can be obtained in a similar way. Let Π be a directed set of partitions of $[0, 1]$ such that $\lim_\pi \operatorname{mesh}(\pi) = 0$. For each $\pi \in \Pi$ we define $U_\pi : DK[0, 1] \to DK[0, 1]$ by $U_\pi(g) := G_\pi = \sum_{I\in\pi} \frac{G(I)}{|I|} \chi_I$, where G is the primitive of g.

Applying Proposition 1 and Lemma 2 we infer $\lim_\pi U_\pi(g) = g$ for every $g \in DK[0, 1]$, and $\|U_\pi\|_A \le 1$. Thus, for each x^* we have

$$\lim_\pi \|U_\pi[(x^*F)'_{ap}] - (x^*F)'_{ap}\|_A = 0\,,$$

uniformly on W_F. Let us fix $\varepsilon > 0$ and π_0 such that

$$\|(x^*F)'_{ap} - U_\pi[(x^*F)'_{ap}]\|_A < \varepsilon/2,$$

for every $\pi \ge \pi_0$ and every $x^* \in B(X^*)$.

So we have, for each $0 < t \le 1$ and each $x^* \in B(X^*)$,

$$\begin{aligned}&\left| x^*F[0, t] - (DK)\int_0^t x^*F_{\pi_0}\, d\lambda \right| \\ &\quad = \left| (DK)\int_0^t (x^*F)'_{ap}\, d\lambda - (DK)\int_0^t U_{\pi_0}[(x^*F)'_{ap}]\, d\lambda \right| < \varepsilon/2.\end{aligned}$$

In particular, we have

$$\|F(I) - (DKP)\int_I F_{\pi_0} d\lambda\| < \varepsilon, \qquad \text{for every} \quad I \in \mathcal{I}.$$

Then it is enough to take $h = F_{\pi_0}$. □

LEMMA 3. *Let $f : [0, 1] \to X$ be an HKP-integrable function (resp. DKP-integrable) and let $F : \mathcal{I} \to X$ be its primitive. Then $F(\mathcal{I})$ is norm relatively compact if and only if the set $W_F = \{x^*f : \|x^*\| \le 1\}$ is a norm (relatively) compact subset of $HK[0, 1]$ (resp. $DK[0, 1]$).*

Proof. Let $T : X^* \to HK[0, 1]$ be defined by $T(x^*) = x^*f$.Then $T^* : BV[0, 1] \to X^{**}$ is such that

$$\langle x^*, T^*\chi_I\rangle = \langle T(x^*), \chi_I\rangle = (HK)\int_I x^*f\, d\lambda = \langle x^*, (HKP)\int_I f\, d\lambda\rangle = \langle x^*, F(I)\rangle.$$

It follows that $T^*\chi_I = F(I)$. But, according to the Schauder Theorem, T is compact if and only if T^* is compact and so the required equivalence follows. □

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COROLLARY 1. *If $f : [0,1] \to X$ is a DKP-integrable function and $F(\mathcal{I})$ is norm relatively compact, then the weak topology $\sigma(HK, BV)$, the norm topology and the topology of convergence in measure coincide on W_F.*

THEOREM 4. *Let $f : [0,1] \to X$ be an HKP-integrable function (resp. Denjoy–Khintchine–Pettis-integrable) and let $F : \mathcal{I} \to X$ be its primitive. Then $F(\mathcal{I})$ is norm relatively compact if and only if there exists a sequence of step functions $f_n : [0,1] \to X$ such that* $\lim_n \|f - f_n\|_A = 0$.

Proof. If the range of F is norm relatively compact, then the approximation is a consequence of Lemma 3 and Proposition 2.

Now assume that a sequence $(f_n)_n$ of step functions is convergent to f in the Alexiewicz norm. That is, given $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\left\| \int_I f(t)\,dt - \int_I f_k(t)\,d(t) \right\| < \varepsilon\,, \quad \text{for every } I \in \mathcal{I}\,. \tag{2}$$

If F_k is the primitive of f_k, then $F_k(\mathcal{I})$ is a relatively compact subset of a finitely dimensional subspace of X. Together with (2) this fact yields the relative compactness of $F(\mathcal{I})$ in X. □

QUESTION 2. Can the sequence of step functions in Theorem 4 be chosen equi-integrable in case of Henstock integrability?

We are going to present now an example of a HKP-integrable function without norme relatively compact range of its integral.

EXAMPLE 2. We will use the function $f : [0,1] \to c_0$ presented in **[10]**.

It is constructed as follows. Consider a sequence of intervals $J_n = [a_n, b_n] \subseteq [0,1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $f : [0,1] \to c_0$ by

$$f(t) = \left(\frac{1}{2|J_{2n-1}|}\chi_{J_{2n-1}}(t) - \frac{1}{2|J_{2n}|}\chi_{J_{2n}}(t) \right)_{n=1}^{\infty}.$$

One could apply Theorem 2 to prove the HKP-integrability of f but the direct proof is much simpler. It can be easily seen that

$$F(I) = \left(\frac{|I \cap J_{2n-1}|}{2|J_{2n-1}|} - \frac{|I \cap J_{2n}|}{2|J_{2n}|} \right)_{n=1}^{\infty} \in c_0$$

is the HKP-integral of f on $I \in \mathcal{I}$. It is obvious that the set $\{F(I) : I \in \mathcal{I}\}$ is not norm relatively compact because the set $\{F(J_{2n-1}) : n \in \mathbb{N}\}$ is discrete.

QUESTION 3. Assume that c_0 cannot be embedded isomorphically into X, can the range of each *DKP* (resp. *HKP*) integral be a norm relatively compact set?

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