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A FURTHER CHARACTERIZATION OF A PROJECTIVE SPECIAL LINEAR GROUP

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Abstract

In this paper we present a characterization of PSL(2, 7) by a condition different from that given in our previous paper.

1. Introduction

In the first place we shall fix our notation. Let G be a finite group and $\pi(G)$ the set of primes each dividing the order of G. Then we denote by $\tau(G)$,

$$\tau(G) = \{p \in \pi(G) \mid [G:M] \text{ is a power of } p, \text{ where } M \text{ is a subgroup of } G\}$$

and by $\psi(G)$, the set of pairs

$$\psi(G) = \{(M,p) \mid p \in \tau(G), p \mid [G:M] \text{ for a maximal subgroup } M \text{ of } G\}.$$

We shall be using the following hypothesis:

(*) (i) *G is simple*

(ii) *Every maximal subgroup of G has index a power of a prime.*

In Adnan (1976), we have been able to show that if the group G satisfies hypothesis (*) together with the condition

(A) *For every $(M,p) \in \psi$, M is q-soluble for some $q \in \tau - \{p\}$,*

then $G \simeq \mathrm{PSL}(2,7)$.

In this paper we replace condition (A) by another condition which is easier to work with, and show that in presence of hypothesis (*), our new condition implies condition (A) and as such $G \simeq \mathrm{PSL}(2,7)$.

Next we proceed to state our main theorem.

MAIN THEOREM. *Let G be a finite group satisfying* (*). *Then $G \simeq PSL(2,7)$ if the following condition (B) holds.*

(B) *There are primes $p, q \in \tau$, $p \neq q$, such that for every $(M, p), (N, q) \in \psi$, $M \cap N$ is disjoint from its conjugates.*

2. Preparatory lemmas

LEMMA 1. (Non-simplicity condition). *Let G be a finite group and H and K be subgroups of G with*

$$[G:H] = p^i \qquad [G:K] = q^j$$

p, q being distinct primes of $\pi(G)$. If $H \cap K$ contains a unique involution x, and if $C_G(x) \subseteq N_G\ (H \cap K)$, then G is not simple.

PROOF. Set $Y = H \cap K$. If $q \neq 2 \neq p$, then Y contains an S_2-subgroup T of G. Since Y contains a unique involution, it follows that T is either cyclic or generalized quaternion and so G is not simple by p. 373 of [2]. Thus we may assume that $q = 2$.

Let $B = O_2(Y)$. Then $B \neq 1$ by hypothesis. Moreover, if S is an S_2 subgroup of H, then $H = YS$. Since $Y \cap S$ is an S_2-subgroup of Y, we have $B \subseteq S$. We conclude that $O_2(H) \neq 1$.

Now let t be an involution such that $t \in Z \cap O_2(H)$, where $Z = Z(S)$. Since $Z \subseteq C_G(x) \subseteq N_G(Y)$, t normalizes Y. If r is an odd prime dividing $|Y|$ and R is an S_r-subgroup of Y, then $[R, t] \subseteq [Y, t] \subseteq Y \cap O_2(H)$. Let z be any non-identity element of R. Then $[z, t] \in O_2(H)$ and since $t \in Z$, t centralises $O_2(H)$, and hence t centralises t^z. Hence $[z, t] = t^z t$ is an involution in Y or 1. By hypothesis $[z, t] \in \langle x \rangle$. Similarly $[z^2, t] \in \langle x \rangle$ and we conclude that $[z, t] = 1$. Therefore t centralises R and thus Y. Since $t \in Z$, we have $Y, S \subseteq C_G(t)$ i.e. $H \subseteq C_G(t)$. Thus $[G: C_G(t)]$ is a power of p and so G is not simple by p. 131 of [2].

LEMMA 2 (Solubility Criterion). *If G is a finite group expressible in the form $G = HQ$ where H is a regular group of automorphisms of some p-group, having no quaternion subgroups and Q is a q-subgroup of G for some prime $q \in \pi(G)$, then G is soluble.*

PROOF. Since H contains no quaternion subgroups, we deduce that H is metacyclic (by [2] p. 258). If $q \in \pi(H)$, then we write $G = KQ$, where K is a Hall q'-subgroup of H and Q, a Sylow q-subgroup of G. Thus we may assume without loss of generality that H is a q'-subgroup of G.

We proceed now by induction on $|G|$. We first show that if N is a non-trivial normal subgroup of G, then N is soluble. If $Q \subseteq N$ then

$N = (N \cap H)Q$ and by induction N is soluble. If $Q \not\subseteq N$ then $L = HN$ is a proper subgroup of G. Also $Q \cap N$ is a Sylow q-subgroup of N and hence of L, and so $L = H(Q \cap N)$. Again by induction L and hence N is soluble. In particular if $G' \subset G$, then G' and hence G is soluble. We may therefore assume that G is perfect. If $O_q(G) \neq 1$, then clearly $\bar{G} = G/O_q(G)$ satisfies the hypothesis of the lemma and so $\bar{G}$ and hence G is soluble. On the other hand if $O_{q'}(G) \neq 1$, then let r be the largest prime dividing $O_{q'}(G)$ and D be an S_r-subgroup of $O_{q'}(G)$. Since $O_{q'}(G) \subseteq H$, $O_{q'}(G)$ is metacyclic and so $D \trianglelefteq O_{q'}(G)$ i.e. $D \triangleleft G$. Thus $G/C_G(D)$ embeds into $\mathrm{Aut}(D)$. Since D is cyclic and G is perfect, we have $G = C_G(D)$. Set $\bar{G} = G/D$. Then $1 \neq D \subseteq Z(G) \cap G'$ and so the Schur multiplier of $\bar{G}$ (see [3], p. 628) is non-trivial. However by Satz ([3], p. 642) the r-part of the Schur multiplier of $\bar{G}$ embeds into the Schur multiplier of R, R being an S_r-subgroup of $\bar{G}$. Since R is cyclic, it follows by Satz in [3] p. 643 that the Schur multiplier of R is 1, a contradiction. Therefore we may assume $O_q(G) = 1 = O_{q'}(G)$ i.e. $F(G) = 1$. Now let $x \in Z(H)$ ($Z(H) \neq 1$ by Satz in [3] p. 506). Then $[G: C_G(x)]$ is a power of q and hence G is not simple by a theorem of Burnside (cf. [2] p. 131). Let N be a normal (non-trivial) subgroup of G. Since N is soluble, we have $1 \neq F(N) \subseteq F(G)$, the last contradiction.

REMARK. In the proof of lemma 2 above, one can use theorem 4.4 (ii) of [2], p. 253 instead of the Schur multiplier and argue that G is not perfect.

LEMMA 3. *Let G be a finite group, then*

(i) *$G = MN$ for two subgroups M and N implies $N_G(M \cap N)$ is factorisable.*

(ii) *If G satisfies (*), and if (M, p) and $(N, q) \in \psi(G)$, for p and q distinct primes in $\tau(G)$, then $G = MN$. If M is Frobenius group then $N_G(M \cap N) \subseteq N$. Conversely if $M \cap N$ is disjoint from its conjugates and if $N_G(M \cap N) \subseteq N$, then M is Frobenius.*

(iii) *Let G satisfy (*), and let (M, p), $(N, q) \in \psi(G)$ for distinct primes p and q in $\tau(G)$ such that $M \cap N$ is disjoint from its conjugates. If M is Frobenius then $M \cap N$ is a Frobenius complement for M and has odd order.*

PROOF. (i) Let us write $H = M \cap N$. If $g \in N_G(H)$, then $g = mn$, for some $m \in M$ and $n \in N$. So

$$H^g = H^{mn} = H.$$

Thus

$$H^m = H^{n^{-1}} \subseteq M \cap N = H \quad \text{i.e.} \quad m, n \in N_G(H).$$

(ii) By (i) we may write $N_G(H) = (M \cap N_G(H))(N \cap N_G(H))$. Since M s Frobenius, it follows by Thompson's theorem (cf. [2] p. 337) that $F(M) \neq 1$. 3y lemma 4 in [1] it follows that $F(M)$ is an S_p-subgroup of G. Since $[M: H]$ s a power of p, we have $M \cap N_G(H) = H$. Thus $N_G(H) = N \cap N_G(H)$ is :ontained in N. Conversely if $N_G(H) \subseteq N$, then as $M \cap N = H$, we conclude $N \cap N_G(H) = H$. Further H is disjoint from its conjugates and so M is Frobenius.

(iii) To show H is a complement for M, we notice that by our hypothesis H is disjoint from its conjugates. Moreover since $[M: H]$ is a power of p and M is Frobenius with kernel an S_p-subgroup of G (cf. [1], lemma 4), we have $M \cap N_G(H) = H$ i.e. H is a complement for M.

To show H has odd order, we notice that if $|H|$ is even, then since H is a complement for M, by the above, H contains a unique involution, say x. By hypothesis, H is disjoint from its conjugates, and so $C_G(x) \subseteq N_G(H)$. By lemma 1, however G could not be simple, thus leading to a contradiction.

To establish our main theorem we proceed to prove in two parts.

(A) *Let* $\mathcal{I} = \{(M, N) | (M, q), (N, p) \in \psi$, *M and N being non-Frobenius*$\}$; *then we establish first that* $\mathcal{I}$ *is empty.*

Proof. If $\mathcal{I}$ is not empty, choose $(M, N) \in \mathcal{I}$, with (M, q), $(N, p) \in \psi$, such that $M \cap N = H$ has maximal order. Let L be a maximal subgroup of G containing $N_G(H)$. Since $[G: H]$ is divisible only by p or q, L has index a power of p or q. We may therefore assume that $(L, q) \in \psi$. Suppose now by way of contradiction that L is Frobenius. Then $H \subseteq N \cap L = K$ where K is a complement of L by part (iii), Lemma 3. Now let R be an S_r-subgroup of G and also contained in H, for $r \in \pi(G)$. Then $\Omega_1(R)$ char H, $\Omega_1(R)$ char K.

Since H and K each is disjoint from its conjugates, we have

$$N_G(H) = N_G(\Omega_1(R)) = N_G(K).$$

By lemma 3 (ii), $N_G(H) = N_G(K) \subseteq N$. By the same lemma 3 (ii) again, we conclude that M is Frobenius contrary to the fact that $(M, N) \in \mathcal{I}$.

On the other hand, if L is non-Frobenius, with $(L, q) \in \psi$, then $(N, L) \in \mathcal{I}$. Since $H \subseteq N \cap L$, maximality of H forces $H = N \cap L$. Since $N_G(H) \subseteq L$, it follows by Lemma 3 (ii) that N is Frobenius — a contradiction. Thus $\mathcal{I}$ is empty.

(B). By (A) using Thompson's theorem ([2] p. 337) we obtain that a maximal subgroup of G has in fact a nontrivial Fitting subgroup. By [1] lemma 4 $\tau = \{p, q\}$. Next let (U, q), $(V, p) \in \psi$. By part (A) above we may assume that U is a Frobenius group. By lemma 3 (ii) $U \cap V$ is a complement of U and has odd order and so U is soluble. Since $V = (U \cap V)Q$, where Q

is a Sylow q-subgroup of G, it follows by lemma 2 that V is soluble also. Thus by the main theorem in [1], we conclude that

$$G \simeq PSL(2, 7).$$

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