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Abstract

On a Fano manifold M we study the supremum of the possible t such that there is a Kähler metric $\omega \in c_1(M)$ with Ricci curvature bounded below by t. This is shown to be the same as the maximum existence time of Aubin's continuity path for finding Kähler–Einstein metrics. We show that on $\mathbf{P}^2$ blown up in one point this supremum is 6/7, and we give upper bounds for other manifolds.

1. Introduction

The problem of finding Kähler–Einstein metrics is a fundamental one in Kähler geometry. After the works of Yau [Yau78] and Aubin [Aub78] what remained is settling the existence question for Fano manifolds. Yau [Yau93] conjectured that in this case the existence is related to stability of the manifold in the sense of geometric invariant theory. Important progress was made by Tian [Tia97], who introduced the notion of K-stability. This was extended by Donaldson to the study of more general constant scalar curvature Kähler metrics (see e.g. [Don01, Don09]). The conjecture relating K-stability to the existence of constant scalar curvature Kähler metrics, now called the Yau–Tian–Donaldson conjecture, is currently a very active field of research. For a survey and many more references, see Phong and Sturm [PS09].

In this paper we study Aubin's [Aub84] continuity method for finding Kähler–Einstein metrics. Given a Kähler metric $\omega \in c_1(M)$, this approach is to find ω_t solving

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega$$

for all $t \in [0, 1]$. For $t = 0$ a solution exists by Yau's theorem. This continuity path has nice properties, an important one being that the Mabuchi energy [Mab86] is monotonically decreasing along the path. This was exploited in [BM87] to show the lower boundedness of the Mabuchi energy. It is also crucial for finding *a priori* estimates using properness of the Mabuchi functional (see [Tia97]).

We are interested in the situation when we cannot solve up to $t = 1$. Clearly understanding this is crucial in the study of obstructions to the existence of Kähler–Einstein metrics. A natural question is what the supremum of the t is for which we can solve the equation. We first show that this is independent of the choice of ω, and is equal to the invariant $R(M)$ that we define by

$$R(M) = \sup_{t\in[0,1]} \{\exists \omega \in c_1(M) \text{ such that } \omega > 0 \text{ and } \mathrm{Ric}(\omega) > t\omega\}.$$

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The proof goes via relating the existence of a solution to properness of a certain functional. For the case $t=1$ this has been done in [Tia97], with a stronger version of properness shown in Phong *et al.* [PSSW08].

The problem then becomes to determine $R(M)$ for manifolds which do not admit Kähler–Einstein metrics. Tian [Tia92] considered the problem of bounding $R(M)$ and obtained the upper bound $R(M_1) \leqslant 15/16$ where M_1 is $\mathbf{P}^2$ blown up in one point. We show that in fact $R(M_1) = 6/7$. More generally we give an upper bound for any Fano manifold which has non-trivial vector fields and non-vanishing Futaki invariant. In § 4 we show that if $R(M)=1$ then the manifold is K-semistable with respect to test-configurations with smooth total space.

2. The definition of $R(M)$

Let M be a Fano manifold, so $c_1(M)>0$. Let us fix a base metric $\eta \in c_1(M)$, and consider a family of metrics

$$\omega_t = \eta + i\partial\overline{\partial}\phi_t.$$

The Mabuchi functional [Mab86] is defined by its variation

$$\frac{d}{dt}\mathcal{M}(\omega_t) = \int_M \dot{\phi}_t(n - S(\omega_t))\omega_t^n,$$

normalised so that $\mathcal{M}(\eta)=0$. Here $S(\omega_t)$ is the scalar curvature. For any Kähler metric $\alpha \in c_1(M)$ we also define the functional $\mathcal{J}_\alpha$ by its variation

$$\frac{d}{dt}\mathcal{J}_\alpha(\omega_t) = \int_M \dot{\phi}_t(\Lambda_{\omega_t}\alpha - n)\omega_t^n,$$

normalised so that $\mathcal{J}_\alpha(\eta)=0$. Here Λ_{ω_t} means the trace with respect to ω_t. The functional $\mathcal{J}_\alpha$ is essentially the same as $I-J$ in terms of Aubin's I, J functionals (see [BM87]). When α is not necessarily in the same Kähler class as ω, it was introduced in [Che00] to study the Mabuchi energy on manifolds with $c_1<0$. See also [SW08, Wei04].

Given any $\omega \in c_1(M)$, Aubin's continuity path for finding Kähler–Einstein metrics is given by

$$\omega_t^n = e^{h_\omega - t\phi_t}\omega^n, \tag{1}$$

where h_ω is the Ricci potential, defined by

$$\mathrm{Ric}(\omega) - \omega = i\partial\overline{\partial}h_\omega,$$

and normalised so that $\int_M e^{h_\omega}\omega^n = \int_M \omega^n$. Equivalently we have $\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega$. For $t=0$ this can be solved by Yau's theorem [Yau78].

Finally we call a functional $\mathcal{F}$ defined on the space of Kähler metrics in $c_1(M)$ *proper* if there exist constants $\epsilon, C>0$ such that

$$\mathcal{F}(\omega) > \epsilon\mathcal{J}_\eta(\omega) - C$$

for all $\omega \in c_1(M)$. Since $\mathcal{J}_\eta$ is the same as the functional $I-J$ in the literature, this notion of properness coincides with the one used in [Tia97].

Theorem 1. *The following are equivalent for* $0 \leqslant t < 1$.

- *We can solve equation (1).*
- *There exists a metric* $\omega \in c_1(M)$ *such that* $\mathrm{Ric}(\omega) > t\omega$.
- *The functional* $\mathcal{M} + (1-t)\mathcal{J}_\omega$ *is proper for any* $\omega \in c_1(M)$.

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In particular we can introduce an invariant $R(M)$ to be the supremum of the possible $t<1$ for which the above statements hold.

The proof of the theorem follows from Lemmas 2, 4 and 5. The statement of the theorem for $t=1$ (the second statement replaced with $\mathrm{Ric}(\omega)=\omega$) follows from the works of Tian [Tia97] and Phong *et al.* [PSSW08]. Note that by the following lemma all the $\mathcal{J}_\omega$ for different ω are equivalent, so by definition they are all proper. The invariant $R(M)$ measures what the smallest multiple of $\mathcal{J}_\omega$ is that we need to add to $\mathcal{M}$ to make it proper.

LEMMA 2. *If α, α' are in the same Kähler class then for all $\omega \in c_1(M)$, we have*

$$|(\mathcal{J}_\alpha - \mathcal{J}_{\alpha'})(\omega)| < C$$

for some constant C independent of ω.

Proof. Let us write $\alpha = \alpha' + i\partial\overline{\partial}\psi$, and $\omega = \eta + i\partial\overline{\partial}\phi$. Writing $\omega_t = \eta + ti\partial\overline{\partial}\phi$ we have

$$\begin{aligned}\frac{d}{dt}(\mathcal{J}_\alpha - \mathcal{J}_{\alpha'})(\omega_t) &= \int_M \phi\Lambda_{\omega_t}(\alpha-\alpha')\omega_t^n \\ &= n\int_M \phi(i\partial\overline{\partial}\psi)\wedge\omega_t^{n-1} \\ &= n\int_M \psi(i\partial\overline{\partial}\phi)\wedge\omega_t^{n-1} \\ &= \frac{d}{dt}\int_M \psi\omega_t^n.\end{aligned}$$

It follows then that

$$(\mathcal{J}_\alpha - \mathcal{J}_{\alpha'})(\omega) = \int_M \psi(\omega^n - \eta^n),$$

which is uniformly bounded in terms of $\sup|\psi|$. □

The following proposition, which follows directly from the work of Chen and Tian [CT08] is the key technical result.

PROPOSITION 3. *If ω satisfies the equation*

$$\mathrm{Ric}(\omega) = s\omega + (1-s)\alpha, \tag{2}$$

where $\alpha \in c_1(M)$ is positive, then the functional $\mathcal{M} + (1-s)\mathcal{J}_\alpha$ is bounded below.

Proof. First note that ω satisfying equation (2) is a critical point of the functional $\mathcal{M} + (1-s)\mathcal{J}_\alpha$. This follows directly from the variational formula

$$\frac{d}{dt}[\mathcal{M}(\omega_t) + (1-s)\mathcal{J}_\alpha(\omega_t)] = \int_M \dot{\phi}_t[sn + (1-s)\Lambda_{\omega_t}\alpha - S(\omega_t)]\omega_t^n$$

and taking the trace of equation (2). The result now follows formally from the convexity of the functional $\mathcal{M} + (1-s)\mathcal{J}_\alpha$ along geodesics in the space of Kähler potentials, studied by Mabuchi [Mab87], Semmes [Sem92] and Donaldson [Don99]. Indeed with some computation we obtain

$$\begin{aligned}\frac{d^2}{dt^2}[\mathcal{M}(\omega_t) + (1-s)\mathcal{J}_\alpha(\omega_t)] = &\int_M (\ddot{\phi}_t - |\overline{\partial}\dot{\phi}_t|^2)[sn + (1-s)\Lambda_{\omega_t}\alpha - S(\omega_t)]\omega_t^n \\ &+ \int_M [|\mathcal{D}\dot{\phi}_t|^2 + (1-s)|\overline{\partial}\dot{\phi}_t|^2_\alpha]\omega_t^n\end{aligned}$$

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where $\mathcal{D}$ is the Lichnerowicz operator and $|\overline{\partial}\dot{\phi}|^2_\alpha = g^{j\bar{q}}\alpha_{p\bar{q}}g^{p\bar{k}}\partial_j\dot{\phi}\partial_{\bar{k}}\dot{\phi}$ in local coordinates, with g being the metric corresponding to ω_t. Now the path ω_t is a geodesic if $\ddot{\phi}_t = |\overline{\partial}\dot{\phi}_t|^2$ so the second derivative of $\mathcal{M} + (1-s)\mathcal{J}_\alpha$ is non-negative along geodesics. If ω is a critical point, and any other metric can be joined to ω using such a geodesic, then the lower boundedness follows from convexity. Unfortunately it is still an open problem whether any two metrics can be joined using geodesics, but we can instead rely on the theory of Chen and Tian [CT08] as follows.

By Yau's theorem [Yau78] we can find a metric $\omega_0 \in c_1(M)$ such that $\mathrm{Ric}(\omega_0) = \alpha$. By the same computation as in Chen [Che00], we have

$$\mathcal{M}(\omega) + (1-t)\mathcal{J}_\alpha(\omega) = D + \int_M \log\frac{\omega^n}{\omega_0^n}\omega^n - t\mathcal{J}_\alpha(\omega),$$

for some constant D. As in Chen–Tian, Theorem 6.1.1. this functional is weakly sub-harmonic on almost smooth solutions of the geodesic equation in the space of Kähler metrics. Then the argument in Theorem 6.2.1. implies that the functional is bounded below on the space of metrics in the first Chern class. □

LEMMA 4. *If there exists a metric ω with $\mathrm{Ric}(\omega) > t\omega$, then the functional $\mathcal{M} + (1-t)\mathcal{J}_\eta$ is proper.*

Proof. Let us write

$$\mathrm{Ric}(\omega) = t\omega + (1-t)\alpha,$$

where α is a positive form in $c_1(M)$. It follows from the previous proposition that the functional $\mathcal{M} + (1-t)\mathcal{J}_\alpha$ is bounded from below. By Lemma 2 it follows that $\mathcal{M} + (1-t)\mathcal{J}_\eta$ is also bounded from below.

In order to show that it is proper, we use a perturbation argument. We want to show that for sufficiently small $\epsilon > 0$ we can find ω' such that

$$\mathrm{Ric}(\omega') - (t+\epsilon)\omega' = (1-t-\epsilon)\alpha.$$

This is just the openness statement in Aubin's continuity method [Aub84]. Then the previous argument implies that $\mathcal{M} + (1-t-\epsilon)\mathcal{J}_\eta$ is bounded below, so

$$\mathcal{M} + (1-t)\mathcal{J}_\eta = \epsilon\mathcal{J}_\eta + (\mathcal{M} + (1-t-\epsilon)\mathcal{J}_\eta) > \epsilon\mathcal{J}_\eta - C,$$

which is what we wanted to prove. □

LEMMA 5. *If the functional $\mathcal{M} + (1-s)\mathcal{J}_\eta$ is proper, then for any metric $\omega \in c_1(M)$ we can find an ω_s such that*

$$\mathrm{Ric}(\omega_s) = s\omega_s + (1-s)\omega,$$

that is, we can solve along the continuity method up to time s.

Proof. This is a slight extension of a result in [Tia97] (see also [BM87]). Using Yau's estimates [Yau78] we only need to show that if the path of metrics $\omega_t = \omega + i\partial\overline{\partial}\phi_t$ satisfies

$$\omega_t^n = e^{h_\omega - t\phi_t}\omega^n \tag{3}$$

for $t < s$, then there is a uniform C^0 bound $\sup|\phi_t| < C$. For this we compute the derivative

$$\frac{d}{dt}[\mathcal{M}(\omega_t) + (1-s)\mathcal{J}_\omega(\omega_t)].$$

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Differentiating equation (3) we get

$$\Delta_t \dot{\phi}_t = -\phi_t - t\dot{\phi}_t,$$

where Δ_t is the Laplace operator of the metric ω_t. Using the formula

$$S(\omega_t) = tn + (1-t)\Lambda_{\omega_t}\omega,$$

we can compute

$$\begin{aligned}
\frac{d}{dt}[\mathcal{M}(\omega_t) + (1-s)\mathcal{J}_\omega(\omega_t)] &= \int_M \dot{\phi}_t[-(1-t)\Lambda_{\omega_t}\omega + (1-s)\Lambda_{\omega_t}\omega - (s-t)n]\omega_t^n \\
&= (s-t)\int_M \dot{\phi}_t \Lambda_{\omega_t}(\omega_t - \omega)\omega_t^n \\
&= (s-t)\int_M \dot{\phi}_t \Delta_t \phi_t \omega_t^n \\
&= (s-t)\int_M (-\phi_t - t\dot{\phi}_t)\phi_t\omega_t^n \\
&= (s-t)\bigg[-\int_M \phi_t^2\omega_t^n + t\int_M \dot{\phi}_t(\Delta_t\dot{\phi}_t + t\dot{\phi}_t)\omega_t^n\bigg] \\
&\leqslant 0,
\end{aligned}$$

as long as $t < s$. Here we have used that $\Delta_t + t$ is a negative operator since $\mathrm{Ric}(\omega_t) \geqslant t\omega_t$.

Since $\mathcal{M} + (1-s)\mathcal{J}_\omega$ is proper (again using Lemma 2 to relate the different $\mathcal{J}$ functionals), we obtain a uniform bound

$$\mathcal{J}_\omega(\omega_t) < C$$

for $t < s$. As in [BM87] this gives the required C^0 estimate. □

3. Bounding the invariant $R(M)$

It is an interesting problem to find bounds on $R(M)$ for a given Fano manifold M. First let us briefly discuss lower bounds. Clearly when M admits a Kähler–Einstein metric then $R(M) = 1$. The converse however is not true. For instance the unstable deformations of the Mukai 3-fold given by Tian [Tia97] have $R(M) = 1$. To see this first recall that the Mukai 3-fold M_0 admits a Kähler–Einstein metric (see Donaldson [Don]), so for any $t < 1$ there is a metric ω_0 on M_0 with

$$\mathrm{Ric}(\omega_0) > t\omega_0.$$

Tian's example is a manifold M such that M_0 has arbitrarily small deformations which are biholomorphic to M (there exists a degeneration of M to M_0). With such small deformations we can obtain a metric ω on M such that $\mathrm{Ric}(\omega) > t\omega$ still holds. Since we can do this for any $t < 1$, this implies that $R(M) = 1$. Alternatively, it is well-known that $R(M) = 1$ if the Mabuchi energy is bounded from below (see [BM87]), and Chen [Che08] showed that this is the case for the manifold M. More generally we have the following.

PROPOSITION 6. *If M is a Kähler–Einstein manifold and M' is a sufficiently small deformation of the complex structure of M, then $R(M') = 1$. More precisely there exists some $\epsilon > 0$ such that $R(M') = 1$ for all M' such that the complex structures of M and M' differ by at most ϵ in the C^k norm for large k (say $k > 4$).*

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Proof. This follows from the proof of the main result in [Szé]. It is shown there that there exists a small ball $B \subset \mathbf{C}^k$ with a linear action of the group of holomorphic automorphisms $\mathrm{Aut}(M)$ on B such that points in a complex analytic subset $Z \subset B$ give all the small deformations of the complex structure of M which have the same first Chern class as M (i.e. Z is a subset of the Kuranishi space [Kur65] of M) and manifolds in the same $\mathrm{Aut}(M)$ orbit are biholomorphic. Moreover the points in Z which are polystable for the action of $\mathrm{Aut}(M)$ (i.e. their orbit is closed in $\mathbf{C}^k$) correspond to deformations of M which admit Kähler–Einstein metrics. Suppose that the small deformation M' corresponds to a point $z \in Z$. Either z is polystable, in which case M' admits a Kähler–Einstein metric, or there exists a polystable point z_0 in the closure of the $\mathrm{Aut}(M)$-orbit of z, such that also $z_0 \in Z$. This z_0 is obtained by minimising the norm over the $\mathrm{Aut}(M)$-orbit of z. Let M_0 be the manifold corresponding to z_0 (it may be that $M_0 = M$), so M_0 admits a Kähler–Einstein metric. Since z_0 is in the closure of the orbit of z, we can realise M' as an arbitrarily small deformation of M_0. The above argument then shows that $R(M') = 1$. □

In addition one can give a lower bound in terms of the alpha invariant $\alpha(M)$ or its equivariant version (see Tian [Tia87]), namely Tian showed that $R(M) \geqslant \alpha(M)\cdot(n+1)/n$ as long as this is no greater than 1, where n is the complex dimension.

There is much less known about upper bounds for $R(M)$. The problem was briefly studied in the paper of Tian [Tia92], and he found some bounds in terms of the tangent bundle. For $\mathbf{P}^2$ blown up in one point he found the upper bound 15/16. In the next section we will show that in fact $R(M_1) = 6/7$ where M_1 is $\mathbf{P}^2$ blown up in one point. For the blowup in two points we show $R(M_2) \leqslant 21/25$.

To obtain upper bounds we can use the recent work of Stoppa [Sto09]. The basic observation is that the equation

$$\mathrm{Ric}(\omega) = t\omega + (1-t)\alpha$$

is a twisted cscK equation (or generalised Kähler–Einstein equation in the terminology of Song and Tian [ST07]). Stoppa gives an obstruction to solving this equation, generalising the slope stability obstruction to the existence of cscK metrics due to Ross and Thomas [RT06]. As we will see, this gives a good bound for $\mathbf{P}^2$ blown up in one point, but for the blowup in two points it does not give anything because it is slope stable (see Panov and Ross [PR09]). So we now give another upper bound which in some sense is more basic. Both are based on constructing sequences of metrics along which $\mathcal{M} + (1-t)\mathcal{J}_\omega$ is not bounded from below for certain t. Stoppa uses a metric degeneration which models deformation to the normal cone, whereas we look at one parameter families of metrics induced by holomorphic vector fields.

Proposition 7. *Fix a metric ω such that $\mathrm{Ric}(\omega) = \alpha$ is a positive form. Let H be a smooth real-valued function on M and suppose that $X = \nabla H$ is a holomorphic vector field. Write $f_t : M \to M$ for the one-parameter group of diffeomorphisms generated by X. Let $\omega_t = f_t^*\omega$. Then*

$$\lim_{t\to\infty} \frac{d}{dt}\mathcal{J}_\alpha(\omega_t) = \int_M H(S(\omega)-n)\omega^n + \lim_{t\to\infty}\int_M (f_t^{-1})^*(\Delta H)\omega^n.$$

Here Δ is the Laplacian with respect to the metric ω.

It follows that

$$\lim_{t\to\infty} \frac{d}{dt}(\mathcal{M}(\omega_t) + (1-s)\mathcal{J}_\alpha(\omega_t)) = s\int_M H(n - S(\omega))\omega^n + (1-s)K\ \mathrm{Vol}(M),$$

where K is the divergence of the vector field X on the submanifold where H achieves its minimum. If this limit is negative, then $\mathcal{M} + (1-s)\mathcal{J}_\alpha$ is not bounded below, and so $R(M) \leqslant s$.

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Proof of Proposition 7. Let $\omega_t = f_t^*\omega$ and write $\omega_t = \omega + i\partial\overline{\partial}\phi_t$. Then one can normalise $\dot{\phi}_t$ so that $\dot{\phi}_t = f_t^* H$. We compute

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_\alpha(\omega_t) &= \int_M \dot{\phi}_t(\Lambda_{\omega_t}\alpha - n)\omega_t^n \\ &= \int_M \dot{\phi}_t[\Lambda_{\omega_t}(\mathrm{Ric}(\omega) - \mathrm{Ric}(\omega_t))]\omega_t^n + \int_M \dot{\phi}_t(S(\omega_t) - n)\omega_t^n.\end{aligned}$$

The second term is simply

$$\int_M H(S(\omega) - n)\omega^n.$$

For the first term we have

$$\begin{aligned}\int_M \dot{\phi}_t[\Lambda_{\omega_t}(\mathrm{Ric}(\omega) - \mathrm{Ric}(\omega_t))]\omega_t^n &= \int_M \dot{\phi}_t \Delta_t \log\frac{\omega_t^n}{\omega^n}\omega_t^n \\ &= \int_M (\Delta_t\dot{\phi}_t)\log\frac{\omega_t^n}{\omega^n}\omega_t^n \\ &= \frac{d}{dt}\int_M \log\frac{\omega_t^n}{\omega^n}\omega_t^n = \frac{d}{dt}\int_M \log\frac{\omega^n}{\omega_{-t}^n}\omega^n.\end{aligned}$$

We have written $\omega_{-t} = (f_t^{-1})^*\omega$. Then $(d/dt)\omega_{-t} = -i\partial\overline{\partial}(f_t^{-1})^* H$ since it is the same as flowing along $-\nabla H$. Therefore we obtain

$$\frac{d}{dt}\int_M -\log\frac{\omega_{-t}^n}{\omega^n}\omega^n = \int_M (f_t^{-1})^*(\Delta H)\omega^n.$$

The first part of the result follows.

For convenience let us assume that $\inf H = 0$. Note first of all that $J\nabla H$ is a Killing field, and so it generates a torus action. In particular H is a component of the moment map for a torus action. It follows that H is a Morse–Bott function with even-dimensional critical manifolds of even index (see McDuff and Salamon [MS98]), and so $H^{-1}(0)$ is a connected complex submanifold, and in addition

$$\lim_{t\to\infty} f_t^{-1}(x) \in H^{-1}(0)$$

for a dense open set in M. For a point $y \in H^{-1}(0)$, the Laplacian $\Delta H(y)$ is the divergence of the vector field X, which is independent of the metric since $X(y) = 0$. It is just given by the total weight of the action on the normal bundle of $H^{-1}(0)$, or alternatively the weight of the action on the anticanonical bundle at y. This is independent of the choice of $y \in H^{-1}(0)$, and we denote it by K. To see that $\Delta H(y)$ does not depend on the point $y \in H^{-1}(0)$, note that $H^{-1}(0)$ is connected and

$$(\Delta H)_p(y) = H_{i\bar{i}p}(y) = H_{ip\bar{i}}(y) - R_{j\bar{j}p\bar{l}}(y)H_l(y) = 0,$$

since ∇H is a holomorphic vector field (so $H_{ip} = 0$) and y is a zero of ∇H. Here we used subscripts for covariant derivatives and $R_{j\bar{j}p\bar{l}}$ is the curvature tensor.

It follows that

$$\lim_{t\to\infty}(f_t^{-1})^*(\Delta H)(x) = K \quad \text{for almost every } x \in M.$$

Since $(f_t^{-1})^*(\Delta H)$ is uniformly bounded, independent of t, it follows that

$$\lim_{t\to\infty}\int_M (f_t^{-1})^*(\Delta H)\omega^n = K\int_M \omega^n.$$

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At the same time

$$\frac{d}{dt}\mathcal{M}(\omega_t) = \int_M \dot{\phi}_t(n - S(\omega_t))\omega_t^n = \int_M H(n - S(\omega))\omega^n,$$

so the proposition follows. □

3.1 $\mathbf{P}^2$ blown up in one point

Let M_1 be $\mathbf{P}^2$ blown up in one point. In this section we prove the following.

Theorem 8. *For $\mathbf{P}^2$ blown up in one point we have $R(M_1) = 6/7$.*

We first show that $R(M_1) \leqslant 6/7$ using twisted slope stability. Since we give an alternative proof, we will be brief. Let us write E for the exceptional divisor. We are using the polarisation $c_1(M_1) = \mathcal{O}(3) - E$. The Seshadri constant of E is then 2. If there is a metric ω with $\mathrm{Ric}(\omega) > t\omega$, then, by taking the trace, we must have

$$S(\omega) - (1-t)\Lambda_\omega \alpha = 2t$$

for some positive form $\alpha \in c_1(M_1)$. According to Stoppa [Sto09, § 5.1] we have

$$\frac{3}{2}\frac{2c_1(M_1).E - 2[(-c_1(M_1) + (1-t)c_1(M_1)).E + E^2]}{2(3c_1(M_1).E - 2E^2)} \geqslant \frac{-(-c_1(M_1) + (1-t)c_1(M_1)).c_1(M_1)}{c_1(M_1)^2}.$$

Computing this and simplifying, we obtain exactly $t \leqslant 6/7$.

Alternatively, for a more self-contained proof we can use Proposition 7. It is easiest to compute in terms of toric geometry. The moment polytope of M_1 has vertices $(0,0)$, $(2,0)$, $(2,1)$, $(0,3)$. We choose $H(x,y) = -x$. Then, for any toric metric $\omega \in c_1(M_1)$, using Donaldson's formula [Don02] we have

$$\int_{M_1} H(2 - S(\omega))\frac{\omega^2}{2!} = 2\int_P H\,d\mu - \int_{\partial P} H\,d\sigma = -\frac{2}{3},$$

where $d\mu$ is the Lebesgue measure on P and $d\sigma$ is a multiple of the Lebesgue measure on each edge of P, as described in [Don02]. In our case $d\sigma = |dy|$ on the vertical edges and $d\sigma = |dx|$ on the remaining edges. To compute the weight we can compute the Laplacian of H in terms of a symplectic potential u on P. It is given by

$$\Delta H = \partial_i(u^{ij}\partial_j H) = \partial_i u^{ij}\partial_j H,$$

where u^{ij} is the inverse of the Hessian of u and we used that fact that H is linear. We want to compute this on the edge where H achieves its infimum, i.e. on the edge joining $(2,0)$ and $(2,1)$. The vector field $\partial_i u^{ij}$ converges to the inward normal ν on the edges, normalised so that the area form $dx \wedge dy$ contracted with ν gives the measure $d\sigma$ on the edge (see Donaldson [Don02]). Hence

$$\Delta H = \nabla_\nu H = -\partial_x H = 1$$

on the edge $x = 2$. We get that the weight $K = 1$, so

$$s\int_{M_1} H(2 - S(\omega))\frac{\omega^2}{2!} + (1-s)K\,\mathrm{Vol}(M_1) = 4 - \frac{14}{3}s.$$

This is negative for $s > 6/7$, in which case the functional $\mathcal{M} + (1-s)\mathcal{J}_\omega$ is not bounded below by Proposition 7, and so $R(M_1) \leqslant 6/7$.

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To show that $R(M_1) \geqslant 6/7$ we explicitly construct metrics with $\mathrm{Ric}(\omega) > t\omega$ for all $t < 6/7$. Again thinking of M_1 as the $\mathbf{P}^1$ bundle $\mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O})$, we will use the momentum construction to obtain metrics on M_1 (for more details on this construction see [HS02]). Let ω_0 be the Fubini–Study metric on $\mathbf{P}^1$, and let h be a Hermitian metric on $\mathcal{O}(-1)$ with curvature form $i\omega_0$. Write $p : \mathcal{O}(-1) \to \mathbf{P}^1$ for the projection map. On the complement of the zero section in the total space of $\mathcal{O}(-1)$ define the metric

$$\omega = p^*\omega_0 + 2i\partial\bar{\partial} f(s),$$

where $s = (1/2) \log |z|_h^2$ and $f(s)$ is a suitably convex function. We change coordinates to $\tau = f'(s)$, which is the moment map for the S^1-action rotating the fibres of $\mathcal{O}(-1)$. Let $I \subset \mathbf{R}$ be the image of τ and let $F : I \to \mathbf{R}$ be the Legendre transform of f. In other words F is defined by the equation

$$f(s) + F(\tau) = s\tau.$$

We then define the *momentum profile* of the metric ω to be

$$\phi(\tau) = \frac{1}{F''(\tau)}.$$

We can compute the Ricci curvature of ω in terms of $\phi(\tau)$. In addition, if ϕ has suitable behaviour at the endpoints of I, then the metric ω can be extended across the zero and infinity sections, and we obtain a metric on M_1. This is summarised in the following proposition. For more details see [HS02] (or also [Szé09]).

Proposition 9. *Let $\phi : [0, 2] \to \mathbf{R}$ be a smooth function such that ϕ is positive on $(0, 2)$, and*

$$\phi(0) = \phi(2) = 0, \quad \phi'(0) = 2, \quad \phi'(2) = -2.$$

Then we obtain a metric $\omega_\phi \in c_1(M_1)$, given in suitable local coordinates by

$$\omega_\phi = (1 + \tau)p^*\omega_0 + \phi(\tau)\frac{i\, dw \wedge d\overline{w}}{2|w|^2},$$

and whose Ricci form is

$$\rho_\phi = \left(2 - \frac{[(1+\tau)\phi]'}{2(1+\tau)}\right)p^*\omega_0 - \phi \cdot \left\{\frac{[(1+\tau)\phi]'}{2(1+\tau)}\right\}' \cdot \frac{i\, dw \wedge d\overline{w}}{2|w|^2},$$

where the primes mean differentiating with respect to τ.

In order to have $\rho_\phi \geqslant t\omega_\phi$ we need to satisfy two inequalities

$$2 - \frac{[(1+\tau)\phi]'}{2(1+\tau)} \geqslant t(1+\tau)$$
$$-\left\{\frac{[(1+\tau)\phi]'}{2(1+\tau)}\right\}' \geqslant t.$$

By integrating once, it is easy to see that the second inequality implies the first one for all $t \leqslant 1$.

Let $t = 6/7$ and let us solve the case of equality in the second inequality. We obtain a ψ such that

$$(1+\tau)\psi(\tau) = 2\tau + \frac{1}{7}\tau^2 - \frac{4}{7}\tau^3.$$

This ψ is positive on $(0, 2)$ and it satisfies the boundary conditions

$$\psi(0) = \psi(2) = 0, \quad \psi'(0) = 2, \quad \psi'(2) = -\frac{10}{7}.$$

Now let $\phi(\tau) = \psi(\tau) + \eta(\tau)$, where η satisfies

$$\eta(0) = \eta(2) = 0, \quad \eta'(0) = 0, \quad \eta'(2) = -\frac{4}{7}.$$

For any $\delta > 0$ we can choose η so that for all τ we have

$$\eta(\tau) \geqslant 0, \quad \eta'(\tau), \eta''(\tau) < \delta.$$

Then $\phi = \psi + \eta$ satisfies the boundary conditions that we want, and

$$\begin{aligned} -\left\{\frac{[(1+\tau)\phi]'}{2(1+\tau)}\right\}' &= \frac{6}{7} - \left\{\frac{[(1+\tau)\eta]'}{2(1+\tau)}\right\}' \\ &= \frac{6}{7} - \frac{1}{2}\eta''(\tau) - \frac{\eta'(\tau)}{2(1+\tau)} + \frac{\eta(\tau)}{2(1+\tau)^2} \\ &> \frac{6}{7} - \delta. \end{aligned}$$

Letting $\delta \to 0$, we find that we can obtain a metric with $\mathrm{Ric}(\omega) \geqslant t\omega$ for all $t < 6/7$, so $R(M_1) \geqslant 6/7$. Note that we would have to analyse the metrics more carefully near $\tau = 0$ and $\tau = 2$ to see whether we have the strict inequality, but clearly $\mathrm{Ric}(\omega) \geqslant t\omega$ is enough for what we want. This completes the proof that $R(M_1) = 6/7$.

Note that in the limiting case $t = 6/7$, the function ψ that we found above defines a singular metric satisfying $\mathrm{Ric}(\omega) \geqslant (6/7)\omega$. The fact that $\psi'(2) = -10/7$ means that the metric has conical singularities with angle $2\sin^{-1}\sqrt{5/7}$ along a line not meeting the exceptional divisor (i.e. along the divisor at infinity in $\mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O})$).

3.2 $\mathbf{P}^2$ blown up in two points

Let M_2 be $\mathbf{P}^2$ blown up in two points. In this section we prove the following proposition.

Proposition 10. *For $\mathbf{P}^2$ blown up in two points we have $1/2 \leqslant R(M_2) \leqslant 21/25$.*

In this case twisted slope stability will not give any obstruction, since M_2 is slope stable (see Panov and Ross [PR09]) and our twisting just makes things more stable (we are adding a proper function to $\mathcal{M}$). However, we can apply Proposition 7. Once again we work in terms of the toric polygon to make the computations easier. The polygon corresponding to M_2 has vertices $(0,0)$, $(2,0)$, $(2,1)$, $(1,2)$, $(0,2)$. We let $H(x,y) = -x - y$. The minimum of H is achieved on the edge where $x + y = 3$ and the normalised inward normal there is

$$-\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

By the same argument as before, this implies that the weight $K = 1$. As before, using Donaldson's formulae we obtain

$$s\int_{M_2} H(2 - S(\omega))\frac{\omega^2}{2} + (1-s)K\,\mathrm{Vol}(M_2) = \frac{7}{2} - \frac{25}{6}s.$$

This is negative if $s > 21/25$, so by Proposition 7 we obtain $R(M_2) \leqslant 21/25$.

To show that $R(M_2) \geqslant 1/2$ we use the α-invariant. According to Song [Son05] the α-invariant for torus invariant Kähler potentials on M_2 is $1/3$. It follows (see [Tia87]) that $R(M_2) \geqslant 1/3 \cdot 3/2 = 1/2$. It would be very interesting to find better bounds on $R(M_2)$.

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4. More general test-configurations

We have seen in Proposition 6 that if M is Kähler–Einstein then $R(M) = 1$ but the converse is not true. In this section we show the following weaker converse.

Theorem 11. *If $R(M) = 1$, then M is K-semistable with respect to test-configurations with smooth total space.*

Remark. In a forthcoming paper with Munteanu [MS] we show that in fact $R(M) = 1$ implies K-semistability.

Before giving the proof we briefly explain K-semistability. A test-configuration χ for M is a flat polarised family $\pi : (\mathcal{M}, \mathcal{L}) \to \mathbf{C}$, such that the following hold.

- π is $\mathbf{C}^*$-equivariant.
- $\mathcal{L}$ is relatively ample.
- We have
$$(\mathcal{M}_t, \mathcal{L}|_{\mathcal{M}_t}) \cong (M, (-K_M)^k)$$
for $t \neq 0$ and some integer $k > 0$.

The central fibre is then a polarised scheme (M_0, L_0), with a $\mathbf{C}^*$-action. This allows us to define the Futaki invariant $F(\chi)$ of the test-configuration, which generalises the classical Futaki invariant in case M_0 is smooth and the $\mathbf{C}^*$-action is generated by a holomorphic vector field. For details see Donaldson [Don02]. The manifold M is called *K-semistable* if $F(\chi) \geqslant 0$ for all test-configurations χ. If, in addition, $F(\chi) = 0$ only for test-configurations where the central fibre is isomorphic to M, we say that M is *K-polystable*. The central conjecture is the following.

Conjecture 12 (Yau–Tian–Donaldson conjecture). *The manifold M admits a Kähler–Einstein metric if and only if M is K-polystable.*

In light of this it is reasonable to conjecture the following.

Conjecture 13. *The Fano manifold M is K-semistable if and only if $R(M) = 1$.*

Our Theorem 11 goes some way in proving the easier direction of this conjecture.

Proof of Theorem 11. Suppose we have a test-configuration for M with total space $\mathcal{M}$. We can realise it as a one parameter group acting on an embedding in projective space. More precisely we have an embedding $F : M \to \mathbf{P}^N$ and a $\mathbf{C}^*$-action on $\mathbf{P}^N$. Choose a Fubini–Study metric ω_{FS} on $\mathbf{P}^N$ which is invariant under S^1, and let H be a Hamiltonian function of this S^1-action, normalised so that $\sup H = 0$. Let us write $f_t : \mathbf{P}^N \to \mathbf{P}^N$ for the gradient flow of ∇H. We then have a family of metrics
$$\omega_t = F^*(f_t^* \omega_{FS})$$
on M and we let $\omega = \omega_0$. Suppose that the Futaki invariant of the test-configuration is negative, i.e. M is not K-semistable. We want to show that $R(M) < 1$. Since the total space of the test-configuration is smooth, according to [PRS08] (see also [PT]) we have
$$\limsup_{t\to\infty} \frac{d}{dt} \mathcal{M}(\omega_t) < 0.$$

We want to show that for suitably small $\epsilon > 0$ we have

$$\limsup_{t\to\infty} \frac{d}{dt}(\mathcal{M}(\omega_t) + \epsilon \mathcal{J}_\omega(\omega_t)) < 0, \tag{4}$$

which will imply that $R(M) \leqslant 1 - \epsilon$.

To show the inequality (4) we compute

$$\frac{d}{dt}\mathcal{J}_\omega(\omega_t) = \int_M \dot{\phi}_t(\Lambda_{\omega_t}\omega - n)\omega_t^n.$$

Note that $\dot{\phi}_t = F^*(f_t^* H)$ and, since $H \leqslant 0$, we have

$$\frac{d}{dt}\mathcal{J}_\omega(\omega_t) \leqslant -n \int_M F^*(f_t^* H)\omega_t^n \leqslant -n \operatorname{Vol}(M) \inf H.$$

Thus the limit as $t \to \infty$ is bounded above, so for suitably small $\epsilon > 0$ we have (4). □

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Gábor Székelyhidi gabor@math.columbia.edu
Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA

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