Note on Binet's Inverse Factorial Series for $\mu(x)$.

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Binet * shewed that the function

$$\mu(x) = \log \Gamma(x) - (x - \tfrac{1}{2}) \log x + x - \tfrac{1}{2} \log 2\pi$$

can be expanded as an inverse factorial series. This note furnishes a new and much simpler proof † of his result, based on a formula which is an analogue ‡ of the Binomial Theorem for factorials. This formula is that, if we denote by $[x]^n$ the ratio

$$\Gamma(1+x)/\Gamma(1+x-n),$$

then

$$[x+h]^m = [x]^m + \binom{m}{1}[x]^{m-1}[h]^1 + \binom{m}{2}[x]^{m-2}[h]^2 + \ldots$$

where $\binom{m}{r}$ denotes the coefficient of x^r in the expansion of $(1+x)^m$.

This formula is valid if $R(x+h+1)>0$. Putting $m=-1$ in this formula, we have, provided $R(x+t)>0$,

$$1 - \frac{x}{x+t} = \sum_{n=1}^{\infty} (-1)^{n+1}[x]^{-n}[t]^n.$$

If we denote by Σ_x the ordinary finite-difference summation operator with respect to x, we easily see that, if $\varpi(x)$ is an arbitrary periodic function of period unity,

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* *Journ. de l'Ecole Polyt.* 16 (1839), 123.

† The usual proof depends on expressing $\mu(x)$ as an integral. See Nielsen: *Handbuch der Theorie der Gammafunktionen;* p. 284 *et seq.*

‡ This is merely the theorem that $F(a, b; c; 1)$ can be expressed as $\Gamma(c)\,\Gamma(c-a-b)/\Gamma(c-a)\,\Gamma(c-b)$ if $R(c-a-b)>0$.

$$2\mu(x)+\varpi(x)=\Sigma_x\left\{2-x\log\frac{x}{x-1}-x\log\frac{x+1}{x}\right\}+\left\{1-x\log\frac{x}{x-1}\right\}$$

$$=\Sigma_x\int_{-1}^{0}\left\{2-\frac{x}{x+t}-\frac{x}{x+t+1}\right\}dt+\int_{-1}^{0}\left\{1-\frac{x}{x+t}\right\}dt$$

$$=\Sigma_x\int_{-1}^{0}\sum_{n=1}^{\infty}(-1)^{n+1}[x]^{-n}\{[t]^n+[t+1]^n\}dt$$

$$+\int_{-1}^{0}\sum_{n=1}^{\infty}(-1)^{n+1}[x]^{-n}[t]^n dt$$

$$=\Sigma_x\int_{-1}^{0}\sum_{n=2}^{\infty}(-1)^{n+1}[x]^{-n}\{[t]^n+[t+1]^n\}dt$$

$$+\int_{-1}^{0}\sum_{1}^{\infty}(-1)^{n+1}[x]^{-n}[t]^n dt$$

$$=\Sigma\int_{-1}^{0}\sum_{n=1}^{\infty}(-1)^n[x]^{-\overline{n+1}}\{[t]^{n+1}+[t+1]^{n+1}\}dt$$

$$+\int_{-1}^{0}\sum_{1}^{\infty}(-1)^{n+1}[x]^{-n}[t]^n dt.$$

Now as Σ_x is a finite difference operator, we can operate inside the signs of integration and summation, and so we obtain

$$2\mu(x)+\varpi(x)=\int_{-1}^{0}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}[x]^{-n}\{[t]^{n+1}+[t+1]^{n+1}+n[t]^n\}dt$$

$$=\int_{-1}^{0}(2t+1)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}[x]^{-n}[t]^n dt.$$

Before we can integrate the right hand side term by term, we must investigate the convergence of the series occurring there.

Now Landau * has shewn that the inverse factorial series $\Omega(x)=\sum_{n=0}^{\infty}n!\,a_n[x-1]^{-\overline{n+1}}$ converges or diverges everywhere with the Dirichlet Series $\Psi(x)=\sum_{n=1}^{\infty}\frac{a_n}{n^x}$ whilst the binomial coefficient series $W(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}[x-1]^n$ converges or diverges everywhere with the Dirichlet Series $\sum_{n=1}^{\infty}\frac{(-1)^n a_n}{n^x}$. Using these results, which

* LANDAU: *Munich Sitzungsberichte* (1906) 36, 151-221.

apply to convergence, uniform convergence and absolute convergence, we see that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}[x]^{-n}[t]^{n}$ converges or diverges everywhere with the series $\sum_{n=1}^{\infty}\frac{1}{n^{x+t+2}}$. But if t lies in the interval $(-1, 0)$ this series converges uniformly with respect to t, provided that $R(x)>0$ Hence if $R(x)>0$, term by term integration is legitimate. By putting $t=-a$, and using the fact that $\mu(x)\to 0$ as $x\to+\infty$, we finally have

$$\mu(x)=\sum_{n=1}^{\infty}\frac{[x]^{-n}}{2n}\int_{0}^{1}(a+1)(a+2)\dots(a+n-1)(2a-1)\,ada$$

provided that $R(x)>0$. This is Binet's result. It should be noticed that the series converges nowhere on the line $R(x)=0$.