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A CLASS OF ALMOST COMMUTATIVE NILALGEBRAS

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1. Introduction. The purpose of this paper is to investigate a class of nonassociative nilalgebras which have absolute zero divisors. If a nilalgebra is nilpotent, it, of course, possesses an absolute zero divisor. For the nilpotence of nonassociative nilalgebras, the situation however becomes quite complicated even in the finite-dimensional case. For example, Gerstenhaber [**3**] has conjectured the nilpotence of commutative nilalgebras. While Gerstenhaber and Myung [**4**] prove that any commutative nilalgebra of dimension $\leqq 4$ in characteristic $\neq 2$ is nilpotent, Suttles [**9**] discovered an example of a 5-dimensional commutative nilalgebra which is solvable but not nilpotent. Thus this is a counterexample to the conjecture of Gerstenhaber. All algebras considered are finite-dimensional over a field and nilalgebras are assumed to be power-associative. If A is a finite-dimensional nilalgebra, it is well-known that $a^{\dim A+1} = 0$ for all $a \in A$. A nonzero element a of an algebra A is called an *absolute zero divisor* if $aA = Aa = 0$. In terms of the right and left multiplications in A, this is to say $R(a) = L(a) = 0$ on A. If A is a commutative nilalgebra, all $R(x)$, $L(x)$ are nilpotent, which is proved by Gerstenhaber [**3**] in characteristic 0 and by Oehmke [**7**] in characteristic >2. In the noncommutative case, this result still holds for many of the well-known noncommutative Jordan nilalgebras in which case the algebras turn out to be nilpotent. However, the situation is quite different for anticommutative algebras (nilalgebras of nil-index 2). In fact, in view of Engel's Theorem, all $R(x)$ are nilpotent in a Lie algebra A if and only if A is nilpotent. A closer look at the example of Suttles reveals the interesting fact that a commutative nilalgebra may not possess an absolute zero divisor. It seems thus quite natural to look for a class of nilalgebras possessing absolute zero divisors from noncommutative nilalgebras where all $R(x)$ and $L(x)$ are nilpotent. In this paper we obtain such a class from "almost" commutative nilalgebras.

For an algebra A, the minus-algebra A^- of A is defined as the same vector space as A but with a multiplication given by $[x, y] = xy - yx$. Then A is said to be Lie-admissible if A^- is a Lie algebra. If a Lie-admissible algebra A is flexible; that is, A satisfies the flexible law $x(yx) = (xy)x$, then all $D(x) \equiv R(x) - L(x)$ are derivations of A; $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in A$. The plus-algebra A^+ of A is defined by $x \cdot y = \frac{1}{2}(xy + yx)$ on the same vector space as A if the characteristic is not 2. Then A is called Jordan-admissible if A^+ is a Jordan algebra, and it is shown in [**8**] that A is flexible Jordan-admissible if and only if A is a noncommutative Jordan algebra. It will

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be worthwhile to point out that flexible Lie-admissible algebras may not be power-associative, while every flexible Jordan-admissible algebra is power-associative in characteristic $\neq 2$. It is not difficult to find such examples, but they seem not to have been shown in a literature. Let L be a Lie algebra over a field Φ of characteristic $\neq 2, 3, 5$. Let $A = L + \Phi e$ be a vector space direct sum of L and a one-dimensional space Φe. For a fixed $\alpha \in \Phi$, we define a product in A by

$$(1) \quad (a + \lambda e)(b + \mu e) = ab + \alpha(\mu a + \lambda b) + \lambda\mu e$$

for $a, b \in L$ and $\lambda, \mu \in \Phi$. One easily checks that A is flexible Lie-admissible, and that $x^2x^2 = x^3x$ for all $x \in A$ [**1**, p. 557] if and only if $2\alpha^3 - 3\alpha^2 + \alpha = 0$, so that A is power-associative if and only if $\alpha = 0, \frac{1}{2}$, or 1.

A noncommutative algebra A is said to be *almost commutative* if A contains a commutative subalgebra of codimension one. Similarly, a nonabelian Lie algebra is called *almost abelian* if it contains an abelian subalgebra of codimension one. An almost abelian Lie algebra is not necessarily nilpotent, as shown by certain solvable Lie algebras; for example, the 3-dimensional solvable Lie algebra L with multiplication $xy = x$, $xz = yz = 0$, where we notice that $B = \Phi y + \Phi z$ is an abelian subalgebra of codimension one, but not an ideal in L. Let L be an almost abelian Lie algebra over a field Φ of characteristic $\neq 2, 3, 5$ and B be an abelian subalgebra of codimension one of L. Then we note that the algebra $A = L + \Phi e$ constructed by (1) is an almost commutative algebra and that $S = B + \Phi e$ is a commutative subalgebra of codimension one but is not an ideal of A. However, in case A is a nilalgebra, we will see that any codimension one subalgebra of A is an ideal, provided all $R(x)$, $L(x)$ are nilpotent in A (this will be the case if all $D(x)$ are nilpotent; for example, A^- is a nilpotent Lie algebra). We now state the main theorem.

THEOREM. *Let A be a finite-dimensional, flexible, strictly power-associative algebra over a field Φ of characteristic $\neq 2$. If A is a nilalgebra such that A^- is an almost abelian, nilpotent Lie algebra, then A contains absolute zero divisors and furthermore the center Z of A^- is an ideal of A.*

We have observed that the condition that A^- is nonabelian and nilpotent is essential in the theorem.

2. Proof of the theorem.

We begin with the following lemma.

LEMMA. *Let A be a finite-dimensional, flexible, strictly power-associative nilalgebra over a field Φ of characteristic $\neq 2$.*

(i) *If x is an element in A such that $D(x)$ is nilpotent then $R(x)$ and $L(x)$ are nilpotent in A.*

(ii) *If S is a subalgebra of codimension one of A such that $D(x)$ is nilpotent in A for all $x \in S$, then S is an ideal of A. In particular, if A is almost commutative, every commutative subalgebra of codimension one is an ideal of A.*

Proof. (i) Consider the commutative nilalgebra A^+ and let $T(x) = \frac{1}{2}(R(x) + L(x))$. Then, if the characteristic is 0, it is shown in [**3**] that $T(x)$ is nilpotent. If the characteristic is greater than 2, then we adjoin an identity to A^+ to get a commutative algebra $(A^+)'$ of degree one. Then Oehmke [**7**] proves that $T(x)$ is nilpotent on $(A^+)'$ and so on A^+ for all $x \in A$ (his proof does not use the simplicity of the algebra). Thus in any case $T(x)$ is nilpotent for all $x \in A$. Using the flexible law $R(x)L(x) = L(x)R(x)$, we have that if $D(x)$ is nilpotent then $R(x) = \frac{1}{2}D(x) + T(x)$ and $L(x) = T(x) - \frac{1}{2}D(x)$ are nilpotent too.

(ii) Let S be a codimension one subalgebra of A. Let a be an element of A but not in S. Suppose that S is not an ideal of A. Then, since S and a span A, we may assume there exists an element $x \in S$ such that $ax \equiv \lambda a \pmod{S}$ for some $\lambda \neq 0$ in Φ. Since S is a subalgebra of A, we have $aR(x)^n \equiv \lambda^n a \pmod{S}$ and $0 \equiv \lambda^n a \pmod{S}$ for some n since $R(x)$ is nilpotent. This forces $\lambda = 0$, a contradiction, and so $ax \in S$ for all $x \in S$. Similarly, we have $xa \in S$ for all $x \in S$ and hence S is an ideal of A.

For the proof of the theorem, let B be a codimension one, abelian subalgebra of A^-. Since A^- is nilpotent, applying the lemma to A^- implies that B is an ideal of A^-. We first show that B is a subalgebra of A. Let $A = \Phi h + B$ be a vector space direct sum. Then $[A, A] = [B, h] \neq 0$ since B is abelian in A^-. Let $x, y \in B$ and let $xy \equiv \alpha h \pmod{B}$. For $g \neq 0$ in $[A, A]$, let $g = [b, h]$ for $b \in B$. Since $D(b)$ is a derivation of A and B is abelian, applying $D(b)$ to $xy \equiv \alpha h \pmod{B}$ implies $0 = \alpha[h, b] = \alpha g$ and $\alpha = 0$. Hence B is a subalgebra of A and is again an ideal of A by the lemma.

Since $D(h)$ induces a nilpotent linear transformation on B, B can be expressed as a direct sum

$$B = M_1 \oplus M_2 \oplus \ldots \oplus M_r$$

of cyclic subspaces M_i in B relative to $D(h)$ such that $n_1 \geqq n_2 \geqq \ldots \geqq n_r$ where $n_i = \dim M_i$ and n_1 is the nil-index of $D(h)$ in B. Let $x_{i,1}, \ldots, x_{i,n_i}$ be a basis of M_i such that $[x_{i,k-1}, h] = x_{i,k}$ and $[x_{i,n_i}, h] = 0$, $k = 2, 3, \ldots, n_i$. Since B is abelian and $[B, h] \neq 0$, the center Z of A^- is contained in B, and hence Z is the centralizer of h in B. Therefore, if we let $x_1 = x_{1,n_1}, \ldots, x_r = x_{r,n_r}$, $x_1, \ldots, x_r$ form a basis of Z. Recalling that B is an ideal of A, $hx_i = x_i h \in B$ and so $[hx_i, h] = h[x_i, h] = 0$. Hence

$$(2) \quad hx_i = x_i h \in Z, \qquad i = 1, 2, \ldots, r.$$

Since $[B, h] \neq 0$, $n_1 \geqq 2$. Let p be such that $n_1 \geqq n_2 \geqq \ldots \geqq n_p \geqq 2$ and $n_i = 1$ if $i > p$. For $x \in B$, if $i \leqq p$ then

$$0 = [x_{i,n_i-1}, xh] = x[x_{i,n_i-1}, h] = xx_i,$$

and similarly $x_i x = 0$ (again recall B is abelian and is an ideal of A). Hence we have

(3) $Bx_i = x_iB = 0, \qquad i = 1, 2, \ldots, p.$

If $j > p$, by (2) we see

$$0 = [x_{i,k}, x_jh] = x_j[x_{i,k}, h] = x_jx_{i,k+1},$$

$$i = 1, 2, \ldots, p \text{ and } 1 \leqq k \leqq n_i - 1.$$

Therefore we have

(4) $x_jx_{i,k} = x_{i,k}x_j = 0, \quad 1 \leqq i \leqq p, \quad 2 \leqq k \leqq n_i, \quad p < j.$

If $i \leqq p$ and $j > p$, by (4)

$$[x_jx_{i,1}, h] = x_j[x_{i,1}, h] = x_jx_{i,2} = 0,$$

and since Z is the centralizer of h in B, this implies that $x_jx_{i,1} = x_{i,1}x_j \in Z$ for $j > p$ and $1 \leqq i \leqq p$. Therefore by (2), (3), and (4) we see that Z is an ideal of A.

Finally, we show that

(5) $h([A, A] \cap Z) = ([A, A] \cap Z)h = 0.$

Let $z \in [A, A] \cap Z$ and let $h^2 \equiv \lambda h \pmod{B}$ for $\lambda \in \Phi$. Then $z = [b, h]$ for $b \in B$ and $[b, h^2] = h[b, h] + [b, h]h = 2zh$, while $[b, h^2] = \lambda[b, h] = \lambda z$. Hence $2zh = \lambda z$ and since $R(h)$ is nilpotent, either $z = 0$ or $\lambda = 0$. In any case, $zh = 0$, thus showing (5). Since $x_1, \ldots, x_p \in [A, A] \cap Z$, it follows from (3) and (5) that $x_1, \ldots, x_p$ are absolute zero divisors of A. This completes the proof of the theorem.

3. Examples. Since any nonabelian nilpotent Lie algebra of dimension $\leqq 4$ is almost abelian and is completely known [**2**, p. 120], the theorem can be used to determine all noncommutative flexible Lie-admissible nilalgebras A of dimension $\leqq 4$ such that A^- is nilpotent. In this case, $\dim A = 3$ or 4 and if $\dim A = 4$ then $\dim Z(A^-) = 1$ or 2. In the theorem, "strict" power-associativity is needed only to show that all $T(x)$ are nilpotent on A. However, if $\dim A \leqq 4$, then, without the condition that A is strict, it is shown in [**4**] that A^+ is nilpotent and so all $T(x)$ are nilpotent. In the following we assume that A is a noncommutative algebra over the field Φ.

(I) A is a flexible nilalgebra such that A^- is a nilpotent Lie algebra of dimension 3 if and only if A is given by the multiplication

$$x^2 = \alpha z, xy = \beta z, yx = (\beta - 1)z, y^2 = \gamma z, \alpha, \beta, \gamma \in \Phi,$$

and all other products are 0.

(II) A is a flexible nilalgebra such that A^- is a nilpotent Lie algebra of dimension 4 and $\dim Z(A^-) = 1$ if and only if A is given by

$$x^2 = \alpha z, xh = -\tfrac{1}{2}y + \beta z, hx = \tfrac{1}{2}y + \beta z,$$

$$yh = -hy = -\tfrac{1}{2}z, h^2 = \gamma z, \qquad \alpha, \beta, \gamma \in \Phi,$$

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and all other products are 0. In this case A is a nilalgebra of nil-index 3 and is a Lie algebra if and only if $\alpha = \beta = \gamma = 0$.

(III) A is a flexible nilalgebra such that A^- is a nilpotent Lie algebra of dimension 4 and $\dim Z(A^-) = 2$ if and only if A is given by

$$x^2 = \alpha y + \beta z, xz = zx = \gamma y, xh = \delta y + \lambda z,$$

$$hx = (\delta + 1)y + \lambda z, zh = hz = \nu y, z^2 = \mu y, h^2 = \sigma y + \tau z,$$

and all other products are 0, and $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \sigma, \tau \in \Phi$ with $\mu\beta^2 = \mu\lambda^2 = \mu\tau^2 = 0$. In this case A is a nilalgebra of nil-index 4 if $\mu = 0$ and of nil-index 3 if $\mu \neq 0$. A is a Lie algebra if and only if $\delta = -\frac{1}{2}$ and all other parameters are 0.

Here we only prove Case (III) and the other cases are entirely similar. In this case A^- has a basis x, y, z, h such that $[h, x] = y$ and all other Lie products are 0 (see [**2**, p. 120]). Then $B = \Phi x + \Phi y + \Phi z$ is an ideal of A^- and $Z = \Phi y + \Phi z$ is the center of A^-. Hence by (5) y is an absolute zero divisor of A. From $[h, x^2] = 2xy = 0$, we obtain $x^2 = \alpha y + \beta z$ and $[h, h^2] = 0$ implies $h^2 = \sigma y + \tau z$. Since $[h, xh] = [h, x]h = yh = 0$, $xh = \delta y + \lambda z$ and $hx = (\delta + 1)y + \lambda z$. Setting $zx = xz = \gamma y + \gamma' z$ (recall Z is an ideal of A), we get that $(xz)x = \gamma' xz$ and since $R(x)$ is nilpotent, $\gamma' = 0$. Similarly $hz = zh = \nu y$. Since $z^3 = 0, z^2 = \mu y$. That $x \in B$ implies $0 = x^2x^2 = (\alpha y + \beta z)^2 = \beta^2 z^2 = \beta^2 \mu y$ and hence $\beta^2\mu = 0$. Since h belongs to the subalgebra $\Phi y + \Phi z + \Phi h$, $h^2h^2 = 0$ implies $\mu\tau^2 = 0$. Similarly we obtain $\mu\lambda^2 = 0$ from $(xh)^2(xh)^2 = 0$. Therefore A has the multiplication table given in (III). Conversely, it can be shown that the algebra A in (III) is a flexible, (power-associative) nilalgebra such that A^- is a nilpotent Lie algebra and $\dim Z(A^-) = 2$.

Incidentally we see that the algebras above are all nilpotent such that all products of any 4 elements in A are 0. In fact, in (I) we get $A^3 = 0$. In Case (II) $A^3 \subseteq \Phi z$ and since z is an absolute zero divisor and $A^2A^2 = 0$, A is nilpotent. In Case (III) $A^3 \subseteq \Phi y$ (again note y is an absolute zero divisor). Also $A^2A^2 \subseteq \Phi \cdot \mu y$, and if $\mu \neq 0, \beta = \lambda = \tau = 0$ and so in any case $A^2A^2 = 0$, thus A is nilpotent. Combining this with the known result [**4**] for the commutative case, we can state

PROPOSITION. *Let A be a flexible, power-associative nilalgebra over a field of characteristic $\neq 2$ such that A^- is a nilpotent Lie algebra. If* $\dim A \leqq 4$ *then A is also nilpotent such that all products of any* 4 *elements in A are* 0.

Therefore, there is no simple nilalgebra of dimension $\leqq 4$ described in the proposition. It is not known whether or not there exists a simple, flexible, Lie-admissible nilalgebra A such that A^- is nilpotent. This question was raised in [**6**] from attempting to classify simple flexible Lie-admissible nil-algebras. We have resolved this for dimension $\leqq 4$ and for the algebra A described in the theorem. The proposition for an arbitrary dimension does not hold as remarked for the commutative case in Introduction. We however conjecture that the algebra A described in the theorem is nilpotent.

A noncommutative nilalgebra may have an absolute zero divisor without being almost commutative. Such an example easily comes from Lie or associative algebras. We close this section with an example of a nonassociative nilalgebra of nil-index 3 that is not almost commutative but has an absolute zero divisor. The following characterization might be interesting.

(IV) Let A be a flexible nilalgebra of dimension $\leqq 4$ over an algebraically closed field Φ of characteristic 0. Then A^- is a nonsolvable Lie algebra if and only if A is one of the following:

(i) the 3-dimensional simple Lie algebra;

(ii) a nonsolvable Lie algebra of dimension 4;

(iii) an algebra of dimension 4 with the multiplication given by

$$xy = z + \tfrac{1}{2}h,\ yx = z - \tfrac{1}{2}h,\ xh = -hx = \tfrac{1}{2}x,\ hy = -yh = \tfrac{1}{2}y,\ h^2 = -z,$$

and all other products are 0. In Case (iii) A is a nilalgebra of nil-index 3.

Proof. Since any Lie algebra of dimension $\leqq 2$ is solvable, $\dim A = 3$ or 4. If $\dim A = 3$, then A^- is the 3-dimensional simple Lie algebra [**5**, p. 14] and hence by [**6**, Theorem 3.1] A is a Lie algebra isomorphic to A^-.

Suppose $\dim A = 4$. Let N be the solvable radical of A^- and $A^- = S \oplus N$ be a Levi-decomposition of A^- where S is a maximal semisimple subalgebra of A^-. Since A^- is not solvable. $\dim N \leqq 3$. It is well-known that there is no semisimple Lie algebra of dimension 1, 2, or 4 in characteristic 0. Thus we have $\dim S = 3$ and $\dim N = 1$. Therefore S is the 3-dimensional simple Lie algebra under $[\ ,\ ]$ and $N = \Phi z$. For any finite-dimensional Lie algebra L of characteristic 0, it is easy to see that if L has one-dimensional radical N, then N is the center of L. Hence Φz is the center of A^-. Let x, y, h be a basis of S such that $[x, h] = x$, $[y, h] = -y$, $[x, y] = h$. Then $H = \Phi z + \Phi h$ is a Cartan subalgebra of A^-, and since H is a (commutative) nil subalgebra of A [**6**, p.81], $u^3 = 0$ for all $u \in H$. Hence it follows from [**6**, Lemma 3.2(i)] that $u^2 \in \Phi z$ for all $u \in H$. Thus $H^2 \subseteq \Phi z$ since H is commutative, and so by the lemma, $Hz = 0$. Let $h^2 = \alpha z$ for $\alpha \in \Phi$. Then $0 = [x, h^2] = h[x, h] + [x, h]h = hx + xh$ and this together with $[x, h] = x$ implies $xh = -hx = \frac{1}{2}x$, and similarly, $hy = -yh = \frac{1}{2}y$. Since Φx and Φy are the root spaces of A^- for H corresponding to the roots 1 and -1, we have $xz = yz = 0$ since $R(z)$ is nilpotent (also see [**6**, p. 80]). Thus z is an absolute zero divisor of A. Let $xy = \beta z + \gamma h$, so $yx = \beta z + (\gamma - 1)h$. Using the foregoing relations, the flexible law $(xy)h - x(yh) + (hy)x - h(yx) = 0$ gives $\beta = -\alpha$ and $\gamma = \frac{1}{2}$. If $\alpha = 0$, A is a nonsolvable Lie algebra. If $\alpha \neq 0$, replace $-\alpha z$ by z to obtain the algebra given in (iii). In this case, it is easy to see that A is a flexible nilalgebra of nil-index 3.

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