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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM ARISING IN ISOTHERMAL AUTOCATALYTIC CHEMICAL KINETICS

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In this paper we consider the questions of existence and uniqueness of solutions to a singular, nonlinear boundary value problem arising from a model problem in isothermal autocatalytical chemical kinetics. The boundary value problem occurs in the construction of a small time asymptotic solution to an initial-boundary value problem (King and Needham [**14**]), and existence and uniqueness for the boundary value problem are required for consistency of this formal asymptotic solution.

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1. Introduction

In a one-dimensional unstirred environment, the study of the isothermal autocatalytic reaction scheme,

$$A \to B \qquad \text{rate} = k[A][B]^p, \tag{1.1}$$

(where A, B are reactant and autocatalyst respectively, $k>0$ is the rate constant and $p>0$ in the reaction order) leads to an examination of the coupled reaction-diffusion initial-boundary value problem,

$$\frac{\partial \alpha}{\partial t} = \frac{\partial^2 \alpha}{\partial x^2} - (\alpha\beta^p)_+, \quad \frac{\partial \beta}{\partial t} = \frac{\partial^2 \beta}{\partial x^2} + (\alpha\beta^p)_+, \quad x, t > 0 \tag{1.2}$$

$$\alpha(x,0) = 1, \quad x \geqq 0, \quad \beta(x,0) = \begin{cases} g(x), & 0 \leqq x \leqq \sigma, \\ 0 \quad , & x > \sigma, \end{cases} \tag{1.3}$$

$$\alpha_x(0,t) = \beta_x(0,t) = 0, \quad t > 0 \tag{1.4}$$

$$\alpha(x,t)\to A(t), \beta(x,t)\to B(t), \quad \text{with} \quad 0\leqq A(t)\leqq 1, 0\leqq B(t)<\infty \quad \text{as} \quad x\to\infty, t>0. \tag{1.5}$$

Here $\alpha(x,t), \beta(x,t)$ are dimensionless concentrations of the reactant and autocatalyst respectively, x is dimensionless distance and t is dimensionless time, with the notation $(\alpha\beta^p)_+$ defined to be,

$$(\alpha\beta^p)_+ = \begin{cases} 0\ , & \alpha\leqq 0 \quad \text{or} \quad \beta\leqq 0 \\ \alpha\beta^p, & \alpha,\beta>0. \end{cases} \tag{1.6}$$

In (1.3), $g(x)>0$ is an analytic function in $0\leqq x\leqq\sigma$, and so $g(x)\sim g_\sigma(\sigma-x)^r$ as $x\to\sigma^-$, for some constant $g_\sigma>0$ and $r\in\mathbb{N}$. Under these conditions it is readily shown (via the scalar maximum principle for parabolic operators) that $\alpha(x,t), \beta(x,t)\geqq 0$ for all $x,t>0$.

For $p\geqq 1$ the initial-boundary value problem (1.2)–(1.5) has been studied extensively by Merkin *et al.* [**19**], Merkin and Needham [**16, 17, 18**], Gray *et al.* [**11**], Billingham and Needham [**4, 5, 6, 7**] and Needham and Merkin [**21**]. An important part of examining this system is a full understanding of the scalar initial-boundary value problem,

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+(u^p)_+, \quad x,t>0, \\ &u(x,0)=\begin{cases} g(x); & 0\leqq x\leqq\sigma \\ 0\ ; & x>\sigma, \end{cases} \\ &u_x(0,t)=0, \quad t>0, \end{aligned}\right\} \qquad I[p]$$

$u(x,t)\to u_\infty(t)$, with $0\leqq u_\infty(t)<\infty$, as $x\to\infty$, $t>0$.

Here $(\cdot)_+$ is defined as in (1.6) and throughout the paper, the notation $(\cdot)_+$ will have the following definition,

$$(f(x))_+ \equiv \begin{cases} f(x), & x\geqq 0 \\ 0\ , & x<0. \end{cases}$$

We will refer to the above problem as $I[p]$. With $p\geqq 1$, $I[p]$ has been studied extensively (see, for example, Fujita [**10**], Bandle and Levine [**1**], Weissler [**24**], Levine [**15**]). For $0<p<1$, the equivalent "sink" problem (with $+(up)_+$ replaced by $-(up)_+$ in $I[p]$) has been considered in detail by Bandle and Stakgold [**2**] and Grundy and Peletier [**12**]. The corresponding source problem $I[p]$ has recently been examined by King and Needham [**14**] and Needham [**20**], who in particular obtain an asymptotic solution to $I[p]$ as $t\to 0^+$, uniform in x, using the method of matched asymptotic expansions. In the course of the analysis in [**14**] the following modified initial-boundary value problem arose,

$$\left.\begin{aligned}&\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial \bar{X}^2}+(U^p)_+, \quad -\infty<\bar{X}<\infty, \quad t>0,\\ &U(\bar{X},0)=\begin{cases} g_\sigma(-\bar{X})^{2/(1-p)}, & \bar{X}<0,\\ 0, & \bar{X}\geqq 0\end{cases}\\ &U(\bar{X},t)\to\begin{cases}(1-p)^{1/(1-p)}t^{1/(1-p)} & \text{as } \bar{X}\to\infty\\ g_\sigma(-\bar{X})^{2/(1-p)} & \text{as } \bar{X}\to-\infty\end{cases}, \quad t>0,\end{aligned}\right\} J[p]$$

Following [**14**], this problem is reduced by the similarity transformation,

$$U(\bar{X},t)=t^{1/(1-p)}V(X), \tag{1.7}$$

with $X=\bar{X}t^{-1/2}$. On substituting from (1.7) into $J[p]$, we are left with the following nonlinear boundary value problem for $V(X)$, namely,

$$V_{XX}+\tfrac{1}{2}XV_X+\left[V^p-\frac{1}{1-p}V\right]_+=0, \quad -\infty<X<\infty, \tag{1.8}$$

$$V(X)\geqq 0 \quad \text{for all} \quad -\infty<X<\infty, \tag{1.9}$$

$$V(X)\to\begin{cases}(1-p)^{1/(1-p)}, & X\to+\infty\\ g_\sigma(-X)^{2/(1-p)}, & X\to-\infty\end{cases}, \tag{1.10}$$

which will henceforth be referred to as BVP. It should be noted that in the reduction of $J[p]$ to BVP, we have replaced $(V^p)_+-(1/(1-p))V$ by $[V^p-(1/(1-p))V]_+$. This is allowable by condition (1.9) and will be convenient in what follows. The details of this problem were not considered in [**14**]; only the asymptotic forms as $X\to\pm\infty$, which were immediately required as matching conditions, were derived. However, for the asymptotic structure derived in [**14**] to be formally complete, we require that for a fixed $0<p<1$, then BVP has a unique solution for each $g_\sigma>0$. It is this existence and uniqueness question for BVP which we consider in the present paper. Related problems in the non-singular case $p>1$ have been considered by Escobedo and Zua Zua [**9**].

We adopt a shooting method, similar in spirit to that used by Berestycki *et al.* [**3**] and Peletier and Serrin [**22**] for radial problems on the half line. This method is adapted for BVP, which is defined on the full line. In particular, we consider a modified boundary value problem $\overline{\text{BVP}}$ (defined in (2.1)–(2.4)) for $u=\hat{u}(X)$, $-\infty<X<\infty$ and establish the following main theorem.

Theorem. *The set of solutions to* $\overline{\text{BVP}}$ *consists of a one-parameter family, which can* *be parametrized by* $\delta>0$. *For each* $\delta>0\ \exists$ *a unique solution of* $\overline{\text{BVP}}$ *if and only if* $v=v_c(\delta)$. *Moreover, that solution can be constructed in terms of solutions to* IVP1, 2 *as*

$$\hat{u}(X)=\begin{cases}\tilde{u}(X,v_c(\delta)), & x\geqq 0\\ \bar{u}(X,v_c(\delta)), & x<0.\end{cases}$$

Here $\tilde{u}$ and $\bar{u}$ are solutions of the initial value problems IVP1,2, (defined in (3.1, 2) and (4.1, 2) respectively) with $v_c(\delta)$ being a critical value of v defined in section 3. This theorem enables existence and uniqueness for BVP to be deduced directly.

2. A modified boundary value problem

We consider in this section a modified form of BVP, namely,

$$U_{XX}+\tfrac{1}{2}XU_X+\left[U^p-\frac{1}{1-p}U\right]_+=0, \quad -\infty<X<\infty, \tag{2.1}$$

$$U(X)\geqq 0 \quad \text{in} \quad -\infty<X<\infty, \tag{2.2}$$

$$U(X)=0[(-X)^{2/(1-p)}] \quad \text{as} \quad X\to-\infty, \tag{2.3}$$

$$U(X)\to(1-p)^{1/(1-p)} \quad \text{as} \quad X\to+\infty, \tag{2.4}$$

which we will refer to as $\overline{\text{BVP}}$. A solution of $\overline{\text{BVP}}$ is to be a solution in the classical sense; that is, a twice continuously differentiable function $U(X)$ satisfying (2.1) on $-\infty<X<\infty$, together with conditions (2.2)–(2.4). We begin by first establishing some general properties concerning $\overline{\text{BVP}}$.

Proposition 2.5. *Let* $U(X)$ *be a solution of equation* (2.1) *in a neighbourhood* N_0 *of* $X=X_0$, *such that at* $X=X_0$, $U(X_0)=U_X(X_0)=0$, *whilst* $U(X)\geqq 0$ *in* N_0; *then,*

(i) $X_0>0\Rightarrow U(X)=0$ *in* $X\geqq X_0$
(ii) $X_0<0\Rightarrow U(X)=0$ *in* $X\leqq X_0$
(iii) $X_0=0\Rightarrow U(X)=0$ *for all* $-\infty<X<\infty$.

Proof. (i) In this case $X_0>0$ and $U(X_0)=U_X(X_0)=0$, with $U(X)\geqq 0$ in N_0. For $X\in N_0$, we now multiply (2.1) by U_X and apply $\int_{X_0}^{X}\cdots ds$, to obtain

$$U_X^2(X)=-\int_{X_0}^{X} sU_s^2(s)\,ds-\frac{2U^{p+1}(X)}{(1+p)}+\frac{U^2(X)}{(1-p)}; \quad X\in N_0, \tag{2.5}$$

after use of the conditions at $X=X_0$. We now take $X>X_0$, and use the mean-value theorem on the first term on the right-hand side of (2.5), to give,

$$U_X^2(X) = -(X - X_0)[\hat{X}U_X^2(\hat{X})] - \frac{2U^{p+1}(X)}{(1+p)} + \frac{U^2(X)}{(1-p)}; \quad X \in N_0, \tag{2.6}$$

with $\hat{X} \in (X_0, X)$. Now, since $0<p<1$, there exists a $\delta>0$, depending upon X_0 and p, such that, from (2.6), $U_X^2(X) \leqq 0 \, \forall X \in [X_0, X_0+\delta]$. Hence $U_X(X) \equiv 0$ in $[X_0, X_0+\delta]$, and as $U(X_0)=0$ and $U(X)$ is continuous in N_0, we conclude that $U(X) \equiv 0$ for $X \in [X_0, X_1]$ for any $X_1 > X_0$, and the results follow.

Parts (ii) and (iii) are established similarly. □

This result can be used to establish the following monotone property for all solutions to $\overline{\text{BVP}}$.

Proposition 2.7. *Let $U(X)$ be a solution of $\overline{\text{BVP}}$, then $U(X)$ is strictly monotone decreasing, with $U(X)>(1-p)^{1/(1-p)}$ for all $-\infty<X<\infty$.*

Proof. From condition (2.2), $U(X) \geqq 0$. However, conditions (2.2)–(2.4) together with Proposition 2.5 lead us to conclude that $U(X)>0$ for all $-\infty<X<\infty$. Now, suppose that $U(X) \ngeqq (1-p)^{1/(1-p)}$ for all $-\infty<X<\infty$, then (via conditions (2.3), (2.4)) $U(X)$ must have a local minimum at $X=X_T$ (say), with $0<U(X_T)<(1-p)^{1/(1-p)}$, $U'(X_T)=0$ and $U''(X_T) \geqq 0$. However, using equation (2.1) we have $U''(X_T)=(1/(1-p))U(X_T)-U^p(X_T)<0$, which gives a contradiction. Hence $U(X) \geqq (1-p)^{1/(1-p)}$ for all $-\infty<X<\infty$. Next suppose that $U(X)$ has a turning point at $X=\bar{X}_T$ (say), then, via (2.1),

$$U''(\bar{X}_T) = \frac{1}{1-p}U(\bar{X}_T) - U^p(\bar{X}_T) \begin{cases} >0, & U(\bar{X}_T)>(1-p)^{1/(1-p)} \\ =0, & U(\bar{X}_T)=(1-p)^{1/(1-p)}. \end{cases}$$

Thus, $U(X)$ can only have a local minimum for all $-\infty<X<\infty$. It then follows from conditions (2.3), (2.4) that $U(X)$ must be strictly monotone decreasing in $-\infty<X<\infty$, after which the above inequality tightens to $U(X)>(1-p)^{1/(1-p)}$ for $-\infty<X<\infty$, as required. □

These results will be revisited at a later stage. We now adopt a shooting technique to obtain the complete family of solutions to $\overline{\text{BVP}}$. This involves the study of two related initial value problems.

3. The initial value problem in $X>0$

In this section we consider the initial value problem,

$$\tilde{u}_{XX} + \tfrac{1}{2}X\tilde{u}_X + \left[\tilde{u}^p - \frac{1}{1-p}\tilde{u}\right]_+ = 0, \quad X>0 \tag{3.1}$$

$$\tilde{u}(0) = (1-p)^{1/(1-p)} + \delta, \quad \tilde{u}_X(0) = -v\delta, \tag{3.2a, b}$$

where $\delta \geqq 0$ and $\nu \geqq 0$, and we henceforth refer to this as IVP1. We first make the following remark.

Remark 3.3. With $\delta = 0$, IVP1 clearly has the global solution $\tilde{u}(X) \equiv (1-p)^{1/(1-p)} \forall X \geqq 0$. It follows that this is unique through an application of the local uniqueness result (Coddington and Levinson, [**8**, Theorem 2.2]).

We now restrict attention to the case when $\delta > 0$, and to proceed, we require the corresponding linearized initial value problem, namely,

$$u_{lXX} + \tfrac{1}{2} X u_{lX} - [u_l - (1-p)^{1/(1-p)}] = 0, \quad X > 0, \tag{3.4}$$

$$u_l(0) = (1-p)^{1/(1-p)} + \delta, \quad u_{lX}(0) = -\nu\delta, \tag{3.5, 6}$$

which we shall refer to as LIVP1. The general solution to equation (3.4) is readily obtained, and conditions (3.5, 6) determine the unique solution to LIVP1 as,

$$u_l(X) = (1-p)^{1/(1-p)} + \delta\nu(1 + \tfrac{1}{2}X^2) \int_X^\infty \frac{e^{-s^2/4}}{(1 + \frac{1}{2}s^2)^2} ds + \delta(1 - \sqrt{\pi}\nu)(1 + \tfrac{1}{2}X^2), \tag{3.7}$$

for all $X \geqq 0$. We note that for $X \gg 1$,

$$u_l(X) \sim (1-p)^{1/(1-p)} + \frac{2\delta\nu e^{-X^2/4}}{X(1 + \frac{1}{2}X^2)} + \frac{\delta}{2}(1 - \sqrt{\pi}\nu)X^2. \tag{3.8}$$

For $\nu = 1/\sqrt{\pi}$, $u_l(X)$ is monotone decreasing in X with $u_l(X) \to (1-p)^{1/(1-p)}$ as $X \to \infty$. However, for $0 \leqq \nu < 1/\sqrt{\pi}$, $u_l(X) > (1-p)^{1/(1-p)}$ for all $X > 0$ and $u_l(X) \sim (\delta/2)(1 - \sqrt{\pi}\nu)X^2$ as $X \to \infty$. For $\nu > 1/\sqrt{\pi}$, $u_l(X)$ is monotone decreasing with $u_l(X) \sim -(\delta/2)(\sqrt{\pi}\nu - 1)X^2$ as $X \to \infty$. We are now able to relate $\tilde{u}(X)$ to $u_l(X)$.

Proposition 3.9. *Let $\tilde{u}(X)$ be a solution to* IVP1 *for $X \in [0, X_e]$ for any $X_e > 0$. Then $\tilde{u}(X) \geqq u_l(X)$ and $\tilde{u}_X(X) \geqq u_{lX}(X) \, \forall X \in [0, X_e]$.*

Proof. Define the linear differential operator, $L[\cdot]$, as $L[w] \equiv w_{XX} + \frac{1}{2} X w_X - (w - (1-p)^{1/(1-p)})$, for any suitably differentiable function $w(X)$. Now, $L[u_l] = 0 \, \forall X \in [0, X_e]$ and $u_l(0) = (1-p)^{1/(1-p)} + \delta$, $u_{lX}(0) = -\nu\delta$. Also,

$$L[\tilde{u}] \equiv \tilde{u}_{XX} + \tfrac{1}{2} X \tilde{u}_X - [\tilde{u} - (1-p)^{1/(1-p)}] = -\left[\tilde{u}^p - \frac{1}{1-p}\tilde{u}\right]_+ - [\tilde{u} - (1-p)^{1/(1-p)}] \geqq 0$$

$\forall -\infty < \tilde{u} < \infty$ and hence $\forall X \in [0, X_e]$. Moreover, $\tilde{u}(0) = u_l(0)$ and $\tilde{u}_X(0) = u_{lX}(0)$; thus we can apply the comparison theorem for initial value problems with linear ordinary differential operators (see, for example, Protter and Weinberger [**23**, Ch. 1, Theorem 13, p. 26]) to $\tilde{u}(X)$ and $u_l(X)$ in $[0, X_e]$ to obtain, $u_l(X) \leqq \tilde{u}(X)$ and $u_{lX}(X) \leqq \tilde{u}_x(X)\ \forall X \in [0, X_e]$, as required. □

We next consider a further initial value problem,

$$u^l_{XX} + \tfrac{1}{2} X u^l_X - N(u^l)_+ = 0, \quad X > 0, \tag{3.10}$$

$$u^l(0) = (1-p)^{1/(1-p)} + \delta, \quad u^l_X(0) = -\nu\delta \tag{3.10, 11}$$

where $N = 1 + \text{Int}(1/1-p)$, and we shall henceforth refer to this initial value problem as LIVP2. The solution to LIVP2 is readily obtained. For $\nu \leqq 2^{2N-1}(N!)^2/(2N)!\sqrt{\pi}$ then $u^l(X)$ is always positive in $X > 0$ and,

$$u^l(X) = (1-p)^{1/(1-p)} + \nu\delta A_{2N}(x) \int_X^\infty \frac{e^{-s^2/4}}{A_{2N}^2(s)} ds + \delta \left\{ 1 - \frac{(2N)!\sqrt{\pi}\nu}{2^{2N-1}(N!)^2} \right\} A_{2N}(X), \quad X > 0 \tag{3.12}$$

where $A_{2N}(X) = \sum_{r=0}^N N![(2r)!(N-r)!]^{-1} X^{2r}$. It is readily shown from (3.12) that in this case $u^l(X) > (1-p)^{1/(1-p)}\ \forall X > 0$. Now, for $\nu > 2^{2N-1}(N!)^2/(2N)!\sqrt{\pi}$, then there is a point $X = X^* > 0$ at which $u^l(X^*) = 0$, with $u^l(X) > 0$ for $0 \leqq X < X^*$ and $u^l(X) < 0$ for $X > X^*$. In this case $u^l(X)$ is given by (3.12) for $0 \leqq X \leqq X^*$, but has the form

$$u^l(X) = u^l_X(X^*) \int_{X^*}^X e^{-((s^2 - X^{*2})/4)} ds \tag{3.13}$$

for $X > X^*$. We note that, in this case,

$$u^l(X) \to u^l_X(X^*) \int_{X^*}^\infty e^{-((s^2 - X^{*2})/4)} ds < 0 \tag{3.14}$$

as $X \to \infty$. In addition we observe from (3.12, 13) that for any $\nu \geqq 0$, there is a constant $K(N)$ such that,

$$u^l(X) < (1-p)^{1/(1-p)} + \delta[1 + K(N)X^{2N}]\ \forall X \geqq 0. \tag{3.15}$$

We can now establish:

Proposition 3.16. *Let $\tilde{u}(X)$ be a solution of* IVP1 *for $X \in [0, X_e]$ for any $X_e > 0$. Then $\tilde{u}(X) \leqq u^l(X)$ and $\tilde{u}_X(X) \leqq u^l_X(X)\ \forall X \in [0, X_e]$.*

Proof. We first observe that if $\tilde{u}(X) \not> 0\ \forall X \in [0, X_e]$ then $\exists$ an $X_0 \in [0, X_e]$ such that $\tilde{u}(X_0) = 0$, $\tilde{u}(X) > 0\ \forall X \in [0, X_0]$ and $\tilde{u}(X) \equiv 0$ or $\tilde{u}(X) < 0\ \forall X \in (X_0, X_e]$. This follows from equation (3.1) and Proposition 2.5. There are now three cases to consider:

(i) First suppose $u^l(X)>0\,\forall\, X\in[0,X_e]$ and $\tilde{u}(X)>0\,\forall\, X\in[0,X_e]$. We then define the linear differential operator, $L[\cdot]$, as $L[w]\equiv w_{XX}+\frac{1}{2}Xw_X-Nw$, for any suitably differential function $w(X)$. Now via LIVP2, $L[u^l]=0\,\forall\, X\in[0,X_e]$. Also

$$L[\tilde{u}]\equiv \tilde{u}_{XX}+\tfrac{1}{2}X\tilde{u}_X-N\tilde{u}=-\left[\tilde{u}^p-\frac{1}{1-p}\tilde{u}\right]-N\tilde{u}\leqq 0$$

$\forall\,\tilde{u}>0$ and hence $\forall\, X\in[0,X_e]$. Moreover, $\tilde{u}(0)=u^l(0)$, $\tilde{u}_X(0)=u^l_X(0)$, thus we can apply the comparison theorem for linear ordinary differential operators, [**23**], to $\tilde{u}(X)$ and $u^l(X)$ in $[0,X_e]$ to obtain $\tilde{u}(x)\leqq u^l(X)$ and $\tilde{u}_X(X)\leqq u^l_X(X)\,\forall\, X\in[0,X_e]$.

(ii) Next suppose $u^l(X)>0\,\forall\, X\in[0,X_e]$, but $\tilde{u}(X)\not>0\,\forall\, X\in[0,X_e]$, and let X_0 be as defined above: For $X\in[0,X_0]$ the result follows from (i) above, whilst for $X\in[X_0,X_e]$ we apply the same argument as in (i) but using the operator $\bar{L}[w]\equiv w_{XX}+\frac{1}{2}Xw$. For $X\in[X_0,X_e]$, $\tilde{u}(X)\leqq 0$, from above. Thus, using equation (3.1), $\bar{L}[\tilde{u}]=0\,\forall\, X\in[X_0,X_e]$. However, in this case $u^l(X)>0\,\forall\, X\in[X_0,X_e]$ so that $\bar{L}[u^l]=Nu^l>0\,\forall\, X\in[X_0,X_e]$. Moreover, $u^l(X_0)\geqq\tilde{u}(X_0)$ and $u^l_X(X_0)\geqq\tilde{u}_X(X_0)$, and so the comparison theorem for linear differential operators, [**12**], gives $\tilde{u}(X)\leqq u^l(X)$ and $\tilde{u}_X(X)\leqq u^l_X(X)\,\forall\, X\in[X_0,X_e]$, as required.

(iii) Finally suppose $u^l(X)\not>0$ on $[0,X_e]$. Then $\exists$ an $X=\bar{X}_0$ such that $u^l(X)>0\,\forall\, X\in[0,\bar{X}_0)$, $u^l(\bar{X}_0)=0$ and $u^l(X)<0\,\forall\, X\in[\bar{X}_0,X_e]$, via (3.12, 13). For $X\in[0,\bar{X}_0)$ the result follows from parts (i) and (ii). Moreover we can deduce that $\tilde{u}(\bar{X}_0)\leqq u^l(\bar{X}_0)=0$ and $\tilde{u}_X(\bar{X}_0)\leqq u^l_X(\bar{X}_0)<0$ from the result on $[0,\bar{X}_0)$ and continuity of $\tilde{u}(X)$, $u^l(X)$ and first derivatives at $X=\bar{X}_0$. These conditions enable us to conclude that (via the first part of this proof and (3.12, 13)) $\tilde{u}(X)$, $u^l(X)\leqq 0\,\forall\, X\in[\bar{X}_0,X_e]$, and so via (3.10), (3.1) $\bar{L}[\tilde{u}]=\bar{L}[u^l]=0\,\forall\, X\in[\bar{X}_0,X_e]$, and the result follows via the comparison theorem.

All cases have now been considered and the proof is complete. □

Remark 3.17. On the interval $X\in[0,X_e]$, for any $X_e>0$, Propositions 3.9, 3.16 show that, $u_l(X)\leqq\tilde{u}(X)\leqq u^l(X)$, $u_{lX}(X)\leqq\tilde{u}_X(X)\leqq u^l_X(X)$, which provide *a priori* bounds on the solution of IVP1.

Having established *a priori* bounds on the solution of IVP1, we are now able to consider (for each $\delta>0$, $v\geqq 0$) global existence and uniqueness of solutions to IVP1.

Proposition 3.18. *For each* $\delta>0$ *and* $0\leqq v\leqq 1/\sqrt{\pi}$ *there exists a unique solution to* IVP1 *with* $X\in[0,X_e]$, *for any* $X_e>0$.

Proof. We first write IVP1 as the equivalent first order system

$$\left.\begin{aligned}&\tilde{u}_X=\tilde{V},\quad \tilde{V}_X=-\tfrac{1}{2}X\tilde{V}-\left[\tilde{u}^p-\frac{1}{1-p}\tilde{u}\right]_+,\quad X\in[0,X_e]\\ &\tilde{u}(0)=(1-p)^{1/(1-p)}+\delta,\quad \tilde{V}(0)=-v\delta.\end{aligned}\right\}\tag{3.19}$$

Now, via Propositions 3.9, 3.16 and 3.17, any solution of (3.19) is *a priori* bounded in $[0, X_e]$ with, for $0 \leqq v \leqq 1/\sqrt{\pi}$,

$$(1-p)^{1/(1-p)} \leqq \tilde{u}(X) \leqq (1-p)^{1/(1-p)} + \delta[1 + K(N)X_e^{2N}], \quad -\frac{\delta}{\sqrt{\pi}} \leqq \tilde{V}(X) \leqq 2N\delta K(N)X_e^{2N-1}. \tag{3.20}$$

Now let $D = \hat{R} \times [0, X_e]$, where $\hat{R}$ is the rectangle described in (3.20), and define $F: D \to \mathbb{R}^2$ as,

$$F(\tilde{u}, \tilde{V}, X) = \left(\tilde{V}, -\tfrac{1}{2}X\tilde{V} - \left[\tilde{u}^p - \frac{1}{1-p}\tilde{u}\right]_+\right).$$

It is clear that F is continuous throughout D. Moreover, since via (3.20) $\tilde{u}$ is bounded away from zero in D, then F is a differentiable function of $(\tilde{u}, \tilde{V})$ throughout D, and hence is Lipschitz continuous in $(\tilde{u}, \tilde{V})$ throughout D. Under these conditions, a repeated application of the local existence and uniqueness theorem (see, for example, Coddington and Levinson [**8**, Ch. 1., Theorem 2.3]) on the intervals $[0,\alpha], [\alpha, 2\alpha], \dots, [(s-1)\alpha, s\alpha]$ (where $\alpha = \min(X_e, b/M)$ with, $b = \frac{1}{2}(1-p)^{1/(1-p)} + 1$, $M = \max|F(\tilde{u}, \tilde{V}, X)| \,\forall (\tilde{u}, \tilde{v}, X) \in [\frac{1}{2}(1-p)^{1/(1-p)}, \frac{3}{2}(1-p)^{1/(1-p)} + \delta(1 + K(N)X_e^{2N})] \times [-(1+\delta/\sqrt{\pi}), 2N\delta K(N)X_e^{2N-1} + 1] \times [0, X_e]$, and $s \in \mathbb{N}$ with $X_e/\alpha \leqq s < X_e/\alpha + 1$) establishes existence and uniqueness on the interval $X \in [0, X_e]$, for any $X_e > 0$. □

Remark 3.19. For the above proof, in the notation of Coddington and Levinson [**8**], the rectangle R used in each local application of [**8**, Theorem 2.3], with initial conditions $(\tilde{u}_0, \tilde{v}_0)$ at X_0, is $|\tilde{u} - \tilde{u}_0| \leqq \frac{1}{2}(1-p)^{1/(1-p)}$, $|\tilde{v} - \tilde{v}_0| \leqq 1$, $|X - X_0| \leqq 1$.

The restriction $0 \leqq v \leqq 1/\sqrt{\pi}$ in Proposition 3.18 can be removed as follows:

Extension 3.20. *For $v > 1/\sqrt{\pi}$ existence can again be established on $[0, X_e]$, for any $X_e > 0$, via the a priori bounds of Propositions* 3.9, 3.16 *and the Cauchy–Peano local existence theorem* ([**8**, Ch. 1, Theorem 1.2]). *However, in this case the lower bound on $\tilde{u}$ is* **negative**, *and so uniqueness cannot be guaranteed immediately as now F is not Lipschitz continuous in $(\tilde{u}, \tilde{v})$ throughout D (D now contains part of the plane $\tilde{u} = 0$). Despite this, uniqueness can still be established.*

Proof (of uniqueness for $v > 1/\sqrt{\pi}$). Suppose $\tilde{u}(X; v)$ is a solution of IVP1 with $v > 1/\sqrt{\pi}$ and $X \in [0, X_e]$. There are two cases to consider,

(i) $\tilde{u}(X; v) > 0 \,\forall X \in [0, X_e]$

Uniqueness follows from applying the local uniqueness result ([**8**, Ch. 1, Theorem 2.3]) at each $X_0 \in [0, X_e]$, as $F(\tilde{u}, \tilde{V}, X)$ is locally Lipschitz continuous at each such point $(\tilde{u}(X_0), \tilde{V}(X_0), X_0)$, since $\tilde{u}(X; v)$ is bounded away from zero.

(ii) $\tilde{u}(X; v) \not> 0 \,\forall X \in [0, X_e]$

In this case $\exists\ X^* \in (0, X_e]$ such that $\tilde{u}(X^*; v) = 0$ and $\tilde{u}(X, v) > 0\ \forall\ X \in [0, X^*)$. Uniqueness for $X \in [0, X^*)$ follows as in (i) above. At $X = X^*$, $\tilde{u}_X(X^*; v) \leqq 0$. With $\tilde{u}_X(X^*; v) = 0$, then via equation (3.1) and Proposition 2.5, we deduce that $\tilde{u}(X; v) \equiv 0$ for $X \in [X^*, X_e]$, and uniqueness follows on this interval. The remaining possibility is that $\tilde{u}_X(X^*; v) < 0$, when for $X \in [X^*, X_e]$, $\tilde{u}(X; v)$ satisfies the initial value problem (via IVP1),

$$\tilde{u}_{XX} + \tfrac{1}{2} X \tilde{u}_X = 0, \quad X \in [X^*, X_e]$$

$$\tilde{u}(X^*; v) = 0, \quad \tilde{u}_X(X^*; v) = -\alpha^*$$

for some $\alpha^* > 0$. This has the unique solution

$$\tilde{u}(X; v) = -\alpha^* \int_{X^*}^{X} e^{-((s^2 - X^{*2})/4)} ds < 0.$$

$\forall\ X \in [X^*, X_e]$, and the result is established. □

The next stage is to examine the closeness of solutions to IVP1 and LIVP1. We begin with:

Lemma 3.21. *Let $\tilde{u}(X)$ and $u_l(X)$ be solutions of* IVP1 *and* LIVP1 *respectively on* $[0, X_e]$, *for any $X_e > 0$, then,*

(i) $0 \leqq H(u_l) - H(\tilde{u}) \leqq \dfrac{1}{1-p}(\tilde{u} - u_l), \quad 0 \leqq v \leqq 1/\sqrt{\pi},$

(ii) $0 \leqq H_l(u_l) - H(u_l) \leqq \Lambda(p)\delta^2, \quad v = 1/\sqrt{\pi},$

for all $X \in [0, X_e]$. Here $H(w) = [w^p - (1/(1-p))w]_+$ and $H_l(w) = -[w - (1-p)^{1/(1-p)}]$, with $\Lambda(p) = \frac{1}{2}p(1-p)^{-1/(1-p)}$.

Proof. (i) Via Proposition 3.9 we have $\tilde{u}(X) \geqq u_l(X)\ \forall\ X \in [0, X_e]$. Moreover $u_l(X) > (1-p)^{1/(1-p)}\ \forall\ X > 0$. Hence $\tilde{u}(X) \geqq (1-p)^{1/(1-p)}\ \forall\ X \in [0, X_e]$. Now, $H(w)$ is strictly monotone decreasing in w for $w > (1-p)^{1/(1-p)}$. Therefore $[H(u_l(X)) - H(\tilde{u}(X))] \geqq 0\ \forall\ X \in [0, X_e]$. In addition, $H(w)$ is also Lipschitz continuous in $w > (1-p)^{1/(1-p)}$ (it is differentiable, with bounded derivative $|H'(w)| \leqq (1/1-p)\ \forall\ w \geqq (1-p)^{1/(1-p)}$). Thus $[H(u_l(X)) - H(\tilde{u}(X))] \leqq (1/1-p)[\tilde{u}(X) - u_l(X)]\ \forall\ X \in [0, X_e]$, as required.

(ii) We note first that when $v = 1/\sqrt{\pi}$, $u_l(X)$ is monotone decreasing in $X \geqq 0$, with $u_l(X) \to (1-p)^{1/(1-p)}$ as $X \to \infty$. Also, in $w \geqq (1-p)^{1/(1-p)}$, $H_l(w) - H(w)$ is positive and monotone increasing. Therefore $0 \leqq H_l(u_l(X)) - H(u_l(X)) \leqq H_l(u_l(0)) - H(u_l(0)) \leqq \Lambda(p)\delta^2\ \forall\ X \in [0, X_e]$, on using $u_l(0) = (1-p)^{1/(1-p)} + \delta$ and Taylor's theorem with remainder. □

Extension 3.22. *The inequality (i) also holds for $v > 1/\sqrt{\pi}$, but only extends to the maximal interval $[0, \hat{X}_0]$, where $\hat{X}_0$ is the unique, positive value of X with $u_l(\hat{X}_0) = [p(1-p)]^{1/(1-p)} < (1-p)]^{1/(1-p)}$. Note that $\hat{X}_0$ depends on δ and v.*

The inequality (ii) holds for $v > 1/\sqrt{\pi}$, but only extends to the maximal interval $[0, \hat{X}_1]$, where $\hat{X}_1$ is the unique, positive value of X with $u_l(\hat{X}_1) = \max\{(1-p)^{1/(1-p)} - \delta,$

$[p(1-p)]^{1/(1-p)}\}$. *This inequality also holds for* $0 \leqq \nu < 1/\sqrt{\pi}$, *but only extends to the maximal interval* $[0, \hat{X}_2]$, *where* $\hat{X}_2$ *is the unique, positive value of* X *with* $u_l(\hat{X}_2) = (1-p)^{1/(1-p)} + \delta$. *Again we note that both* $\hat{X}_1$ *and* $\hat{X}_2$ *will depend on* δ *and* ν.

We next write IVP1 and LIVP1 as equivalent first order systems,

$$\left.\begin{aligned} \tilde{u}_X = \tilde{v}, \tilde{v}_X = -\tfrac{1}{2}X\tilde{v} - H(\tilde{u}) \quad ; \quad X > 0 \\ u_{lX} = v_l, v_{lX} = -\tfrac{1}{2}Xv_l - H_l(u_l); \quad X > 0 \end{aligned}\right\} \tag{3.23}$$

subject to $u_l(0) = \tilde{u}(0) = (1-p)^{1/(1-p)} + \delta$, $v_l(0) = \tilde{v}(0) = -\nu\delta$. On defining $W(X) = (\tilde{u}(X) - u_l(X), \tilde{v}(X) - v_l(X))^T$, we readily find from (3.23) that $W(X)$ satisfies the following initial value problem

$$W_X = A(X)W + g(W), \quad W(0) = 0, \quad X > 0, \tag{3.24}$$

where

$$A(X) = \begin{pmatrix} 0 & 1 \\ 0 & -\tfrac{1}{2}X \end{pmatrix}, \quad g(W) = (0, H_l(u_l) - H(\tilde{u}))^T. \tag{3.25}$$

The initial value problem (3.24) is equivalent to the integral equation,

$$W(X) = \int_{s=0}^{s=X} B(X)B^{-1}(s)g(W(s))\,ds, \quad X > 0, \tag{3.26}$$

where $B(X)$ is a fundamental matrix for the system $Y_X = A(X)Y$, and can be taken as,

$$B(X) = \begin{pmatrix} 1 & \sqrt{\pi}\, erfc(\tfrac{1}{2}X) \\ 0 & -e^{-X^2/4} \end{pmatrix}. \tag{3.27}$$

On substitution into (3.26) using (3.25) and (3.27) we arrive at,

$$W(X) = \int_{s=0}^{s=X} [\sqrt{\pi}[erf(\tfrac{1}{2}X) - erf(\tfrac{1}{2}s)], e^{-(1/4X^2)}]^T e^{(1/4s^2)} \times [H_l(u_l(s)) - H(\tilde{u}(s))]\,ds, \quad X > 0,$$

which leads directly to the inequality,

$$\begin{aligned} |W(X)| \leqq \int_{s=0}^{s=X} \{\sqrt{\pi}[erf(\tfrac{1}{2}X) - erf(\tfrac{1}{2}s)] + e^{-(1/4X^2)}\} e^{(1/4s^2)} \\ \times |H_l(u_l(s)) - H(\tilde{u}(s))|\,ds, \quad X > 0, \end{aligned} \tag{3.28}$$

We can now establish:

Proposition 3.29. *Let* $\tilde{u}(X)$ *and* $u_l(X)$ *be solutions of* IVP1 *and* LIVP1 *respectively. Then for any* $\delta>0$, $v\geqq 0$,

$$\left.\begin{array}{l}|\tilde{u}(X)-u_l(X)|\\ |\tilde{u}_X(x)-u_{lX}(X)|\end{array}\right\}\leqq\tfrac{1}{2}\Lambda(p)\delta^2X_e(X_e+2)\exp\left\{\frac{(X_e+1)}{(1-p)}X\right\} \tag{3.29}$$

for all $X\in[0,X_e]$ (*where* X_e, *when necessary, is restricted to those values allowable for Lemma* 3.21 *to hold*).

Proof. From (3.28) we have immediately that

$$|W(X)|\leqq\int_{s=0}^{s=X}[(X-s)+1]\,|H_l(u_l(s))-H(\tilde{u}(s))|\,ds,\quad X>0. \tag{3.30}$$

Now, for $X\in[0,X_e]$ (with X_e, if necessary, restricted so that Lemma 3.21 holds) we have, via Lemma 3.21 and (3.22),

$$\begin{aligned}0\leqq H_l(u_l)-H(\tilde{u})&=[H_l(u_l)-H(u_l)]+[H(u_l)-H(\tilde{u})]\\&\leqq\frac{1}{(1-p)}(\tilde{u}-u)+\Lambda(p)\delta^2,\end{aligned} \tag{3.31a}$$

$\forall s\in[0,X]\subseteq[0,X_e]$. Thus, using (3.31a) in (3.30) we arrive at,

$$|W(X)|\leqq\int_{s=0}^{s=X}[(X-s)+1]\left\{\frac{1}{(1-p)}|W(s)|+\Lambda(p)\delta^2\right\}ds,$$

$\forall X\in[0,X_e]$. This leads to,

$$|W(X)|\leqq\frac{(X_e+1)}{(1-p)}\int_{s=0}^{s=X}|W(s)|\,ds+\Lambda(p)\delta^2(\tfrac{1}{2}X_e^2+X_e), \tag{3.31b}$$

$\forall X\in[0,X_e]$. It is now straighforward to apply the Gronwall inequality (see for example, Hirsch and Smale [**13**, Ch. 8, §4]) to (3.31b), to obtain,

$$|W(X)|\leqq\tfrac{1}{2}\Lambda(p)\delta^2X_e(X_e+2)\exp\left\{\frac{(X_e+1)}{(1-p)}X\right\},$$

$\forall X\in[0,X_e]$, as required. □

Remark 3.32. For any finite (allowable) X_e, Proposition 3.29 implies that $|\tilde{u}(X)-u_l(X)|$, $|\tilde{u}_X(X)-u_{lX}(X)|=0(\delta^2)$ uniformly on $X\in[0,X_e]$ as $\delta\to 0^+$ for fixed $v\geqq 0$.

We next make use of Proposition 3.29 to examine the behaviour of the solution to IVP1 with varying $\nu \geqq 0$, at a fixed $\delta \geqq 0$. We first recall that, for $\nu > 1/\sqrt{\pi}$, then $u_l(X)$ is monotone decreasing in X with $u=0$ at $X = X_c(\nu,\delta)$ where, from (3.7), (3.8),

$$X_c(\nu,\delta) \sim \begin{cases} \dfrac{2(1-p)^{1/(1-p)}}{\delta\sqrt{\pi}(\nu - 1/\sqrt{\pi})} & \text{as} \quad \nu \to \dfrac{1^+}{\sqrt{\pi}} \\ \dfrac{(1-p)]^{1/(1-p)}}{\delta \nu} & \text{as} \quad \nu \to \infty. \end{cases} \tag{3.33}$$

Recall also that for $\nu > 1/\sqrt{\pi}$, $\hat{X}_1(\nu,\delta)$ is defined so that $u_l(\hat{X}_1) = \text{Max}\,\{(1-p)^{1/(1-p)} - \delta, (p(1-p))^{1/(1-p)}\}$, and in this case Proposition 3.29 applies for $X \in [0, \hat{X}_1(\nu,\delta)]$. Hence, applying Proposition 3.29 at $X = \hat{X}_1(\nu,\delta)$ we obtain

$$\tilde{u}(\hat{X}_1(\nu,\delta)) - (1-p)^{1/(1-p)} \leqq \max\{-\delta, -(1-p)^{1/(1-p)}(1-p^{1/(1-p)})\}$$
$$+ \tfrac{1}{2}\Lambda(p)\delta^2 \hat{X}_1(\hat{X}_1 + 2)\exp\left\{\frac{(\hat{X}_1+1)\hat{X}_1}{(1-p)}\right\}, \tag{3.34}$$

We also note, via (3.7), that,

$$\hat{X}_1(\nu,\delta) \to \begin{cases} \infty & \text{as} \quad \nu \to 1^+/\sqrt{\pi} \\ 0 & \text{as} \quad \nu \to \infty, \end{cases} \tag{3.35}$$

with $\hat{X}_1(\nu,\delta)$ being a monotone decreasing function of $\nu > 1/\sqrt{\pi}$. Therefore, for a fixed $\delta > 0$, we observe from (3.34), (3.35) that there exists a $\nu = \nu_u(\delta) > 1/\sqrt{\pi}$ (with $\nu_u(\delta) \to 1^+/\sqrt{\pi}$ as $\delta \to 0^+$) such that,

$$\tilde{u}(\hat{X}_1(\nu,\delta)) < (1-p)^{1/(1-p)} \,\forall\, \nu \in (\nu_u(\delta), \infty). \tag{3.36}$$

Moreover (via equation (3.1), the only turning point of $\tilde{u}$ with $0 < \tilde{u} < (1-p)^{1/(1-p)}$ can be a local maximum) we may also infer that,

$$\tilde{u}_X(\hat{X}_1(\nu,\delta)) < 0 \,\forall\, \nu \in (\nu_u(\delta), \infty). \tag{3.37}$$

Thus, using (3.36, 37) and equation (3.1), it is clear that for each $\nu \in (\nu_u(\delta), \infty)$, then $\tilde{u}(X)$ is monotone decreasing for $0 < X < X^*(\nu)$ $(X^*(\nu) > X_c(\nu))$ with $\tilde{u}(X^*(\nu)) = 0$. For $X > X^*(\nu)$ we have (from (3.1) directly),

$$\tilde{u}(X) = u_X(X^*(\nu)) \int_{X^*(\nu)}^{X} e^{-(s^2 - X^{*2})/4}\, ds,$$

with,

$$\tilde{u}(X) \to u_X(X^*(v)) \int_{X^*(v)}^{\infty} e^{-(s^2 - X^{*2})/4} \, ds \leqq 0,$$

as $X \to \infty$.

We now consider the case when $\underline{0 \leqq v < 1/\sqrt{\pi}}$. In this case recall that $u_l(X) > (1-p)^{1/(1-p)}$ in $X > 0$ and has a single turning point, which is a local minimum, with $u_l(X) \to \infty$ as $X \to \infty$. Moreover, there exists a unique point $X = \hat{X}_2(v, \delta)$ with $u_l(\hat{X}_2) = (1-p)^{1/(1-p)} + \delta$, and Proposition 3.29 holds for $X \in [0, \hat{X}_2]$. We also observe that (via (3.7)),

$$\hat{X}_2(v, \delta) \to \begin{cases} \infty & \text{as} \quad v \to 1^-/\sqrt{\pi} \\ 0 & \text{as} \quad v \to 0^+. \end{cases} \tag{3.38}$$

Thus, for fixed $\delta > 0$, $\tilde{u}(\hat{X}_2) > u_l(\hat{X}_2) > (1-p)^{1/(1-p)}$ and $\tilde{u}_X(\hat{X}_2) > u_{lX}(\hat{X}_2) > 0$, via Proposition 3.9, for all $v \in [0, 1/\sqrt{\pi})$. These conditions imply (using (3.1)) that $\tilde{u}(X)$ is monotone increasing in $X > \hat{X}_2(v, \delta)$ with $\tilde{u}(X) \to \infty$ as $X \to \infty, \forall v \in [0, 1/\sqrt{\pi})$. We have thus established:

Lemma 3.39. *For any $\delta > 0$, then,*

(i) *with $v \in (v_u(\delta), \infty)$, $\tilde{u}(X)$ is monotone decreasing with $\tilde{u}(X) \to \tilde{u}_\infty \leqq 0$ as $X \to \infty$. Here $v_u(\delta) > 1/\sqrt{\pi} \, \forall \delta > 0$, with $v_u(\delta) \to 1^+/\sqrt{\pi}$ as $\delta \to 0^+$.*

(ii) *with $v \in [0, 1/\sqrt{\pi})$, $\tilde{u}(X) > (1-p)^{1/(1-p)} \, \forall X > 0$. Moreover, $\tilde{u}(X)$ is monotone increasing in $X > \hat{X}_2(v)$, and $\tilde{u}(X) \to \infty$ as $X \to \infty$.*

In what follows we regard $\delta > 0$ as fixed and write $\tilde{u}(X) = \tilde{u}(X, v)$ as we wish to explore the dependence of $\tilde{u}$ on the parameter $v \geqq 0$.

Lemma 3.40. *Let $I_\delta = \{v \in \mathbb{R}^+ \cup \{0\} : \tilde{u}(X, v) \geqq (1-p)^{1/(1-p)} \, \forall X \geqq 0\}$, then $I_\delta = [0, v^*(\delta)]$ for some $1/\sqrt{\pi} \leqq v^*(\delta) \leqq v_u(\delta)$.*

Proof. We have already shown that $[0, 1/\sqrt{\pi}) \subseteq I_\delta$. Thus $\inf(I_\delta) = 0$ and putting $\sup(I_\delta) = v^*(\delta)$, then, $1/\sqrt{\pi} \leqq v^*(\delta) \leqq v_u(\delta)$. We now show that I_δ is connected. Suppose that $v_1 \in I_\delta$ ($v_1 > 0$), then $\tilde{u}_1(X) \equiv \tilde{u}(x, v_1) \geqq (1-p)^{1/(1-p)} \, \forall X \geqq 0$. Also let $0 < v_0 < v_1$ with $\tilde{u}_0(X) \equiv \tilde{u}(X, v_0)$. From equation (3.1),

$$\tilde{u}_0'' + \tfrac{1}{2} X \tilde{u}_0' + \left(\tilde{u}_0^p - \frac{1}{1-p} \tilde{u}_0 \right)_+ = \tilde{u}_1'' + \tfrac{1}{2} X \tilde{u}_1' + \left(\tilde{u}_1^p - \frac{1}{1-p} \tilde{u}_1 \right)_+ = 0$$

$\forall X \geqq 0$, and $\tilde{u}_0(0) = \tilde{u}_1(0)$, $\tilde{u}_0'(0) > \tilde{u}_1'(0)$. Thus, via the nonlinear comparison theorem for ordinary differential operators (Protter and Weinberger [**23**, Ch. 1, §9, Theorem 23]) we have $\tilde{u}_0(X) \geqq \tilde{u}_1(X) \, \forall X \geqq 0$ and therefore $v_0 \in I_\delta$. We conclude that I_δ is connected. Finally, we must show that $v^*(\delta) \in I_\delta$ (and hence that $I(\delta)$ is closed). If we suppose that

$v^*(\delta)\notin I_\delta$, then $\exists$ an $\bar{X}>0$ such that $\tilde{u}(\bar{X},v^*)<(1-p)^{1/(1-p)}$. However, $\tilde{u}(\bar{X},v)\geqq(1-p)^{1/(1-p)}\,\forall\,0\leqq v<v^*(\delta)$, and so $\tilde{u}(\bar{X},v)$ cannot be continuous in v at $v=v^*(\delta)$. This contradicts continuous dependence at $X=\bar{X}$ of the solution of IVP1 on initial conditions (Coddington and Levinson [**8**, Ch. 1, §7, Theorem 7.1]), and hence $v^*(\delta)\in I_\delta$. The result follows. □

For each $\delta>0$, we now consider the solution of IVP1 with $v=v^*(\delta)$.

Lemma 3.41. *The solution $\tilde{u}(X,v^*)$ of* IVP1 (*for any $\delta>0$*) *is monotone decreasing in $X\geqq 0$ and has $\tilde{u}(X,v^*)\to(1-p)^{1/(1-p)}$ as $X\to\infty$.*

Proof. Suppose $\tilde{u}(X,v^*)\to\infty$ as $X\to\infty$, then $\exists$ an $X^*>0$ such that $\tilde{u}(X^*,v^*)>1+\delta+(1-p)^{1/(1-p)}$. However $\forall\,v>v^*(\delta)$, $\tilde{u}(X^*,v)<\delta+(1-p)^{1/(1-p)}$. This contradicts continuity of $\tilde{u}(X^*,v)$ on v at $v=v^*(\delta)$. Therefore we conclude that $\tilde{u}(X,v^*)$ must remain bounded as $X\to\infty$. Since $\tilde{u}(X,v^*)\geqq(1-p)^{1/(1-p)}$, then (via (3.1)) $\tilde{u}(X,v^*)$ can have at most one turning point in $X>0$, which must be a local minimum. We suppose that $\tilde{u}(X,v^*)$ has a local minimum at $X=X_m>0$ with $\tilde{u}(X_m,v^*)\geqq(1-p)^{1/(1-p)}$. Then $\tilde{u}(X,v^*)$ is monotone increasing and bounded above in $X>X_m$, so $\tilde{u}(X,v^*)\to u_\infty$ as $X\to\infty$, with $u_\infty>(1-p)^{1/(1-p)}$. However, this is not compatible with equation (3.1), and we conclude that $\tilde{u}(X,v^*)$ is monotone decreasing in $X\geqq 0$. Thus $\tilde{u}(X,v^*)\to u_\infty$ as $X\to\infty$ with now $(1-p)^{1/(1-p)}\leqq u_\infty<(1-p)^{1/(1-p)}+\delta$. Equation (3.1) then gives immediately $u_\infty=(1-p)^{1/(1-p)}$, as required. □

Remark 3.42. It follows from Lemmas 3.40, 3.41 that for all $v\in(v^*(\delta),\infty)$, then $\tilde{u}(X,v)$ is monotone decreasing in X, with $\tilde{u}(X,v)\to\tilde{u}_\infty\leqq 0$ as $X\to\infty$. Note also that $v^*(\delta)\to 1^+/\sqrt{\pi}$ as $\delta\to 0^+$ (via Lemma 3.40).

At present we have shown that for any $\delta>0$, there is at least one value of v, given by $v=v^*(\delta)$, such that the solution of IVP1 is asymptotic to $(1-p)^{1/(1-p)}$ as $X\to\infty$. We now determine that $v=v^*(\delta)$ is the only value of v for which the solution of IVP1 has this property.

Lemma 3.43. *Let $J_\delta=\{v\in\mathbb{R}^+\cup\{0\}:\tilde{u}(X,v)\to(1-p)^{1/(1-p)}$ as $X\to\infty\}$, then $J_\delta=[v_*(\delta),v^*(\delta)]$ for some $1/\sqrt{\pi}\leqq v_*(\delta)\leqq v^*(\delta)$.*

Proof. From Remark 3.42 we have immediately that $J_\delta\subseteq I_\delta$, and, via Lemma 3.41, $\sup(J_\delta)=v^*(\delta)\in J_\delta$. Let $v_*(\delta)=\inf(J_\delta)$, then $v_*(\delta)\geqq 1/\sqrt{\pi}$ via Lemma 3.39. To demonstrate that J_δ is connected, we follow the proof of Lemma 3.40 and use the nonlinear comparison theorem for ordinary differential operators, [**23**]. Finally to show that $v_*(\delta)\in J_\delta$ we again follow the proof of Lemma 3.40, using continuous dependence of the solution of IVP1 on v at fixed X, [**8**]. □

We note from Lemma 3.43 and Remark 3.42 that $v_*(\delta)\to 1^+/\sqrt{\pi}$ as $\delta\to 0^+$. Moreover we are able to show that for each $\delta>0$, J_δ has just one element.

Lemma 3.44. *For each $\delta>0$, we have $v^*(\delta)=v_*(\delta)$.*

Proof. Suppose $\exists$ a $\delta>0$ such that $v^*(\delta)\neq v_*(\delta)$. Then, by definition $v_*(\delta)<v^*(\delta)$ and $\exists$ values $v=v_0,v_1$ with $v_*(\delta)<v_0<v_1<v^*(\delta)$. Let $\tilde{u}_1(X)=\tilde{u}(X,v_1)$ and $\tilde{u}_0(X)=\tilde{u}(X,v_0)$, then, via the nonlinear comparison theorem, [**23**], it is readily deduced that,

$$\psi(X)\geqq 0\ \forall\ X\geqq 0, \tag{3.45}$$

where $\psi(X)\equiv\tilde{u}_0(X)-\tilde{u}_1(X)$ in $X\geqq 0$. Moreover, using initial conditions $(\tilde{u}_0(0)=\tilde{u}_1(0)=(1-p)^{1/(1-p)}+\delta)$ and since $v_0,v_1\in J_\delta$, then,

$$\psi(0)=0, \quad \psi(X)\to 0^+ \quad \text{as} \quad X\to\infty. \tag{3.46}$$

Also, $\psi'(0)=-\delta(v_0-v_1)>0$ and so $\exists$ an $X_+>0$ such that,

$$\psi(X)>0\ \forall\ X\in(0,X_+). \tag{3.47}$$

The conditions (3.45–47) imply that $\exists$ a point $X=X^T>0$ where $\psi(X)$ has a local maximum. Thus,

$$\psi(X^T)>0, \quad \psi'(X^T)=0, \quad \psi''(X^T)\leqq 0. \tag{3.48}$$

Now as both $\tilde{u}_1(X)$ and $\tilde{u}_0(X)$ are solutions of IVP1 with $v=v_1,v_0$ respectively, then $\psi(X)$ satisfies the following,

$$\psi''+\tfrac{1}{2}X\psi'=\frac{1}{(1-p)}[\tilde{u}_0(X)-\tilde{u}_1(X)]-[\tilde{u}_0^p(X)-\tilde{u}_1^p(X)], \tag{3.49}$$

in $X>0$. We now consider $X=X^T$. Since $\psi(X^T)>0$, then, $\tilde{u}_0(X^T)>\tilde{u}_1(X^T)>(1-p)^{1/(1-p)}$ (via definition of J_δ). Thus, using the mean value theorem,

$$\tilde{u}_0^p(X^T)-\tilde{u}_1^p(X)=p\xi^{p-1}[\tilde{u}_0(X^T)-\tilde{u}_1(X^T)],$$

where $\xi\in(\tilde{u}_1,\tilde{u}_0)$. Hence,

$$\begin{aligned}\tilde{u}_0^p(X^T)-\tilde{u}_1^p(X^T)&<\frac{p}{(1-p)}[\tilde{u}_0(X^T)-\tilde{u}_1(X^T)]\\ &<\frac{p}{(1-p)}[\tilde{u}_0(X^T)-\tilde{u}_1(X^T)].\end{aligned} \tag{3.50}$$

Next, evaluating (3.49) at $X=X^T$, using (3.50), we arrive at, $\psi''(X^T)=(1/(1-p))\{\tilde{u}_0(X^T)-\tilde{u}_1(X^T)\}-\{\tilde{u}_0^p(X^T)-\tilde{u}_1^p(X^T)\}>0$, which contradicts the last of (3.48). We conclude that $v^*(\delta)\equiv v_*(\delta)\ \forall\ \delta>0$, as required. □

In the light of the above lemma we introduce the notation $v_c(\delta)=v_*(\delta)=v^*(\delta)$. We can now state:

Theorem 3.51. *For each $\delta>0$ the solution of* IVP1 *is such that $\tilde{u}(X,v)\to(1-p)^{1/(1-p)}$ as $X\to\infty$ if and only if $v=v_c(\delta)$. Moreover, $\tilde{u}(X,v_c(\delta))$ is monotone decreasing in $X\geqq 0$, and $v_c(\delta)\geqq 1/\sqrt{\pi}\,\forall\,\delta>0$, with $v_c(\delta)\to 1^+/\sqrt{\pi}$ as $\delta\to 0^+$.*

Proof. Follows directly from Lemmas 3.39–3.44. □

The above theorem concludes our analysis of IVP1.

4. The initial value problem in $X<0$

In this section we consider the initial value problem,

$$\bar{u}_{XX}+\tfrac{1}{2}X\bar{u}_X+\left[\bar{u}^p-\frac{1}{1-p}\bar{u}\right]_+=0,\quad X<0, \tag{4.1}$$

$$\bar{u}(0)=(1-p)^{1/(1-p)}+\delta,\quad \bar{u}_X(0)=-v\delta, \tag{4.2}$$

with δ, $v\geqq 0$, which we henceforth refer to as IVP2. Again, we can make the following remark.

Remark 4.3. With $\delta=0$, IVP2 has the global solution $\bar{u}(X)\equiv(1-p)^{1/(1-p)}\,\forall\,X\leqq 0$. It follows that this is unique ([**8**, Ch. 1, Theorem 2.2]), from application of the local uniqueness theorem.

To proceed further we re-write IVP2 in terms of $\zeta=-X$,

$$\bar{u}_{\zeta\zeta}+\tfrac{1}{2}\zeta\bar{u}_\zeta+\left[\bar{u}^p-\frac{1}{1-p}\bar{u}\right]_+=0,\quad \zeta>0, \tag{4.4}$$

$$\bar{u}(0)=(1-p)^{1/(1-p)}+\delta,\quad \bar{u}_\zeta(0)=v\delta, \tag{4.5}$$

which we refer to as $\overline{\text{IVP2}}$. This now falls into the same class as IVP1 (with v replaced by $-v$), and we have the following:

Theorem 4.6. *For each $\delta>0$ and $v\geqq 0$,* IVP2 *has a unique solution in $X<0$. Moreover, this solution is monotone decreasing in X with $\bar{u}(X)\to+\infty$ as $X\to-\infty$.*

Proof. We work with the equivalent problem $\overline{\text{IVP2}}$ in $\zeta>0$, with solution $\bar{u}(\zeta)$. We define $\bar{u}_l(\zeta)$ and $\bar{u}^l(\zeta)$ as before, except we replace conditions (3.6), (3.11) by $\bar{u}_{l\zeta}(0)=$

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$\bar{u}^l_\zeta(0)=v\delta$. Similarly following the proofs of Propositions 3.9, 3.16, we readily establish that on any interval $[0,\zeta_e]$, $\bar{u}_l(\zeta)$ and $\bar{u}^l(\zeta)$ provide lower and upper bounds on $\bar{u}(\zeta)$ respectively. These *a priori* bounds then enable existence and uniqueness for $\overline{\text{IVP2}}$ to be established in $[0,\zeta_e]$ for any $\zeta_e>0$. Now, since $\bar{u}(0)>(1-p)^{1/(1-p)}$ and $\bar{u}_\zeta(0)\geqq 0$, an examination of equation (4.4) establishes directly that $\bar{u}(\zeta)$ is monotone increasing in $\zeta>0$. Moreover, since $\bar{u}(\zeta)\geqq\bar{u}_l(\zeta)$ in $\zeta>0$, then $\bar{u}(\zeta)\to\infty$ as $\zeta\to\infty$, as required. □

We are able to use the information that $\bar{u}(X)\to\infty$ as $X\to-\infty$ to obtain the asymptotic form,

$$\bar{u}(X)\sim A(\delta,v)(-X)^{2/(1-p)} \quad \text{as} \quad X\to-\infty, \tag{4.7}$$

with $A(\delta,v)>0$ for any $\delta>0$, $v\geqq 0$. We can now return to the original problem BVP.

5. The boundary value problem BVP

We first return to $\overline{\text{BVP}}$. Through Proposition 2.7, we observe that any solution, $U(X)$, to $\overline{\text{BVP}}$ has $U(0)>(1-p)^{1/(1-p)}$ and $U_X(0)<0$. Thus we may write for any solution to $\overline{\text{BVP}}$,

$$U(0)=(1-p)^{1/(1-p)}+\delta, \quad U_X(0)=-v\delta, \tag{5.1}$$

for some δ, $v>0$. This leads us to:

Theorem 5.2. *There is a bijection between solutions to* $\overline{\text{BVP}}$ *and those pairs* $(\delta,v)\in\mathbb{R}^+\times\mathbb{R}^+$ *for which* IVP1 *has a solution* $\tilde{u}(X,v)$ *with* $\tilde{u}(X,v)\to(1-p)^{1/(1-p)}$ *as* $X\to\infty$.

Proof. Let $S\subseteq\mathbb{R}^+\times\mathbb{R}^+$ be defined by,

$$S=\{(\delta,v):\text{IVPI has a solution at }(\delta,v)\text{ with }\tilde{u}(X,v)\to(1-p)^{1/(1-p)}\text{ has }X\to\infty\},$$

and,

$$B=\{\hat{u}:\mathbb{R}\to((1-p)^{1/(1-p)},\infty):\hat{u}(X)\text{ is a solution of }\overline{\text{BVP}}\}.$$

Now define the mapping $T:B\to S$ by $T[\hat{u}(X)]=(\delta,v)$, where,

$$\delta=\hat{u}(0)-(1-p)^{1/(1-p)}, \quad v=\frac{-\hat{u}_X(0)}{[\hat{u}(0)-(1-p)^{1/(1-p)}]}.$$

Clearly, T is well-defined. We must now show that T is one-one and onto.

(i) *One-one*

Suppose $\hat{u}_1(X)$ and $\hat{u}_2(X)\in B$ and $T[\hat{u}_1(X)]=T[\hat{u}_2(X)]$. Then, by definition of T, $\hat{u}_1(0)=\hat{u}_2(0)$ and $\hat{u}_{1X}(0)=\hat{u}_{2X}(0)$. Thus, in $X\geqq 0$, both $\hat{u}_1(X)$ and $\hat{u}_2(X)$ satisfy IVP1 with the same $v, \delta > 0$ (see (5.1)). Uniqueness follows from Proposition 3.18 and Extension 3.20, and so, $\hat{u}_1(X) \equiv \hat{u}_2(X)$ in $X \geqq 0$. Similarly, Theorem 4.6 shows that $\hat{u}_1(X) \equiv \hat{u}_2(X)$ in $X < 0$. Hence $\hat{u}_1(X) = \hat{u}_2(X)\ \forall X \in \mathbb{R}$ and T is one-one.

(ii) *Onto*

Let $(\hat{\delta}, \hat{v}) \in S$, then we define,

$$\hat{u}(X) = \begin{cases} \tilde{u}(X, \hat{v}), & X \geqq 0 \\ \bar{u}(X, \hat{v}), & X < 0. \end{cases}$$

Now, via Theorem 3.51, since $\tilde{u}(X, \hat{v}) \to (1-p)^{1/(1-p)}$ as $X \to \infty$, then $\tilde{u}(X, \hat{v})$ is monotone decreasing in X with $\tilde{u}(X, \hat{v}) > (1-p)^{1/(1-p)}\ \forall X > 0$. Also, via Theorem 4.6, $\bar{u}(X, \hat{v})$ is monotone decreasing in $X < 0$ with $\bar{u}(X, \hat{v}) \to \infty$ as $X \to -\infty$ and has asymptotic form (4.7). Therefore $\hat{u}(X) \in B$ and so T is onto. □

We note that $S = \{(\delta, v): v = v_c(\delta), \delta > 0\}$, via Theorem 3.51.

Remark 5.3. The correspondence of Theorem 5.2 relates solutions of $\overline{\text{BVP}}$ uniquely to points in the positive quadrant of the (δ, v) plane.

Theorem 5.4. *The set of solutions to* $\overline{\text{BVP}}$ *consists of a one-parameter family, which can be parametrized by* $\delta > 0$. *For each* $\delta > 0\ \exists$ *a unique solution to* $\overline{\text{BVP}}$ *if and only if* $v = v_c(\delta)$. *Moreover, that solution can be constructed in terms of solutions to* IVP1, 2 *as,*

$$\hat{u}(X) = \begin{cases} \tilde{u}(X, v_c(\delta)), & X \geqq 0 \\ \bar{u}(X, v_c(\delta)), & X < 0. \end{cases} \tag{5.5}$$

Proof. The proof follows from Theorem 5.2, 3.51. □

Remark 5.6. From definition 5.5, we readily deduce that at any fixed $X \in \mathbb{R}$, $\hat{u}(X, \delta)$ is a continuous function of $\delta \geqq 0$. In addition we observe that with $\delta = 0$ in (5.15) then $\hat{u}(X, 0) \equiv (1-p)^{1/(1-p)}\ \forall X \in \mathbb{R}$, via Remarks 3.3, 4.3. Hence for fixed $X \in \mathbb{R}$ $\hat{u}(X, \delta) \to (1-p)^{1+/(1-p)}$ as $\delta \to 0^+$.

In the remaining part of the paper, we relate solutions of $\overline{\text{BVP}}$ to solutions of BVP. We begin with:

Proposition 5.7 *The function* $\chi(\delta) \equiv \delta v_c(\delta)$, *for* $\delta > 0$, *is non-decreasing. Moreover* $\chi(\delta) \geqq (1/\sqrt{\pi})\delta$ *in* $\delta > 0$, *and* $\chi(\delta) \sim (1/\sqrt{\pi})\delta$ *as* $\delta \to 0^+$.

Proof. Suppose that $\delta_1 > \delta_0 > 0$ and that $\chi(\delta_1) < \chi(\delta_0)$. Hence $\delta_1 v_1 < \delta_0 v_0$, where $v_1 = v_c(\delta_1)$, $v_0 = v_c(\delta_0)$. Then, via the nonlinear comparison theorem, **[23]**, the solution of IVP1 with $\delta = \delta_0$, $v = \delta_1 v_1 / \delta_0$ has $\tilde{u}(X) \to (1-p)^{1/(1-p)}$ as $X \to \infty$ (as it is bounded above by the solution to IVP1 with $\delta = \delta_1$, $v = v_1$ and bounded below by the solution to IVP1

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with $\delta=\delta_0$, $v=v_0$). Thus $\delta_1 v_1/\delta_0 \in J_{\delta_0}$. But $v_0 \in J_{\delta_0}$ and $v_0 \neq \delta_1 v_1/\delta_0$, which contradicts Lemmas 3.43, 3.44. Therefore $\chi(\delta_1) \geqq \chi(\delta_0)$ and the results follows. The final part follows from Theorem 3.51. □

The family of solutions to $\overline{\text{BVP}}$, (5.5), have the following behaviour as $X \to -\infty$,

$$\hat{u}(X,\delta) \sim \Phi(\delta)(-X)^{2/(1-p)} \quad \text{as} \quad X \to -\infty, \tag{5.8}$$

via (4.7), with $\Phi(\delta)=A(\delta, v_c(\delta))$, and $\Phi(\delta)>0$ for all $\delta>0$. We can establish the following properties of $\Phi(\delta)$:

Proposition 5.9. *In $\delta>0$, $\Phi(\delta)$ is monotone increasing with $\Phi(\delta)\to 0^+$ as $\delta \to 0^+$ and $\Phi(\delta)\to\infty$ as $\delta\to\infty$.*

Proof. Continuous dependence of $\hat{u}(X,\delta)$ on δ (initial conditions) establishes the continuity of $\Phi(\delta)$ with $\delta>0$. Also at $\delta=0$ (via Remark 5.6), $\hat{u}(X,\delta)\equiv 0\,\forall X\in\mathbb{R}$, and continuity of $\hat{u}(X,\delta)$ on δ for fixed X at $\delta=0$ (Remark 5.6) requires $\lim_{\delta\to 0^+}\Phi(\delta)=0$. Now, for $\delta_1>\delta_0>0$ we have (via the nonlinear comparison theorem, [**23**], in $X<0$, and Proposition 5.7) $\hat{u}(X,\delta_1)>\hat{u}(X,\delta_0)$ and $[-\hat{u}_X(X,\delta_1)]\geqq[-\hat{u}_X(X,\delta_0)]\,\forall X<0$. Thus, via (5.8) $\Phi(\delta_1)>\Phi(\delta_0)$, as required. We next show that for a given $\delta>0$, $\hat{u}_X(X,\delta)$ is monotone decreasing in $X<0$. Suppose that $\hat{u}_X(X,\delta)$ has a turning point in $X<0$, at $X=X_T$ say, then $\hat{u}_{XX}(X_T,\delta)=0$, and so, via equations (5.5), (4.1), $\hat{u}_X(X_T,\delta)= X^{T-1}[(1/(1-p))\hat{u}(X_T,\delta)-\hat{u}^p(X_T,\delta)]_+>0$ as $\hat{u}(X_T,\delta)>(1-p)^{1/(1-p)}$. However, via Theorem 4.6 and (5.5), $\hat{u}_X(X_T,\delta)<0$, which gives a contradiction. Hence $\hat{u}_X(X,\delta)$ is monotone in $X<0$, and is monotone decreasing via (5.8). It now follows directly from Proposition 4.7 (noting that $\hat{u}_X(0,\delta)=-\chi(\delta)$) and (5.8) that $\Phi(\delta)\to\infty$ as $\delta\to\infty$. This completes the proof. □

Finally we are able to return to our original boundary value problem BVP given by (1.8)–(1.10). We have:

Theorem 5.10. *For each $g_\sigma>0$,* BVP *has a unique solution. Moreover that solution is given by $V=\hat{u}(X,\delta)$, where δ is the unique, positive, root of the equation $\Phi(\delta)-g_\sigma=0$.*

Proof. The proof follows directly from properties of $\hat{u}(X;\delta)$, $\delta>0$. □

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