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UNITARIES IN SIMPLE ARTINIAN RINGS

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1. Let R be a 2-torsion free simple artinian ring with involution*. The element u of R is said to be *unitary* if u is invertible with inverse u^*. In this paper we shall be concerned with the subalgebras W of R over its centre Z such that $uWu^* \subseteq W$, for all unitaries u of R. We prove that if R has rank superior to 1 over a division ring D containing more than 5 elements and if R is not 4-dimensional then any such subalgebra W must be one of the trivial subalgebras 0, Z or R, under one of the following extra finiteness assumptions: W contains inverses, W satisfies a polynomial identity, the ground division ring D is algebraic, the involution is a conjugate-transpose involution such that D equipped with the induced involution is generated by unitaries. D. Handelman has given an example of an infinite dimensional division ring D with involution *not* generated by its unitaries (or even the bounded elements). Using Handelman's example one can show that there are instances of the considered rings R having arbitrary rank for which certain subalgebras W need not be trivial.

Our arguments revolve about centralizers and use the specific action of an assortment of unitaries. The exploitation of the latter elements requires in some instances heavy calculations we could not avoid. Since the behavior of reasonable subalgebras W (e.g. the linear span of all unitaries, the subalgebra generated by the symmetric idempotents, or even the subalgebra of all "bounded" elements) can be wild (in all fairness) a method for attacking the problem by showing that such subalgebras are preserved by commutation with respect to the skews is not generally sufficient.

The paper proceeds as follows: We are given a 2-torsion free simple artinian ring R with *. We shall turn R into the ring of $n \times n$ matrices over the corresponding division ring D, $R = D_{n\times n}$, and will deal for most of the paper (§ 1–§ 6) with an involution *, which is *canonical transpose*, that is, there is an involution – on D (*induced* involution) and n non-zero corresponding symmetrics $q_i = \bar{q}_i$ of D such that * coincides with the mapping

$$\underline{x} = [x_{ij}] \in R \rightarrow \text{Diag}\,\{q_1, \ldots, q_n\}[\bar{x}_{ji}]_{ij}\,\text{Diag}^{-1}\,\{q_1, \ldots, q_n\};$$

where Diag $\{q_1, \ldots, q_n\}$ is an $n \times n$ diagonal matrix with diagonal coefficients respectively $q_1, \ldots, q_n$. We proceed to the following basic definitions.

Definition 1. Let R be a ring with * and with unity 1 considered as an algebra over its centre Z.

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a) Recall that $u \in R$ is *unitary* if u is invertible with inverse precisely u^*.
b) The subalgebra W of R *contains inverses* if for each $x \in W$ which is invertible, $x^{-1} \in W$.
c) The subalgebra W of R is *invariant* if for each unitary u of R, $uWu^* \subseteq W$.

There now follow examples of invariant subalgebras W successively without condition b), commutative but not central, commutative not central and not diagonal.

Example 1. Let Δ be a division algebra with centre the complex numbers and involution $-$. The element a of Δ is called *bounded* if for some sequence $x_i \in \Delta$ commencing at a, $\sum x_i\bar{x}_i = k$, where k is a real number. D. Handelman has given an example of an infinite dimensional division algebra Δ with an involution $-$ such that (i) $\sum x_i\bar{x}_i = 0$ implies $x_i = 0$, (ii) the subset B of bounded elements is properly contained in Δ [**2**]. Since B is a subalgebra of Δ, it follows that the linear span of the unitaries of Δ is not equal to Δ (for every unitary is evidently bounded). Take R to be $\Delta_{n\times n}$, $n > 1$. Clearly each unitary matrix u of R has coefficients in B (for $\underline{u} \cdot \underline{u}^* = 1$ gives $\sum u_{ki}\bar{u}_{ki} = 1$, so, $u_{ki} \in B$). If W is the subalgebra generated by all the unitaries of R this is evidently invariant and since $W \subseteq B_{n\times n}$, $W \neq R$ follows. Notice that from (i) follows that for every $\underline{x} \in R$, $1 + \underline{x}\,\underline{x}^*$ is not a divisor of zero. Since R is artinian, the latter element is then an invertible element for all $\underline{x} \in R$, thus answering in the negative question #13 of [**5**]. Also, since every symmetric idempotent (or projection) e has coefficients in B, we see that the subalgebra generated by the projections is proper.

Example 2. If $R = (\mathrm{GF}(5))_{2\times 2}$ with ordinary transpose then the diagonal matrices form an invariant subalgebra. For R of rank 3 over GF(5), the diagonal matrices will form, again, an invariant subalgebra under the involution

$$*: [x_{ij}] \rightarrow \mathrm{Diag}\,\{1, 1, 2\}[x_{ji}]_{i,j}\,\mathrm{Diag}^{-1}\,\{1, 1, 2\}.$$

Example 3. Let D be any field *containing more than* 5 *elements*, and let $R = D_{2\times 2}$ with the involution

$$*: [x_{ij}] \rightarrow \mathrm{Diag}\,\{1, q\}[x_{ji}]_{i,j}\,\mathrm{Diag}^{-1}\,\{1, q\},$$

where $0 \neq q$ is a fixed element. Here $\underline{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is unitary if and only if $a = d$, $c = -qb$, or $a = -d$, $c = qb$; where $a^2 + qb^2 = 1$. It can be verified that the subset W of all matrices of the form $\begin{bmatrix} x & y \\ -qy & x \end{bmatrix}$ is the unique non-trivial invariant subalgebra. We proceed to

Definition 2. If R is as in Example 3 and if W is as in Example 3, we shall call W an *R-special* subalgebra.

To be able to exploit elementarily the invariance property one must have in hand a fairly general procedure for constructing unitary matrices of rank at least 2. When $\underline{x} \in R$ is an invertible normal matrix, that is, $\underline{x} \cdot \underline{x}^* = \underline{x}^* \cdot \underline{x}$, then $\underline{x}^{-1} \cdot \underline{x}^*$ is unitary. For $\underline{x} = 1 + \underline{k}$, where $\underline{k}$ is a skew matrix, this gives a so-called Cayley unitary. These constructions hinge on the invertibility of $\underline{x}$. In this connection we state a criterion for which we could not find a reference.

Criterion. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be over the division ring D. Assume that b, cd, $ac^{-1} - bd^{-1}$, and $d^{-1}c - b^{-1}a$ are non zero. Then the matrix is invertible with inverse

$$\begin{bmatrix} (1 - c^{-1}db^{-1}a)^{-1} & 0 \\ 0 & (1 - b^{-1}ac^{-1}d)^{-1} \end{bmatrix} \begin{bmatrix} -c^{-1}db^{-1} & c^{-1} \\ b^{-1} & -b^{-1}ac^{-1} \end{bmatrix}.$$

Start with an arbitrary $x \in D$. Now

$$\underline{k} = \begin{bmatrix} 0 & x \\ -q_2\bar{x}q_1^{-1} & 0 \end{bmatrix}$$

is a skew matrix of $R = D_{2\times 2}$. This gives a Cayley unitary matrix assuming all is well for the invertibility conditions in the criterion. Since this construction will occur frequently we state

Definition 3. Let $R = D_{2\times 2}$ with canonical transpose, and let $x \in D$. The following matrix $\underline{u} = \underline{u}(x)$ is unitary provided the expressions involved are defined:

$$\underline{u} = \begin{bmatrix} (1 + xq_2\bar{x}q_1^{-1})^{-1} & 0 \\ 0 & (1 + q_2\bar{x}q_1^{-1}x)^{-1} \end{bmatrix} \begin{bmatrix} 1 - xq_2\bar{x}q_1^{-1} & -2x \\ 2q_2\bar{x}q_1^{-1} & 1 - q_2\bar{x}q_1^{-1}x \end{bmatrix}.$$

2. Centralizers. We are aiming at the determination of the invariant subalgebras which happen to be centralizers of some invariant subalgebra (or subset). We shall prove the following key theorem, which is the only result in this paper true for any ground division ring D.

THEOREM 1. *Let R be a 2-torsion free simple artinian ring of rank 2 over a division ring D with a canonical transpose involution *. Suppose that W is an invariant subalgebra with at least one matrix having distinct diagonal coefficients. Then the centralizer V of W in R must be diagonal* (*e.g. consists entirely of diagonal matrices*).

We set up additional notations and make some remarks we will be using for the proof of Theorem 1 (and elsewhere). Let $-_{(i)}: x \in D \rightarrow \bar{x}^{(i)} = q_i\bar{x}q_i^{-1}$. This is an involution on D, co-gredient, by construction, to $-$. Denote by $K^{(-)}$ (resp. $K^{(i)}$) the additive subgroup of skews under $-$ (resp. $-_{(i)}$), and by $S^{(-)}$ (resp. $S^{(i)}$) the corresponding subgroup of symmetrics. Let $\mathring{S}^{(-)}$ (resp. $\mathring{S}^{(i)}$) be the additive subgroup of elements anti-commuting with $K^{(-)}$ (resp. $K^{(i)}$).

Remark 1a) Either $\mathring{S}^{(-)} = 0$ or $\mathring{S}^{(-)}$ is a 2-dimensional subspace of D. In the latter case, D is a 4-dimensional division ring with involution of the first kind with $S \supseteq \mathring{S}^{(-)}$, a 3-dimensional subspace. Conversely any 4-dimensional division ring with $S^{(-)} \not\subseteq Z$ and with involution of the first kind has a 2-dimensional subspace $\mathring{S}^{(-)}$. Consequently $K^{(-)}$ is one-dimensional.

b) If $K^{(-)}$ is commutative but D is not then $\mathring{S}^{(-)}$ is 2-dimensional.

c) $\mathring{S}^{(-)}$ and $\mathring{S}^{(i)}$ have the same dimension.

Remark 2a) If u_i is unitary under $-_{(i)}$, then Diag $\{u_1, u_2\}$ is a unitary of R.

b) If W is an invariant additive subgroup then $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ are in W (use Diag $\{1, -1\}$).

Proof of Theorem 1. There are two disjoint cases.

Case (I). $K^{(-)}$ is non commutative. Pick $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ with $a \neq b$ (Remark 1*b*)) and let $\begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix} \in V$. It is to be shown that $t = t' = 0$. Commuting the latter matrices, we get $at = tb$ and $t'a = bt'$. If then a and b are central, we are finished. From $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ follows

$$\begin{bmatrix} a - u_1 a u_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in W$$

for u_1 any unitary under $-_{(1)}$. By the preceding, $a = u_1 a u_1^{-1}$, meaning that a commutes with all unitaries under $-_{(1)}$, and hence a commutes with all skews under $-_{(1)}$, placing a in Z. Similarly $b \in Z$. Therefore $t = t' = 0$. This shows that V is diagonal (Remark 1b)).

Case (II). $K^{(-)}$ is commutative. The above argument evidently works in case D is commutative. Thus we may assume that D is non-commutative so that D is 4-dimensional with a 2-dimensional subspace $\mathring{S}^{(-)}$ (Remark 1b)). Suppose that V is not diagonal.

IIa) It is to be shown that every matrix in V is of the form $\begin{bmatrix} x & t \\ -qt & x \end{bmatrix}$ where $q \neq 0$ is a central element, $q = q_2^{-1}q_1$ (so that $-_{(1)} = -_{(2)}$), $x \in Z + K^{(1)}$, and $t \in \mathring{S}^{(1)}$.

a)-1. It can be verified that all symmetrics in the *-subalgebra generated by W (or V) are central (Proof: Observing that the centre Z of D is infinite we can turn R into an algebraic algebra over A then use Lanski [**4**] to get that W is preserved with respect to commutation with the skews of R; since the latter elements generate R and since R has no homomorphic images of rank 3, we can apply [**1**]).

a)-2. There are distinct non-zero skews k_i under $-_{(i)}$ with $k_1^2 = k_2^2$ and $\begin{bmatrix} 0 & k_1 - k_2 \\ \delta(k_1 - k_2) & 0 \end{bmatrix} \in W$ for some $\delta \in Z$, $\delta \neq 0$. To show this select $\underline{x} =$

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ with $a \neq b$ and $0 \neq \begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix} \in V$. As in Case (I), we get that a (resp. b) commutes with $K^{(1)}$ (resp. $K^{(2)}$) and hence with the maximal subfield $Z + K^{(1)}$ (resp. $Z + K^{(2)}$), placing a in $Z + K^{(1)}$ (resp. b in $Z + K^{(2)}$). Since $\underline{x} + \underline{x}^* \in Z$, a and b have then equal central terms. Thus we may assume that a and b are skews. Then $\underline{x}^2 \in Z$ gives $a^2 = b^2$. Consequently $0 \neq a, b$. It remains to show that $\begin{bmatrix} 0 & a - b \\ \delta(a-b) & 0 \end{bmatrix} \in W$ for some $\delta \in Z$, $\delta \neq 0$. Notice that $q_1^{-1}a$ (resp. $q_2^{-1}b$) is skew under $-$. Since $K^{(-)}$ is one-dimensional there is $\lambda' \neq 0 \in Z$ such that $q_1^{-1}a = \lambda' q_2^{-1}b$, so $q = q_2 q_1^{-1} = \lambda ba$ for some $\lambda \neq 0 \in Z$. Now q is symmetric, and hence ba is symmetric under $-_{(1)}$. By symmetry, ab is symmetric under $-_{(1)}$, so $ab + ba$ is symmetric under $-_{(1)}$. Since $a^2 \in Z$, $ab + ba$ commutes with the skew a, placing it in Z. Choose $z \in Z$ with

$$1 + \lambda z^2 a^2 \neq 0,\ 1 + z^2 q \neq 0,\ 1 + \lambda z^2 q \neq 0, \quad \text{and} \quad z^2\lambda(ab + ba) + \lambda^2 z^4 a^2 b^2 \neq 0.$$

Let $\underline{u} = \underline{u}(z)$ be the unitary corresponding to z (Definition 3, § 1). Conjugating $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ by $\underline{u}$ gives a matrix of W whose off-diagonal entries will be

$$\begin{aligned} c &= 2z(1 + z^2q)^{-1}(1 - z^2q)(a - b)(1 - z^2q)(1 + z^2q)^{-1} \\ d &= 2\lambda z a^2 (1 + z^2q)^{-1}(b(1 - z^2q) - (1 - z^2q)a)(1 + z^2q)^{-1}. \end{aligned}$$

Thus $(1 + \lambda z^2 a^2)^{-1}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in W$. Since W evidently contains inverses we get that $(1 + \lambda z^2 a^2)\begin{bmatrix} 0 & d-1 \\ c^{-1} & 0 \end{bmatrix} \in W$, provided $b(1 - z^2q) - (1 - z^2q)a \neq 0$, which we may assume replacing if necessary z by $2z$. A simplification of the expressions $(1 + \lambda z^2a^2)d^{-1}$ and $(1 + \lambda z^2a^2)c^{-1}$ using the relations $(a - b)a = -b(a - b)$ and $a(a - b) = -(a - b)b$ shows that they can be written in the form $z_1(a - b)^{-1}$ and $z_2(a - b)^{-1}$ with $0 \neq z_i \in Z$. From this $\begin{bmatrix} 0 & a - b \\ z_3(a-b) & 0 \end{bmatrix} \in W$ for some $z_3 \neq 0 \in Z$.

a)-3. $q = q_2 q_1^{-1}$ is necessarily central, so that $-_{(1)}$ and $-_{(2)}$ coincide. Since W contains the non-zero off diagonal matrix $\begin{bmatrix} 0 & a - b \\ \delta(a-b) & 0 \end{bmatrix}$, we can reverse the roles of W and V. Exactly as in the previous step a)-1, we get that every diagonal matrix $\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} \in V$ is such that a', b' have equal central parts and square equal skew parts. It can be assumed that not all diagonal matrices of V are central and consequently there are $\lambda_i \in Z$, $\lambda_i \neq 0$, with

$$\begin{bmatrix} \lambda_1 a & 0 \\ 0 & \lambda_2 b \end{bmatrix} \in V.$$

Since $(\lambda_1 a)^2 = (\lambda_2 b)^2$ and $a^2 = b^2$, we get $\lambda_2 = \pm\lambda_1$, so $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or $\begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} \in V$. On commutation with $\begin{bmatrix} 0 & a-b \\ \delta(a-b) & 0 \end{bmatrix} \in W$, we get $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in V$. As in step a)-2, this gives

$$\begin{bmatrix} 0 & a+b \\ \delta_1(a+b) & 0 \end{bmatrix} \in V \quad \text{for some } \delta_1 \neq 0 \in Z.$$

On commutation with $\begin{bmatrix} 0 & a-b \\ \delta(a-b) & 0 \end{bmatrix} \in W$, we see that $(a+b)(a-b) = 0$, whence $b = -a$. Then $q = \lambda ba = -\lambda a^2 \in Z$, as desired.

a)-4. We show now that every matrix in V is of the prescribed form. For if $\begin{bmatrix} x & t \\ t' & y \end{bmatrix} \in V$, it was shown above that $x, y \in Z + K^{(1)} = Z + K^{(2)}$, with x, y having equal central terms and equal skew terms, so $x = y$. Choose any central element $\epsilon \neq 0$ with $1 + \epsilon q^2 \neq 0$. Form $\underline{u} = \underline{u}(\epsilon)$, take the $\underline{u}$-conjugate of $\begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix}$ and equate the diagonal coefficients of the resulting matrix. This gives $t' = -qt$. Taking the conjugate of this by Diag $\{u_1, 1\}$, u_1 any unitary under $-_{(1)}$, gives $tu_1^{-(1)} = u_1 t$, all u_1, if and only if $t \in \mathring{S}^{(1)} = \mathring{S}^{(2)}$.

IIb). It is to be shown that if V is of the form in IIa), V can not be invariant, thus yielding a contradiction. To make the computations slightly less complex we shall assume here that * is the conjugate-transpose involution, e.g. $q_1 = q_2 = q = 1$, so, $-_{(1)} = - = -_{(2)}$ are the same involution on D. Start with the matrix

$$\underline{m} = \begin{bmatrix} z+\sigma & t(1+\tau) \\ t(\tau-1) & z-\sigma \end{bmatrix},$$

where $z \in Z$, $\sigma, \tau \in K^{(-)}$, and $t \in \mathring{S}^{(-)}$. It can be verified that $\underline{m}$ is a normal invertible matrix for $0 \neq t$. Thus $\Omega = \underline{m}^* \cdot \underline{m}^{-1}$ is a unitary matrix. Explicitly,

$$\Omega = \begin{bmatrix} \alpha_1 & -t_1 \\ t_2 & \bar{\alpha}_1 \end{bmatrix} \begin{bmatrix} \beta_1 & 0 \\ 0 & \bar{\beta}_1 \end{bmatrix}$$

where

$$t_1 = (1-\tau)^{-2}((1+\tau)(z+\sigma)t + (1-\tau)(z-\sigma)t^{-1}),$$
$$t_2 = (1+\tau)^2((1-\tau)(z-\sigma)t + (1+\tau)(z+\sigma)t^{-1}),$$
$$\alpha_1 = t^{-2}(z-\sigma)(1-\tau)^{-2}(z+\sigma) - (1+\tau)(1-\tau)^{-1}$$
$$= t^{-2}(z^2-\sigma^2)(1-\tau)^{-1}((1-\tau)^{-1}(z+\sigma) - (1+\tau)),$$
$$\beta_1 = (1 + t^{-2}(\sigma+z)^2(1-\tau)^2)^{-1}.$$

When $\tau = -z^{-1}\sigma$ the expressions above simplify (somewhat) and become:

$$t_1 = z(z - \sigma)(z + \sigma)^{-1}(t + t^{-1}); \quad t_2 = z(z + \sigma)(z - \sigma)^{-1}(t + t^{-1}),$$
$$\alpha_1 = t^{-2}(z^2 - \sigma^2)(z^2(z + \sigma)^{-1} - (z - \sigma)/(z + \sigma))$$
$$\beta_1 = (1 + t^{-2}(z + \sigma)^2 z^{-2}(z + \sigma)^2)^{-1} = (1 + (zt)^{-2}(z + \sigma)^4).$$

Take the Ω-conjugate of $\begin{bmatrix} 0 & t_0 \\ -t_0 & 0 \end{bmatrix} \in V$ where $t_0 \neq 0 \in \mathring{S}^{(-)}$. Equating the diagonal coefficients of the resulting matrix and eliminating we get that t_0 commutes with $t_1\alpha_1 - \alpha_1 t_2$. Since $t_1\alpha_1 - \alpha_1 t_2 \in K^{(-)} + Z$, this forces $t_1\alpha_1 - \alpha_1 t_2 \in Z$. Now t_1 is of the form $x \cdot (t + t^{-1})$, where $x \in Z + K^{(-)}$, $t + t^{-1} \in \mathring{S}^{(-)}$. Consequently α_1, which also belong to $Z + K^{(-)}$, satisfies the relation

$$t_1\alpha_1 = x \cdot (t + t^{-1})\alpha_1 = x\bar{\alpha}_1(t + t^{-1}) = \bar{\alpha}_1 x(t + t^{-1}) = \bar{\alpha}_1 t_1.$$

Thus $\bar{\alpha}_1(t_1 - t_2) \in Z$. Assuming that $t_1 - t_2 \neq 0$, we get $\bar{\alpha}_1(t_1 - t_2) = (t_1 - t_2)\alpha_1 = (t_1 - t_2)\bar{\alpha}_1$, resulting in $\alpha_1 = \bar{\alpha}_1$. But if $\alpha_1 = \bar{\alpha}_1$ or if $t_1 = t_2$ it can be easily shown that Z is finite, a contradiction. This shows that every off diagonal matrix in V is zero, and consequently V is necessarily diagonal. With this the proof of Theorem 1 is completed.

3. Special invariant subalgebras. We have already given examples of commutative non-central invariant subalgebras (see Example 2, § 1) for the ground division ring GF(5). Henceforth, we shall assume that the ground division ring D contains more than 5 elements, thus eliminating Example 2. We now prove the following theorem.

THEOREM 2. *Let R be as in Theorem 1 with a ground division ring containing more than 5 elements. Let W be an invariant subalgebra.*

(i) *If W is diagonal then either $W = 0$ or Z.*

(ii) *If all matrices in W have equal diagonal coefficients then either $W = 0$ or Z, or W is an R-special subalgebra (Definition 2, § 1).*

Proof. (i). Pick $x \neq 0 \in D$ with $1 \pm xq_2\bar{x}q_1^{-1} \neq 0$. Form the unitary $\underline{u} = \underline{u}(x)$ and form $\underline{u} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \underline{u}^*$ where $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$. Equating the diagonal coefficients of the resulting matrix we get readily that either $a = b = 0$ or both a and $b \neq 0$. If u_1 is a unitary under $-_{(1)}$, then

$$\begin{bmatrix} a - u_1 a u_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in W$$

giving $a = u_1 a u_1^{-1}$ and similarly $b = u_2 b u_2^*$, all unitaries under $-_{(2)}$. If then the skews under $-$ do not always commute we get that $a, b \in Z$, and by the preceding, $a = b$, so $W \subseteq Z$ follows. If, on the other hand, the skews always commute, choose z such that $1 \pm z^2 q \neq 0$, let the unitary $\underline{u} = \underline{u}(z)$ act on $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ by conjugation, and require that the resulting matrix has zero off diagonal entries to get $a = b$. Consider the unitaries $\underline{u}_1 = \underline{u}(sz)$, $\underline{u}_2 = \underline{u}(2sz)$ associated to a given symmetric s in D under $-$, and require that the off-diagonal coefficients of the $\underline{u}_i$-conjugates of $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ are zero. A simple calculation shows that $as = sa$, and hence $a \in Z$, giving $W \subseteq Z$. Since W is a subalgebra, $W = 0$ or Z follows.

(ii) Let $W \neq 0, Z$. By (i), we may assume that W_{12} or $W_{21} \neq 0$; here W_{ij} is the subset consisting of elements occurring in the $i \times j$ coefficient of some member of W. We claim that if $0 \neq \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \in W$, then both c and d are non-zero. For if say, $c = 0$, then given any $\begin{bmatrix} 0 & c' \\ d' & 0 \end{bmatrix} \in W$ we have

$$\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & c' \\ d' & 0 \end{bmatrix} = \begin{bmatrix} cd' & 0 \\ 0 & dc' \end{bmatrix} \in W,$$

so $cd' = dc' = 0$, resulting in $c' = 0$. This means that $W_{12} = 0$. Choose $0 \neq x \in D$ with $1 + xq_2\bar{x}q_1^{-1} \neq 0$ (using cardinality and some commutativity techniques), form the conjugate of $\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}$ under the corresponding unitary $\underline{u} = \underline{u}(x)$, and observe that the 1×2-coefficient of the resulting matrix is $-4xdx$, so $xdx = 0$, whence $d = 0$, a contradiction. Our claim is established. We show now that W_{12} anti-commutes with $K^{(1)}$ (the additive subgroup of skews under the involution $-_{(1)}$). Given $0 \neq \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$, we have $\begin{bmatrix} cd & 0 \\ 0 & dc \end{bmatrix} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}^2 \in W$, so $cd = dc$. For u_1 a unitary under $-_{(1)}$,

$$\begin{bmatrix} 0 & uc \\ du^{-1} & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & 1 \end{bmatrix} \in W.$$

Then

$$\begin{bmatrix} cdu^{-1} & 0 \\ 0 & duc \end{bmatrix} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} \begin{bmatrix} 0 & uc \\ du^{-1} & 0 \end{bmatrix} \in W.$$

Consequently $cdu^{-1} = duc = dcu^{-1}$, whence $cu^{-1} = uc$ for all unitaries u. This means that c anti-commutes with $K^{(1)}$, for all $c \in W_{12}$. Suppose that $K^{(1)} \neq 0$. By Remarks 1, § 2, either $W_{12} = 0$ or D has 2-dimensional subspace $\mathring{S}^{(1)}$. As in § 2 (case IIb)) this forces $W_{12} = 0$. This shows that $K^{(1)} = 0$, so $- = -_{(1)}$ is the identity mapping and D must be a field. Thus W is an R-special subalgebra (Example 2).

Combining Theorems 1 and 2 we get the following corollary.

COROLLARY. *Every invariant subalgebra $W \neq 0$, Z which is not R-special has for centralizer precisely Z.*

4. Subalgebras containing inverses and related conditions. Start with an arbitrary invariant subalgebra $W \neq 0$, Z, which is not R-special. If we adjoin the centre Z we get an invariant subalgebra $W_1 \supsetneqq Z$. By Corollary 1, W_1 has centre Z. Moreover if N is the nil radical of W_1, then N is an invariant subalgebra. Directly or using the relation $N^2 = 0$ we obtain $N = 0$, and W_1 is then semi-prime. To be able to use the double centralizer theorem much more is needed; namely, W_1 should be a simple finite dimensional subalgebra. If, however, we assume that W satisfies a polynomial identity we may proceed. For then W_1 is PI with centre the field Z. By a result of L. Rowen, W_1 is simple and, by Kaplansky's theorem, W_1 is finite dimensional. Since $W_1 \supsetneqq Z$ and has centralizer Z, $W_1 = R$ follows. Since $W_1 = W + Z = R$, W is then a Lie ideal. By [**3**, p. 57], $W = R$ follows. We have shown the following theorem.

THEOREM 3. *Assume that W is a* PI *subalgebra not R-special, which is invariant. Then W is a trivial subalgebra:* 0, Z, *or* R.

COROLLARY. *If R has a finite dimension larger than* 4, *every invariant subalgebra is trivial.*

The Corollary above allows us to proceed to the infinite dimensional case. We then show the following theorem.

THEOREM 4. *Let W be a non R-special invariant subalgebra containing inverses. Then W is a trivial subalgebra.*

Proof. The corner subalgebra W_{ii} is a subalgebra of the division ring D with involution $-_{(i)}$, $i = 1, 2$, inheriting the properties of W; that is, invariant and containing inverses. Thus W_{ii} is an invariant division subalgebra. Since we may assume that R, and hence D, are infinite dimensional (Corollary to Theorem 3), we can then use [**3**, p. 201] to get that $W_{ii} = 0$, Z, or D. We proceed to show that either $W_{ii} \subseteq Z$, $i = 1, 2$ (in which case $W \subseteq Z$) or $W_{ii} = D$ (in which case $W = R$).

If, say, $W_{11} \subseteq Z$, we must show that $W_{22} \subseteq Z$. Let $0 \neq s = \bar{s}$ be a symmetric of D with involution $-$, and suppose that $sq_2sq_1^{-1} \neq -1$. Given an arbitrary b in W_{22} there is $a \in W_{11}$ with $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$. Then take the unitary $\underline{u} = \underline{u}(s)$, conjugate the latter matrix by this, and require that the 1×1-entry of the resulting matrix is central; namely,

$$(1 + sq_2sq_1^{-1})\{(1 - sq_2sq_1^{-1})a(1 - sq_2sq_1^{-1}) - 4sbq_2sq_1^{-1}\} \times (1 + sq_2sq_1^{-1}) \in Z.$$

Commuting this with $sq_2sq_1^{-1}$ we get

$$[sbs^{-1}, sq_2sq_1^{-1}] = [sbs^{-1}, sq_2sq_1^{-1}ss^{-1}] = 0,$$

so $[b, q_2sq_1^{-1}s] = 0$. If, on the other hand, $sq_2sq_1^{-1} = 1$, then $q_2sq_1^{-1}s = 1$, so $[b, q_2sq_1^{-1}s] = 0$ for all $s = \bar{s}$ and all $b \in W_{21} = D$. Thus $q_2sq_1^{-1}s \in Z$ for all

s. From this we get that D is finite-dimensional, which is excluded. This shows that $W_{22} \subseteq Z$.

If $W_{ii} \subseteq Z$, $i = 1, 2$, there is $a \neq b$ with $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ (otherwise W is as in (ii) of Theorem 2). Then $\begin{bmatrix} a-b & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & b-a \end{bmatrix} \in W$ giving readily $\underline{e} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\underline{e}' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$. Now there must be $0 \neq \begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix} \in W$ (otherwise W is as in (i) of Theorem 2). By an argument used in the proof of (ii) of Theorem 2, t and t' are non-zero. Then $\begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix} \underline{e} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \in W$, and similarly $\begin{bmatrix} 0 & 0 \\ t' & 0 \end{bmatrix} \in W$. Then

$$\begin{bmatrix} 0 & 0 \\ t' & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & t't \end{bmatrix} \in W$$

forces $t't \in Z$. Given a unitary u_1 under $-_{(1)}$, we have $\begin{bmatrix} 0 & tu_1 \\ 0 & 0 \end{bmatrix} \in W$. Premultiplying by $\begin{bmatrix} 0 & 0 \\ t' & 0 \end{bmatrix}$ we get $t'(tu_1) = (t't)u_1 \in Z$. Since $0 \neq t't \in Z$, we derive that $u_1 \in Z$ for all unitaries u_1 under $-_{(1)}$, which forces D to be finite-dimensional, contrary to the choice of D. This shows that $W_{11} = W_{22} = D$.

Pick $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$ with $a \notin Z$. There is a unitary u_1 under $-_{(1)}$ such that $a - u_1 a u_1^{-1} \neq 0$. Then

$$\begin{bmatrix} a - u_1 a u_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{in } W$$

gives $\underline{e} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$, and similarly $\underline{e}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in W$. Consequently $W \supseteq \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$. Choose $0 \neq \begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix}$ with $t \neq 0$. As observed earlier, $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ t' & 0 \end{bmatrix}$ are in W. For $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in W$, $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & ty \\ 0 & 0 \end{bmatrix} \in W$. Therefore $W_{12} = D$ and similarly $W_{21} = D$. Thus $W = R$, as required.

To conclude this section we show a companion theorem to Theorem 4.

THEOREM 5. *If D is a 2-torsion free division ring with involution $-$, which is generated by its unitaries, then for $R = D_{2\times 2}$ with $*$ the conjugate-transpose involution, then every invariant subalgebra W is 0, Z or R.*

Proof. As in the proof of Theorem 4, it can be shown that if $Z \subsetneq W$, then neither $W_{11} \subseteq Z$ nor $W_{22} \subseteq Z$. Thus W contains a pair of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$ with $a \notin Z$ and $b' \notin Z$. By an obvious conjugate argument we derive that $\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix} \in W$ for some $\alpha, \beta \notin Z$. Given $\begin{bmatrix} 0 & t \\ t' & 0 \end{bmatrix} \in W$ with $t \neq 0$, we have $\begin{bmatrix} 0 & \alpha t \\ 0 & 0 \end{bmatrix} \in W$, and consequently $\begin{bmatrix} 0 & \alpha t u_1 \\ 0 & 0 \end{bmatrix} \in W$, for all unitaries u_1 under the induced involution $-$. Since W_{12} is a subspace it follows that $W_{12} \supseteq \alpha t \cdot B$ where B is the linear span of the unitaries. Since by hypothesis $B = D$, $W_{12} \supseteq \alpha t B = \alpha D = D$, so $W_{12} = D$, and similarly $W_{21} = D$. This gives, as in Theorem 4, that $W = R$.

5. Higher rank. We shall carry over the forgoing results to the matrix rings R of rank larger than 2. For such ranks it will be shown that the invariant commutative subalgebras W are central. We introduce some notation and make some remarks dealing with this generality.

Notation 1. Given a collection of integers $i_1 < i_2 < \ldots < l$, $l \leqq n - 2$, let $R_{\{i_1,\ldots,i_l\}}$ be the $n - l \times n - l$ matrices over D equipped with the involution $*_{\{i_1,\ldots,i_l\}}$ whose corresponding symmetrics are obtained by discarding from the sequence $q_1, \ldots, q_n$ the subsequence $q_{i_1}, \ldots, q_{i_l}$. For $\underline{x} \in R$, $\underline{x}^{\{i_1,\ldots,i_l\}}$ stands for the matrix of $R_{\{i_1,\ldots,i_l\}}$ obtained by discarding all the lines $i_1, \ldots, i_l$. It is convenient to view a matrix $\underline{y}$ of $R_{\{i_1,\ldots,i_l\}}$ as the matrix $\underline{x}^{\{i_1,\ldots,i_l\}}$ for $\underline{x}$ chosen without ambiguity. Clearly there is a unique such $\underline{x}$ having for diagonal entries the diagonal entries of $\underline{y}$ occurring in the same order of succession but in position outside $\{i_1 x i, \ldots, i_l x i_l\}$, all other diagonal entries of $\underline{x}$ being 1. The corresponding rows of $\underline{y}$ appear in $\underline{x}$ in positions outside the columns $i_1, \ldots, i_l$ of $\underline{x}$, all other off-diagonal positions being filled with zeros. We refer to such a matrix $\underline{x}$ as the *augmentation* of $\underline{y}$.

Remark 4. We observe that for any subalgebra W of R, $W_{\{i_1,\ldots,i_l\}}$ is a subalgebra of $R_{\{i_1,\ldots,i_l\}}$. Since the augmentation of a unitary matrix $\underline{u}$ of $R_{\{i_1,\ldots,i_l\}}$ is a unitary matrix of R, it follows that the invariance property is inherited. Finally an easy exploitation of the symmetries Diag $\{1, \ldots, 1, -1, 1, \ldots, 1\}$ gives that for any invariant subalgebra W of R, W is closed under the following operations: Nullification of a sequence of symmetrically placed off-diagonal entries or, on the contrary, nullification of all entries but the latter sequence of symmetrically placed entries. We proceed to:

THEOREM 6. *Let R be a 2-torsion free simple artinian ring over a division ring containing more than 5 elements with a canonical transpose involution *. Any invariant subalgebra $W \neq 0, Z$ of R has centralizer $V = Z$ provided the rank of R is greater than* 2.

Proof. First let us show that if all matrices $\underline{x}$ in W have equal diagonal coefficients then $W \subseteq Z$. For rank $n = 3$ we proceed as follows: Since the subalgebra $W_{\{3\}}$ inherits the properties of W we get, using Theorem 2, (ii) § 3, that the 1×2 and 2×1 entries in $\underline{x}$ are of the form x_{12} and ϵx_{21} with $\epsilon \neq 0$. Let

$$\underline{y} = \begin{bmatrix} 0 & x12 & 0 \\ \epsilon x_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $\underline{y} \in W$. Squaring this we get a diagonal matrix whose coefficients are ϵx_{12}^2, ϵx_{12}^2, 0. By the choice of W, $x_{21} = x_{12} = 0$ follows. Repeating for $W_{\{1\}}$ and $W_{\{2\}}$ we obtain that W is diagonal. Using Theorem 2, (i), we get that $W \subseteq Z$. An induction on the rank $n \geqq 3$ gives the general property.

Next we show the same conclusion on W where the hypothesis above is required on just $W_{\{n\}}$. For let $i \neq n$. We claim that for every $\underline{x} \in W$, $x_{in} = x_{ni} = 0$. To see this consider the matrix $\underline{y} \in W$ obtained by nullifying all other entries. Now $\underline{x} \cdot \underline{y}$ has for diagonal entries other than the $n \times n$ entry 0 or $x_{in} \cdot x_{ni}$. By hypothesis $x_{in} \cdot x_{ni} = 0$. Suppose, say, that $x_{ni} \neq 0$. Given any $\underline{b} \in W$, let $\underline{z}$ be the matrix of W obtained by nullifying all but the $i \times n$ and $n \times i$ entries of $\underline{b}$. Then $\underline{z} \cdot \underline{y}$ has for diagonal entries other than the $n \times n$ entry 0 or $b_{in} \cdot x_{ni}$. Consequently $b_{in} = 0$. This shows that $W_{in} = 0$. As in many earlier instances this gives that $W_{ni} = W_{in} = 0$. This shows that every matrix $\underline{x}$ of W is a diagonal sum of $\underline{x}^{\{n\}}$ and a scalar matrix x_{nn}. Consequently $W_{\{1,\ldots,n-2\}}$ is a diagonal invariant subalgebra. By Theorem 2, (i), the latter subalgebra is central. Thus W satisfies the hypothesis above, which completes this part.

Now let $W \neq 0, Z$ be any invariant subalgebra. Then $W \not\subseteq Z$. By the above, $W_{\{i\}}$ cannot have the above condition on the diagonal coefficients, $i = 1, \ldots, n$. Using induction on n we may suppose that $n = 3$. By Corollary to Theorem 2, the centralizer $V_{(i)}$ of $W_{\{i\}}$ in $R_{\{i\}}$ is the centre Z. We claim that $V_{\{i\}} \subseteq V_{(i)}$. For if $\underline{x}, \underline{y} \in W$, V then the matrices $\underline{x}'$ and $\underline{y}'$ obtained by nullifying all, say, $i \times n$ and $n \times i$ entries but the $n \times n$ entry are again in W and V. Now

$$[\underline{x}', \underline{y}'] = 0 = [\underline{x}_{\{i\}}, \underline{y}_{\{i\}}] \oplus [x_{33}, y_{33}]$$

forces $[\underline{x}_{\{i\}}, \underline{y}_{\{i\}}] = 0$. Thus $\underline{y}_{\{i\}} \in V_{(i)}$. Therefore $V_{\{i\}} \subseteq V_{(i)} = Z$ for all i, and so $V = Z$, proving the theorem.

Just as Theorem 2 was used to derive Theorem 3, so Theorem 6 can be used to describe the subalgebras W satisfying a polynomial identity. Thus,

THEOREM 7. *If W is an invariant* PI *subalgebra, not R-special (in particular if W is non-commutative or if R has rank superior to* 2), *then* $W = 0$, Z, *or* R. *In particular, if R is of finite dimension superior to* 4, *then every invariant subalgebra W is* 0, Z *or* R.

As was observed earlier, Theorem 7 allows us to take an infinite dimensional division ring D. Turning to arbitrary subalgebras W containing inverses we get, using [**3**, p. 201], that their corner subalgebras W_{ii} are either Z or D. Exactly as in a previous situation we can show that either all the latter subalgebras are precisely Z, in which case $W \subseteq Z$, or all these coincide with D. The following induction step, whose proof is left as an exercise to the reader, makes it clear that Theorems 4 and 5 carry over for an arbitrary rank.

Induction Step. Suppose that the division ring D has the following property: (P_n) Every invariant subalgebra W of R of rank n over D, such that the corner subalgebras (resp. off corner subalgebras) are D, coincides with R. Then D has property (P_{n+1}).

We have shown the following theorems:

THEOREM 8. *If W is non R-special and contains inverses, then* $W = 0, Z$, *or* R.

THEOREM 9. *If * is the conjugate-transpose and if the ground division ring D is generated by unitaries under the ground involution, then every invariant subalgebra W is as in the conclusion of Theorem* 8.

There is one clear-cut class of division rings D such that D is generated by unitaries with respect to any given involution; namely, the algebraic division rings of dimension > 4. For such division rings we have:

THEOREM 10. *If R is any 2-torsion free simple artinian ring whose ground division ring is algebraic of dimension* > 4, *every invariant subalgebra W is* 0, *Z or R.*

6. General involution. In dealing with a general involution on a simple artinian ring R we quote Jacobson's dichotomy asserting that * is either canonical transpose or symplectic. Of these two cases only the latter one must be studied. Then the ground division ring D is necessarily a field and the rank of R over D is even. For rank 2, * will be the mapping

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For rank $2m$, we turn R into the $m \times m$ matrices over the simple ring $\Delta = D_{2\times 2}$, take for the ground involution the preceding mapping, which we denote by $-$, and for involution *, the conjugate-transpose relative to $-$:

$$[\underline{a}_{i,j}]_{i,j} \rightarrow [\bar{\underline{a}}_{j,i}]_{i,j};\quad i,j = 1,\ldots,m.$$

Our objective for this concluding section is to show that every invariant subalgebra of R must be trivial. Some remarks will simplify our work.

Remark 5 (Referee). Every invariant subalgebra (or just subring) W is preserved with respect to commutation with all skews.

Remark 6. W is a simple finite dimensional subalgebra. If $W \neq R$, all symmetrics in (the *-subalgebra generated by) W must be central.

Remark 7. W is closed under the following operations: Nullification of symmetrically placed off diagonal block-entries or, on the contrary, nullification of the other entries.

Because of the relevance of Remark 5 to the case in question and the elegant tool it provides in general, we give a justification. Notice that every skew $\underline{k}$ of R is a sum of square-zero skews and commutators of these. If σ is square-zero skew, then using the unitary $1 \pm \sigma$ we can then commutate σ with the given invariant subalgebra W. By Jacobi's identity, one can then commutate W with commutators of the σ's. Consequently one can commutate $\underline{k}$ with W.

As a result of Remark 5, if R has dimension larger than 16, then by [**3**, p. 219], $W = 0$, Z or R, as desired. In other words, if the rank m of R over Δ is larger than 2, we are done. At any rate, applying [**1**, Theorem] (clearly R is generated by skews and cannot be 3×3 matrices over a field), we get Remark 6.

The proof we proceed to give for $m = 1, 2$ is patterned after the previous study. By Remark 6 and the double centralizer theorem, all we must show is that the centralizer V of W is trivial. If, by way of contradiction, $V \neq Z$, R, then $W \neq R$, so V and W have all their symmetrics central.

Case $R = \Delta$. Notice that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are unitaries of R. Any subspace preserved by conjugation of these unitaries is then closed under the following operations

1) Replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

2) Replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} a+c & (b+d)-(a+c) \\ c & d-c \end{bmatrix}$

3) Replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} a-b & b \\ (a+c)-(b+d) & b+d \end{bmatrix}$

and their immediate consequences:

4) Replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$

5) Replacing $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\begin{bmatrix} 0 & d-(a+c) \\ 0 & -2c \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ a-(b+d) & 2b \end{bmatrix}$.

If for each matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$, $a = d$, then by the relations above $b = c = 0$ follow, so $W \subseteq Z$. If, on the other hand, for each matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$, $b = c$, then by relation 5) $a + c = d$ and $b + d = a$, giving $b + c = 0$, so $b = c = 0$, and $W \subseteq Z$ would follow. This shows that for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$, $b \neq c$, so that

$0 \neq \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix} \in W$. Using the latter relation we derive that every matrix in V has equal diagonal coefficients, so $V \subseteq Z$.

Case $R = \Delta_{2\times 2}$. First, it must be shown that if there is a non-central diagonal matrix in W, then $V = Z$. For let $\begin{bmatrix} \underline{a} & 0 \\ 0 & \underline{b} \end{bmatrix}$ be such a matrix. If $\underline{a} \notin Z$, there is a unitary $\underline{u}$ of Δ with $\underline{a} - \underline{u}\underline{a}\underline{\bar{u}} = \underline{c} \neq 0$ (by the preceding case). Then $\begin{bmatrix} \underline{c} & 0 \\ 0 & 0 \end{bmatrix} \in W$. Similarly if $\underline{b} \notin Z$, $\begin{bmatrix} 0 & 0 \\ 0 & \underline{d} \end{bmatrix} \in W$, for some $\underline{d} \neq 0$. This shows that we may assume $\underline{a} \neq \underline{b}$. If $\begin{bmatrix} 0 & \underline{t} \\ \underline{t}' & 0 \end{bmatrix} \in V$ is non-zero, commute it with $\begin{bmatrix} \underline{a} & 0 \\ 0 & \underline{b} \end{bmatrix}$ and $\begin{bmatrix} \underline{a} - \underline{u}\underline{a}\underline{\bar{u}} & 0 \\ 0 & 0 \end{bmatrix}$. This gives $\underline{a}\underline{t} = \underline{t}\underline{b}$ and $(\underline{a} - \underline{u}\underline{a}\underline{\bar{u}}) \cdot \underline{t} = \underline{t}' \cdot (\underline{a} - \underline{u}\underline{a}\underline{\bar{u}}) = 0$. The latter relations remain true for $\underline{a} - \underline{u}\underline{a}\underline{\bar{u}} = \underline{c}$ replaced by the invariant subalgebra generated by $\underline{c}$. By the preceding $c = 0$, giving $\underline{a} \in Z$. Similarly $\underline{b} \in Z$. In view of the relation $\underline{a}\underline{t} = \underline{t}'\underline{b}$, this gives $(\underline{a} - \underline{b})\underline{t} = 0 = \underline{t}'(\underline{a} - \underline{b})$, a contradiction. We conclude that V is diagonal, so is central. For let $\begin{bmatrix} \underline{f} & 0 \\ 0 & \underline{g} \end{bmatrix} \in V$. Pass to $\begin{bmatrix} \underline{f} - \underline{u}\underline{f}\underline{\bar{u}} & 0 \\ 0 & 0 \end{bmatrix}$, then to the conjugate of this with respect to the unitary $1 + \sigma$ where σ is square-zero. Nullification of the off-diagonal blocks gives $\underline{f} = \underline{u}\underline{f}\underline{\bar{u}}$, so $\underline{f} \in Z$. Similarly $\underline{g} \in Z$. Since $\begin{bmatrix} \underline{f} & 0 \\ 0 & \underline{g} \end{bmatrix}$ is then a symmetric of V, $\begin{bmatrix} \underline{f} & 0 \\ 0 & \underline{g} \end{bmatrix} \in Z$ follows.

Next suppose that all diagonal matrices of W are central. By the above if $W \nsubseteq Z$, there is $0 \neq \begin{bmatrix} 0 & \underline{b} \\ \underline{c} & 0 \end{bmatrix} \in W$. Taking the involute of the latter matrix we get that $\underline{b} + \underline{\bar{c}} = 0$, so $\underline{c} = -\underline{\bar{b}}$. Taking the conjugate with respect to $1 + \sigma$ we get that $\sigma\underline{c} - \underline{b}\sigma = -\underline{c}\sigma + \sigma\underline{b} \in Z$, and consequently $\sigma(\underline{c} + \underline{b})\sigma = \sigma(\underline{b} - \underline{\bar{b}})\sigma = 0$. The latter relation holds for $b - \bar{b}$ replaced by the invariant span of this element, forcing $\underline{b} = \underline{\bar{b}}$, so the original matrix is $\begin{bmatrix} 0 & \underline{b} \\ \underline{b} & 0 \end{bmatrix}$, with $\underline{b}$ symmetric. An obvious conjugate argument gives that $\underline{b}$ anti-commutes with the skews of Δ, resulting in $\underline{b} = 0$, which completes the proof.

The forgoing results combine in the final

THEOREM 11. *Let R be a 2-torsion free simple artinian ring with *. Suppose that R has rank larger than 1 over a division ring D containing more than 5 elements and that W is a subalgebra of R preserved with respect to conjugation by all the unitaries of R.*

a) *If R is not 4-dimensional and if W is* PI *or contains inverses then W is trivial: $W = 0$, Z or R.*

b) *If * is the conjugate-transpose and if the division ring D equipped with the induced involution is generated by unitaries, or if D is algebraic, then W is as in* a).

We conclude with some questions.

Question 1. If a division ring D is generated by its unitaries with respect to the involution $-$, must it be generated by the unitaries with respect to any co-gredient involution?

Question 2. Does Theorem 11 extend to the case of characteristic 2?

Question 3. Does Theorem 11, a) carry over to simple rings R?

A positive answer to Question 1 would make part b) valid for any involution. As for Question 2, a serious difficulty arises from the lack of symmetries Diag $\{1, \ldots, 1, -1, \ldots, 1\}$.

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