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ON LINEAR FUNCTIONAL EQUATIONS WITH NONPOLYNOMIAL C^∞ SOLUTIONS

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It is known (cf. M. A. McKiernan [6]) that the only measurably bounded solutions f of the equations

$$(1) \qquad \sum_{i=1}^{m} \mu_i f(x+\alpha_i t) = f(x),$$

where $x \in R^n$, $t \in R$, α_i $(i=1,\ldots,m)$ span the space R^n, $\sum_{i=1}^{m} \mu_i = 1$, and $\sum_{i \in I} \mu_i \neq 0$ for any $I \subset \{1,\ldots,m\}$, are polynomials. The degree of these polynomials and the dimension of the solution space can be estimated by numbers depending on m and n. (For estimates and other details concerning equations (1) see [1], [2], [3], [4], [5], [6].)

A natural generalization of equations (1) are the equations

$$(2) \qquad \sum_{i=1}^{m} \mu_i f(x+\varphi_i(t)) = f(x)$$

which, similarly to equations (1), can be considered as mean value conditions for the function f. In some special cases when $\varphi_i(t) \neq \alpha_i t$ also all the continuous solutions of equations (2) are polynomials (cf. [7]).

Now, the question arises whether continuous solutions f of equations (2) are always polynomials. The answer to this question is negative as the example of the equation

$$(3) \qquad \frac{1}{2} f\left(u+\ln\left(1-\frac{1}{2}\exp\frac{t^2}{1+t^4}\right)+\ln 2,\, v\right)+\frac{1}{2} f\left(u,\, v+\frac{t^2}{1+t^4}\right) = f(u, v)$$

which is satisfied by the function $f(u, v) = e^{u+v}$, and the following theorem show.(1)

THEOREM. *Suppose that*

(1) $\chi(t), v(t) \in C^2$ *in an open interval* $\Delta \subset R$,

(2) *there exists an* $\alpha \in \Delta$ *such that* $\chi(\alpha) = v(\alpha) = 0$ *and* $\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}' =_{df} \chi'(\alpha)v'(\alpha) \neq 0$.

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(1) Obviously, to answer our question, it suffices to show only one equation of the form (2) possessing a noncontinuous polynomial solution f. However, there exists another reason to formulate a theorem like this one presented here: to show an example of a functional equation with several unknown functions such that one of them characterizes an elementary function of two variables although the choice of the others is almost arbitrary.

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If $\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}'' - \overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}' =_{df} \chi'(\alpha)v''(\alpha) - \chi''(\alpha)v'(\alpha) = 0$, *then nonconstant continuous solutions* f *of the equation*

$$\mu_1 f(u+\chi(t), v) + \mu_2 f(u, v+v(t)) = f(u, v), \tag{4}$$

wherein $\mu_1 + \mu_2 = 1$, $\mu_1 > 0$, $\mu_2 > 0$, *can exist if and only if*

$$v(t) = A\chi(t), \tag{5}$$

where $A \neq 0$.

If (5) *holds, then the most general continuous solution* f *of equation* (4) *has the form*

$$f(u, v) = \alpha\left(u - \frac{\mu_1}{\mu_2 A} v\right) + \gamma, \tag{6}$$

where α, γ *are arbitrary constants and* A *is the same as in* (5).

If $\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}'' - \overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}' \neq 0$, *nonconstant continuous solutions* f *of equation* (4) *exist if and only if*

$$\chi(t) = A[\ln(1 - \mu_2 e^{Bv(t)}) - \ln \mu_1], \tag{7}$$

where A *is a constant different from zero and the function* $v(t)$ *and the constant* $B \neq 0$ *satisfy the condition*

$$Bv(t) < -\ln \mu_2 \quad \text{for every } t \in \Delta. \tag{8}$$

In this case the most general continuous solution f *of equation* (4) *has the form*

$$f(u, v) = K e^{(1/A)u + Bv} + L, \tag{9}$$

where K, L *are arbitrary constants and* A, B *are the same as in* (7).

Proof. It is easy to see that condition (5) implies that $\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}'' - \overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}' = 0$ and elementary computations allow us to verify that the functions (6) satisfy equation (4).

Similarly, condition (7) implies that

$$\chi'(\alpha) = -AB\frac{\mu_2}{\mu_1} v'(\alpha),$$

$$\chi''(\alpha) = -AB\frac{\mu_2}{\mu_1^2}[\mu_1 v''(\alpha) + Bv'(\alpha)^2]$$

and therefore

$$\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}'' - \overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}' = AB^2\frac{\mu_2}{\mu_1^2} v'(\alpha)^3 \neq 0.$$

(In fact, the constants A, B in condition (7) are different from zero and assumption (2) guarantees that $v'(\alpha) \neq 0$.) Elementary computations verify that the functions (9) satisfy equation (4).

Now, we have to show that, under assumptions (1), (2), nonconstant continuous solutions f of equation (4) exist only if either (5) or (7) holds and that in the first case the only possible continuous solutions f of equation (4) are given in (6) and in the second case the only possible continuous solutions f of equation (4) are those given in (9).

We shall establish first the fact that every continuous solution f of equation (4) is a function of class C^∞. To do this we apply the following theorem, being a simple consequence of the results of [8]:

If $\varphi_i(t) \in C^2$ in an open interval $\Delta \subset R\,(i=1,\ldots,m)$, there exists an $\alpha \in \Delta$ such that $\varphi_i(\alpha)=0$ for $i=1,\ldots,m$ and if the vectors $\overset{\alpha}{\varphi}{}_i' = \varphi_i'(\alpha)\,(i=1,\ldots,m)$ span the space R^n, then all the continuous solutions f of equation (2), *where $\sum_{i=1}^m \mu_i = 1$, $\mu_i > 0$ for $i=1,\ldots,m$ and $x \in R^n$, are functions of class C^∞.*

Assumptions (1) and (2) guarantee that the assumptions of the above theorem are satisfied and therefore every continuous solution f of equation (4) is a function of class C^∞.

Differentiating equation (4) with respect to t and setting $t=\alpha$ yields

$$\mu_1 \overset{\alpha}{\chi}{}' f_u + \mu_2 \overset{\alpha}{v}{}' f_v = 0. \tag{10}$$

Differentiating (10) with respect to u yields

$$\mu_1 \overset{\alpha}{\chi}{}' f_{uu} + \mu_2 \overset{\alpha}{v}{}' f_{uv} = 0 \tag{11'}$$

and differentiating with respect to v yields

$$\mu_1 \overset{\alpha}{\chi}{}' f_{uv} + \mu_2 \overset{\alpha}{v}{}' f_{vv} = 0. \tag{11''}$$

Differentiating equation (4) twice with respect to t and setting $t=\alpha$ yields

$$\mu_1 (\overset{\alpha}{\chi}{}')^2 f_{uu} + \mu_2 (\overset{\alpha}{v}{}')^2 f_{vv} + \mu_1 \overset{\alpha}{\chi}{}'' f_u + \mu_2 \overset{\alpha}{v}{}'' f_v = 0. \tag{12}$$

Multiplying (12) by $\overset{\alpha}{\chi}{}'$ and $\overset{\alpha}{v}{}'$ we obtain

$$\mu_1 (\overset{\alpha}{\chi}{}')^3 f_{uu} + \mu_2 \overset{\alpha}{\chi}{}' (\overset{\alpha}{v}{}')^2 f_{vv} + \mu_1 \overset{\alpha}{\chi}{}' \overset{\alpha}{\chi}{}'' f_u + \mu_2 \overset{\alpha}{\chi}{}' \overset{\alpha}{v}{}'' f_v = 0 \tag{13'}$$

and

$$\mu_1 (\overset{\alpha}{\chi}{}')^2 \overset{\alpha}{v}{}' f_{uu} + \mu_2 (\overset{\alpha}{v}{}')^3 f_{vv} + \mu_1 \overset{\alpha}{\chi}{}'' \overset{\alpha}{v}{}' f_u + \mu_2 \overset{\alpha}{v}{}'\,{}^{\alpha''} f_v = 0. \tag{13''}$$

In view of (10) and (11′) equation (13′) can be written as

$$-\mu_2 (\overset{\alpha}{\chi}{}')^2 \overset{\alpha}{v}{}' f_{vu} + \mu_2 \overset{\alpha}{\chi}{}' (\overset{\alpha}{v}{}')^2 f_{vv} - \mu_2 \overset{\alpha}{\chi}{}'' \overset{\alpha}{v}{}' f_v + \mu_2 \overset{\alpha}{\chi}{}' \overset{\alpha}{v}{}'' f_v = 0$$

i.e. in view of $\mu_2 > 0$,

$$\frac{\partial}{\partial v}\left[-af_u + bf_v + sf\right] = 0, \tag{14'}$$

where

$$\begin{aligned} a &=_{df} (\overset{\alpha}{\chi}{}')^2 \overset{\alpha}{v}{}', \\ b &=_{df} \overset{\alpha}{\chi}{}' (\overset{\alpha}{v}{}')^2, \\ s &=_{df} \overset{\alpha}{\chi}{}'\,{}^{\alpha''} - \overset{\alpha}{\chi}{}'' \overset{\alpha}{v}{}'. \end{aligned} \tag{15}$$

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Similarly, making use of (10), (11″) and the assumption $\mu_1>0$, we can write (13″) as

$$\frac{\partial}{\partial u}[af_u-bf_v-sf] = 0. \tag{14″}$$

Now, it follows from (14′) and (14″) that

$$af_u-bf_v-sf = k, \tag{16}$$

where k is a constant.

In the case $s=0$ equations (10) and (16) form a set of two linear equations with constant coefficients in two unknowns f_u, f_v. The determinant of the coefficients matrix is equal to

$$-(\mu_1+\mu_2)(\overset{\alpha}{\chi}')^2(\overset{\alpha}{v}')^2 = -(\overset{\alpha}{\chi}'\overset{\alpha}{v}')^2 \neq 0$$

in view of the assumptions $\overset{\alpha}{\chi}'\overset{\alpha}{v}'\neq 0$ and $\mu_1+\mu_2=1$.

Therefore, solving our set of linear equations, one can conclude that $f_u=\alpha$ and $f_v=\beta$, i.e.

$$f(u, v) = \alpha u+\beta v+\gamma, \tag{17}$$

where α, β, and γ are constants.

Substituting (17) into (4) we obtain the condition

$$\mu_1\alpha\chi(t)+\mu_2\beta v(t) \equiv 0. \tag{18}$$

Conditions (17) and (18) allow us to establish the following fact:

If $s=\overset{\alpha}{\chi}'\overset{\alpha}{v}''-\overset{\alpha}{\chi}''\overset{\alpha}{v}'=0$, then nonconstant continuous solutions f of equation (4) exist if and only if condition (5) is satisfied and the most general continuous solution f of equation (4) has the form (7).

Let us consider the case $s=\overset{\alpha}{\chi}'\overset{\alpha}{v}''-\overset{\alpha}{\chi}''{}^{\alpha'}\neq 0$.

Multiplying equation (16) by μ_2 and taking into account (10) and (15) we obtain

$$\mu_2af_u+\mu_1af_u-\mu_2sf = \mu_2k$$

i.e.

$$af_u-\mu_2sf = \mu_2k \tag{19′}$$

since $\mu_1+\mu_2=1$.

Similarly, multiplying (16) by μ_1 and taking into account (10), (15), and the condition $\mu_1+\mu_2=1$, we obtain

$$-bf_v-\mu_1sf = \mu_1k. \tag{19″}$$

Notice that equation (19′) with a fixed v and equation (19″) with a fixed u are both ordinary linear differential equations with constant coefficients.

Since the most general solution of the equation

$$pv'(u)+qv(u) = r,$$

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where $pq \neq 0$,(²) has the form

$$v(u) = \frac{r}{q} + K e^{-(q/p)u},$$

where K is an arbitrary constant, equations (19′) and (19″) imply

$$f(u, v) = -\frac{k}{s} + K(v) e^{csu} \tag{20′}$$

and

$$f(u, v) = -\frac{k}{s} + \tilde{K}(u) e^{-dsv}, \tag{20″}$$

where

$$c = \frac{\mu_2}{a}, \qquad d = \frac{\mu_1}{b}, \tag{21}$$

and $K(v)$, $\tilde{K}(u)$ are some functions of v and u.

By (20′) and (20″)

$$K(v) e^{csu} = \tilde{K}(u) e^{-dsv}.$$

Setting in the last equality $u=0$ and next $v=0$ one can conclude that

$$K(v) = K e^{-dsv}, \qquad \tilde{K}(u) = K e^{csu},$$

where K is a constant, and therefore the function $f(u, v)$ has to have the form

$$f(u, v) = K e^{s(cu-dv)} + L, \tag{22}$$

where c, d, s are defined by (21) and (15), and K, L are arbitrary constants (L may be arbitrary since $L=-k/s$ and k was arbitrary).

Substituting (22) into (4) we obtain the condition

$$\mu_1 e^{cs\chi(t)} + \mu_2 e^{-dsv(t)} = 1.$$

Hence

$$\chi(t) = \frac{1}{cs} [\ln (1-\mu_2 c^{-dsv(t)}) - \ln \mu_1],$$

i.e. the function $\chi(t)$ has the form

$$\chi(t) = A[\ln (1-\mu_2 c^{Bv(t)}) - \ln \mu_1],$$

where $\mu_2 e^{Bv(t)} < 1$ for every $t \in \Delta$.

The last condition is equivalent to condition (8). Of necessity $AB \neq 0$ since if either $A=0$ or $B=0$, then $\chi(t)=$constant, which contradicts to assumption (2).

(²) The condition $pq \neq 0$ is satisfied for both considered equations since $s \neq 0$ and the assumption $\overset{\alpha}{\chi}{}'^{\alpha'} \neq 0$ implies that $a \neq 0$ and $b \neq 0$.

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Thus we have proved that in the case $s=\overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}''-\overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}'\neq 0$ all the continuous solutions of equation (4) have the form (22), and the conditions (7) and (8) hold.

To finish the proof we have to show that (22) and (9) coincide.

It was shown in the beginning of this proof that

$$\overset{\alpha}{\chi}{}' = -AB\frac{\mu_2}{\mu_1}\overset{\alpha}{v}{}'$$

and

$$s = \overset{\alpha}{\chi}{}'\overset{\alpha}{v}{}''-\overset{\alpha}{\chi}{}''\overset{\alpha}{v}{}' = AB^2\frac{\mu_2}{\mu_1^2}(\overset{\alpha}{v}{}')^3.$$

Therefore, taking into account (21) and (15), we obtain after simple computations $cs=1/A$ and $-ds=B$ which means that (21) and (9) really coincide.

REMARK. The theorem gives the most general solution f, χ, v of equation (4) in the class of continuous f, and χ, $v \in C^2$ having a zero in common and strictly monotonic in a neighbourhood of this zero.

It is interesting that for any fixed v (or χ) satisfying the conditions mentioned above one can find χ (or v) so that there exist nonconstant solutions f of equation (4), but f is then either linear or exponential. It is not possible to choose χ and v so that equation (4) is satisfied by a linear and by an exponential function f simultaneously. Therefore this equation can be used to characterize linear and exponential functions of two variables.

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