PERMUTABILITY OF SEMILATTICE CONGRUENCES ON LATTICES

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Many authors have studied lattice congruences on lattices, but it seems that there are few studies concerning semilattice congruences on lattices. However, it seems that the semilattice congruences on lattices are closely connected with their structure. In this paper, we shall study the characterizations of modular, distributive, and relatively complemented lattices by the permutability of semilattice congruences.

We can obtain the dual statements of the following discussion, but we shall not write them as a rule to avoid double descriptions.

1. Preliminaries. Let L be a lattice. An equivalence relation θ on L is called a join-semilattice congruence on L or simply a join-congruence on L if and only if for any elements a, b, c and d in L,

$$a \equiv_\theta b \text{ and } c \equiv_\theta d \text{ imply } a \cup c \equiv_\theta b \cup d$$

($a \equiv_\theta b$ denotes the fact that a is in the relation θ to b). A join-congruence on L which is a meet-congruence on L is called a lattice congruence on L or simply an l-congruence on L.

Let S be a subset of L. Then it is clear that there exists a least join-congruence on L under which all the elements in S are contained in the same congruence class. We simply say that this is the join-congruence on L generated by S and denote it by $\theta(S, \cup)$. Dually, we can define the meet-congruence on L generated by S and denoted by $\theta(S, \cap)$. A join-congruence on L generated by an ideal of L is called a lower join-congruence on L, and a meet-congruence on L generated by a dual ideal of L is called a lower meet-congruence on L. Especially, a join-congruence on L generated by a principal ideal of L is called a principal join-congruence on L, and a meet-congruence on L generated by a dual principal ideal of L is called a principal meet-congruence on L.

We denote by b/a the closed interval $\{x | a \leqslant x \leqslant b\}$, and we denote by $b/$ and $/a$ the principal ideal $\{x | x \leqslant b\}$ and the dual principal ideal $\{x | x \geqslant a\}$, respectively.

THEOREM 1. *Let I be any ideal of a lattice L. Then, in order that two elements a and b in L are congruent under the lower join-congruence $\theta(I, \cup)$ on L, it is necessary and sufficient that*
(1) *there exists an element n in I satisfying $a \cup n = b \cup n$.*

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If I is a principal ideal $m/$, then the condition (1) *may be replaced by*

$$a \cup m = b \cup m. \tag{2}$$

Proof. Let ϕ be such a relation on L that $a \equiv_\phi b$ if and only if there exists an element n in I satisfying $a \cup n = b \cup n$. Then it is easily verified that ϕ is a join-congruence on L. If x and y are contained in I, then $x \cup y \in I$ and $x \cup (x \cup y) = y \cup (x \cup y)$; thus, $x \equiv_\phi y$. Hence, all the elements in I are contained in the same congruence class of ϕ. Since $\theta(I, \cup)$ is the least join-congruence under which all the elements in I are contained in the same congruence class, we have $\theta(I, \cup) \leqslant \phi$, i.e., if a and b are congruent under $\theta(I, \cup)$, then there exists an element n in I satisfying $a \cup n = b \cup n$. This completes the proof of the necessity.

Conversely, let a and b be elements in L such that there exists an element n in I satisfying $a \cup n = b \cup n$. Then we have

$$a = a \cup (a \cap n) \equiv_{\theta(I,\cup)} a \cup n = b \cup n \equiv_{\theta(I,\cup)} b \cup (b \cap n) = b.$$

Hence a and b are congruent under $\theta(I, \cup)$. This completes the proof of the sufficiency.

Finally, if I is a principal ideal $m/$, then it can easily be verified that the conditions (1) and (2) are equivalent, completing the proof.

From the above theorem, we obtain immediately:

COROLLARY 1. *Let I be any ideal of a lattice L. Then I is a congruence class of the lower join-congruence $\theta(I, \cup)$ on L.*

Let θ and ϕ be two equivalence relations on a lattice L. θ and ϕ are said to be *permutable* if and only if θ and ϕ satisfy the following condition for any elements a, b, and c in L:

(3) If $a \equiv_\theta b \equiv_\phi c$, then there exists an element d in L satisfying $a \equiv_\phi d \equiv_\theta c$.

It is obvious that ϕ and θ are permutable if and only if θ and ϕ are permutable. We say that θ is *up-permutable* to ϕ, or that ϕ is *down-permutable* to θ, if and only if θ and ϕ satisfy the following condition for any elements a, b, and c in L:

(4) If $a \leqslant b \leqslant c$ and $a \equiv_\theta b \equiv_\phi c$, then there exists an element d in L satisfying $a \equiv_\phi d \equiv_\theta c$.

It is obvious that if θ and ϕ are permutable, then θ is up- and down-permutable to ϕ. As for the converse of this fact, we can assert the following theorem concerning l-congruences:

THEOREM 2. *Let θ and ϕ be two l-congruences on a lattice L. If θ is up- and down-permutable to ϕ, then θ and ϕ are permutable.*

Proof. Suppose that a, b, and c are elements in L satisfying $a \equiv_\theta b \equiv_\phi c$. Then

$$a \equiv_\theta a \cup b \equiv_\phi a \cup b \cup c \equiv_\theta b \cup c \equiv_\phi c.$$

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Hence there exists an element x satisfying $a \equiv_\phi x \equiv_\theta a \cup b \cup c$, because θ is up-permutable to ϕ. Similarly there exists an element y satisfying

$$a \cup b \cup c \equiv_\phi y \equiv_\theta c.$$

Hence, we have

$$a = a \cap (a \cup b \cup c) \equiv_\phi x \cap y \equiv_\theta (a \cup b \cup c) \cap c = c.$$

Therefore, θ and ϕ are permutable, completing the proof.

2. Characterizations of modular, distributive, and relatively complemented lattices.

THEOREM 3. *Let L be a lattice. Then the following three conditions are equivalent:*
(5) *L is modular.*
(6) *Any two lower join-congruences on L are permutable.*
(7) *Any principal join-congruence on L is up-permutable to any principal join-congruence on L.*

The implication (6) $\Rightarrow$ (7) is obvious. We have to prove that (5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (5).

Proof of (5) $\Rightarrow$ (6). Suppose that L is modular. Let $\theta(I, \cup)$ and $\theta(J, \cup)$ be any two lower join-congruences on L, and let a, b, and x be elements in L satisfying $a \equiv_{\theta(I,\cup)} x \equiv_{\theta(J,\cup)} b$. Then by Theorem 1, there exist elements $n \in I$ and $m \in J$ satisfying $a \cup n = x \cup n$ and $x \cup m = b \cup m$. Hence we have the following calculation if we put $y = (a \cup m) \cap (b \cup n)$:

$$\begin{aligned} y \cup m &= [(a \cup m) \cap (b \cup n)] \cup m = (a \cup m) \cap (b \cup n \cup m) \\ &= (a \cup m) \cap (x \cup n \cup m) = (a \cup m) \cap (a \cup n \cup m) \\ &= a \cup m. \end{aligned}$$

Similarly, we have $y \cup n = b \cup n$. Hence, by Theorem 1,

$$a \equiv_{\theta(J,\cup)} y \equiv_{\theta(I,\cup)} b.$$

Therefore, $\theta(I, \cup)$ and $\theta(J, \cup)$ are permutable, completing the proof.

Proof of (7) $\Rightarrow$ (5). Let m, n, and a be elements in L satisfying $a \geqslant n$, $a \cup m = n \cup m$, and $a \cap m = n \cap m$. Then it is sufficient to prove that $a = n$. By Theorem 1,

$$n \cap m \equiv_{\theta(n/,\cup)} n \equiv_{\theta(m/,\cup)} a.$$

Hence, by (7), there exists an element x in L satisfying

$$n \cap m \equiv_{\theta(m/,\cup)} x \equiv_{\theta(n/,\cup)} a.$$

Theorem 1 now yields

$$m = (n \cap m) \cup m = x \cup m \quad \text{and} \quad x \cup n = a \cup n = a.$$

Hence, $x \leqslant m$ and $x \leqslant a$; whence, $x \leqslant a \cap m$. Therefore,

$$a = x \cup n \leqslant (a \cap m) \cup n = (n \cap m) \cup n = n.$$

Thus, $a = n$, completing the proof.

THEOREM 4. *Let L be a lattice. Then the following three conditions are equivalent:*

(8) *L is distributive.*

(9) *Any lower join-congruence on L is down-permutable to any lower meet-congruence on L.*

(10) *Any principal join-congruence on L is down-permutable to any principal meet-congruence on L.*

The implication (9) $\Rightarrow$ (10) is obvious.

Proof of (8) $\Rightarrow$ (9). Suppose that L is distributive. Let $\theta(I, \cup)$ be any lower join-congruence on L, and let $\theta(J, \cap)$ be any lower meet-congruence on L where J is a dual ideal of L. Now let a, b, and c be elements in L such that $a \leqslant b \leqslant c$ and that $a \equiv_{\theta(J,\cap)} b \equiv_{\theta(I,\cup)} c$. Then by Theorem 1, there exist elements $m \in J$ and $n \in I$ satisfying $a \cap m = b \cap m$ and $b \cup n = c \cup n$. Hence,

$$\begin{aligned} a \cup n \geqslant (m \cap a) \cup n = (m \cap b) \cup n \geqslant m \cap (b \cup n) \\ = m \cap (c \cup n) \geqslant m \cap c. \end{aligned}$$

This yields

$$[(a \cup n) \cap c] \cap m = (a \cup n) \cap (c \cap m) = c \cap m.$$

On the other hand,

$$[(a \cup n) \cap c] \cup n = (a \cup n) \cap (c \cup n) = a \cup n.$$

Hence, by Theorem 1,

$$a \equiv_{\theta(I,\cup)} (a \cup n) \cap c \equiv_{\theta(J,\cap)} c.$$

Therefore, $\theta(I, \cup)$ is down-permutable to $\theta(J, \cap)$, completing the proof.

Proof of (10) $\Rightarrow$ (8). Let a, b, and c be any elements in L. Then by Theorem 1,

$$b \cap c \equiv_{\theta(/c,\cap)} b \equiv_{\theta(a/,\cup)} a \cup b.$$

Hence, by (10), there exists an element x in L satisfying

$$b \cap c \equiv_{\theta(a/,\cup)} x \equiv_{\theta(/c,\cap)} a \cup b.$$

Theorem 1 now implies that

$$a \cup (b \cap c) = a \cup x \quad \text{and} \quad x \cap c = (a \cup b) \cap c.$$

Hence, $a \cup (b \cap c) \geqslant (a \cup b) \cap c$. Therefore L is distributive, completing the proof.

THEOREM 5. *Let L be a lattice. Then the following three conditions are equivalent:*
(11) *L is relatively complemented.*
(12) *Any join-congruence on L is up-permutable to any meet-congruence on L.*
(13) *For any element a in L, the principal join-congruence* $\theta(a/, \cup)$ *on L is up-permutable to the principal meet-congruence* $\theta(/a, \cap)$ *on L.*

The implication (12) $\Rightarrow$ (13) is obvious.

Proof of (11) $\Rightarrow$ (12). Let θ and ϕ be any join-congruence and any meet-congruence on L, respectively. Now suppose that a, b, and c are elements in L satisfying $a \leqslant b \leqslant c$. By (11), there exists a complement d of b in the interval c/a; i.e., there exists d such that $b \cap d = a$ and $b \cup d = c$. Hence, if $a \equiv_\theta b \equiv_\phi c$, then

$$a = b \cap d \equiv_\phi c \cap d = d = a \cup d \equiv_\theta b \cup d = c.$$

Therefore θ is up-permutable to ϕ, completing the proof.

Proof of (13) $\Rightarrow$ (11). Suppose that a, b, and c are elements in L satisfying $a \leqslant b \leqslant c$. Then by Theorem 1,

$$a \equiv_{\theta(b/,\cup)} b \equiv_{\theta(/b,\cap)} c.$$

Hence, by (13), there exists an element d in L satisfying

$$a \equiv_{\theta(/b,\cap)} d \equiv_{\theta(b/,\cup)} c.$$

Therefore, by Theorem 1, $a \cap b = d \cap b$ and $d \cup b = c \cup b$; whence $a = d \cap b$ and $d \cup b = c$. That is, d is a complement of b in c/a. This completes the proof.

From Theorems 2 and 5, we can immediately obtain the following theorem by R. P. Dilworth (**1**, Theorem 4.2).

COROLLARY 2. *If L is a relatively complemented lattice, then any two l-congruences on L are permutable.*

We denote by $\lambda(L)$ the set of all lower join-congruences and of all lower meet-congruences on a lattice L, and by $\pi(L)$ the set of all principal join-congruences and of all principal meet-congruences on L.

COROLLARY 3. *Let L be a lattice. Then the following three conditions are equivalent:*
(14) *L is relatively complemented and modular.*
(15) *Any lower join-congruence on L is up-permutable to any member of* $\lambda(L)$.
(16) *Any principal join-congruence on L is up-permutable to any member of* $\pi(L)$.

Proof. (14) $\Rightarrow$ (15) and (16) $\Rightarrow$ (14) follow from Theorems 3 and 5. (15) $\Rightarrow$ (16) is obvious.

COROLLARY 4. *Let L be a lattice. Then the following three conditions are equivalent:*

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(17) *L is relatively complemented and distributive.*
(18) *Any two members of* $\lambda(L)$ *are permutable.*
(19) *Any principal join-congruence on L is up- and down-permutable to any principal meet-congruence on L.*

Proof. It is easily verified by Theorem 1 that if L is distributive, then every member of $\lambda(L)$ is an l-congruence on L. Hence, $(17) \Rightarrow (18)$ follows from Corollary 2. $(19) \Rightarrow (17)$ can easily be obtained from Theorems 4 and 5. $(18) \Rightarrow (19)$ is obvious.

Reference

1. R. P. Dilworth, *The structure of relatively complemented lattices*, Ann. Math., *51* (1950), 348–359.

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