A QUEUEING SYSTEM WITH MOVING AVERAGE INPUT PROCESS AND BATCH ARRIVALS

C. PEARCE

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1. Introduction

In a recent paper by P. D. Finch and myself [1], the solution for the limiting distribution of a moving average queueing system was obtained. In this paper the system is generalised to the case of batch arrivals in batches of size $r > 1$.

The queueing system considered is a single server queue in which

(i) batches arrive at the instants $0 = A_0 < A_1 < A_2 < \cdots$, with time interval between the mth and $(m+1)$th batches

$$A_m - A_{m-1} = U_m + \beta U_{m-1}, \qquad m \geqq 1,\ \beta \geqq 0, \tag{1.1}$$

where $\{U_m\}$ is a sequence of identically and independently distributed non-negative random variables with common distribution function

$$A(x) = P(U_m \leqq x), \qquad m \geqq 0,\ x \geqq 0,$$

such that

$$\int_0^\infty x dA(x) < \infty,$$

and

(ii) the service time of the pth customer, $1 \leqq p \leqq r$, in the qth batch is s_{qr+p}, where $\{s_m\}$ is a sequence of identically and independently distributed non-negative random variables, distributed independently of the sequence $\{U_m\}$, with common distribution function

$$P(S_m \leqq x) = 1 - \exp(-\mu x), \qquad x \geqq 0,\ \mu \geqq 0.$$

If P_j^m, $j \geqq 0$, $m \geqq 1$, is the probability that the batch arriving at A_m finds exactly j customers in the system, then it follows from the results of Finch [2] that

$$P_j = \lim_{m \to \infty} P_j^m, \qquad j \geqq 0,$$

exists provided that

$$\mu(1+\beta) \int_0^\infty x dA(x) > r. \tag{1.2}$$

The system can be considered as one with individual arrivals at the instants A_m and an Erlang service distribution of order r. The probabilities Q_0^m and Q_j^m, $j > 0$, $m \geqq 1$, of the individual arriving at A_m finding 0 or j customers in the system then correspond to P_0^m and $P_{(j-1)r+1}^m + \cdots + P_{jr}^m$ of this paper, and similarly for the limiting probabilities.

2. Definitions and notation

We use capital letters to denote random variables and corresponding lower case letters to denote particular values taken by random variables. We denote the $(n+1)$-tuple $(u_0, u_1, \cdots, u_n)$ by $u^{(n)}$ and the corresponding vector random variable $(U_0, U_1, \cdots, U_n)$ by $U^{(n)}$.

$P_j(u^{(n)})$, $j \geqq 0$, denotes the conditional probability, given $U^{(n)} = u^{(n)}$, that the batch arriving at A_n finds exactly j customers in the system. Thus $EP_j(U^{(n)})$, where E denotes expectation, is the (unconditional) probability that the $(n+1)$th batch finds exactly j customers in the system.

We use the following notation:

$$\psi(\alpha) = E \exp(-\mu\alpha U_m) = \int_0^\infty \exp(-\mu\alpha u)dA(u),$$

$$k_j(x, y) = [\{\mu(y+\beta x)\}^j/j!] \exp\{-\mu(y+\beta x)\}, \qquad j \geqq 0,$$

$$K_j(x, y) = \sum_{i=j}^{\infty} k_i(x, y),$$

$$P(u^{(n)}; z) = \sum_{i=0}^{\infty} P_i(u^{(n)})z^i, \qquad |z| \leqq 1,$$

$$k(x, y; z) = \sum_{i=0}^{\infty} k_i(x, y)z^i = \exp\{-(1-z)\mu(y+\beta x)\},$$

$$c_i(u^{(n+1)}) = \sum_{j=0}^{\infty} P_j(u^{(n)})k_{j+i+r}(u_n, u_{n+1}), \qquad i \geqq 0,$$

$$P^*(s; z; n) = E[P(U^{(n)}; z) \exp(-sU_n)]$$

$$= \sum_{i=0}^{\infty} P_i^*(s, n)z^i, \qquad |z| \leqq 1, \quad \mathrm{Re}\, s \geqq 0,$$

$$c_i^*(s; n) = E[c_i(U^{(n)}) \exp(-sU_n)], \qquad \mathrm{Re}\, s \geqq 0.$$

Under the assumption (1.2), it follows from Rouché's theorem that the equation

$$z^r = \psi[(1+\beta)(1-z)] \tag{2.1}$$

has exactly r roots T_j, $j = 1, 2, \cdots, r$, inside the unit circle. We shall assume that these r roots are distinct.

3. Fundamental equations

It follows from the exponential form of the service time distribution that the queue lengths at the instants A_m-0, $m = 0, 1, 2, \cdots, n+1$ form a Markov chain, and that

$$
(3.1)\quad
\begin{aligned}
P_1(u^{(n+1)}) &= \sum_{i=0}^{\infty} P_i(n^{(n)})k_{r-1+i}(u_n, u_{n+1}), & n &\geqq 0,\\
P_2(u^{(n+1)}) &= \sum_{i=0}^{\infty} P_i(u^{(n)})k_{r-2+i}(u_n, u_{n+1}). & n &\geqq 0\\
&\cdots\\
P_{r-1}(u^{(n+1)}) &= \sum_{i=0}^{\infty} P_i(u^{(n)})k_{i+1}(u_n\ \ u_{n+1}). & n &\geqq 0,\\
P_j(u^{(n+1)}) &= \sum_{i=0}^{\infty} P_{j-r+i}(u^{(n)})k_i(u_n, u_{n+1}), & n &\geqq 0,\ j \geqq r.
\end{aligned}
$$

Since $P(u^{(n+1)}); 1) = 1 = k(x, y; 1)$, it follows at once that

$$P_0(u^{(n+1)}) = \sum_{i=0}^{\infty} P_i(u^{(n)})K_{r+i}(u_n, u_{n+1}), \qquad n \geqq 0.$$

Forming $z^r k(u_n, u_{n+1}; z)P(u^{(n)}; z)$ and noting that

$$(3.2)\qquad \sum_{i=0}^{\infty} c_i(u^{(n+1)}) = P_0(u^{(n+1)}),$$

we obtain from the above equations

$$P(u^{(n+1)}; z) = \sum_{i=0}^{\infty} (1-z^{-i})c_i(u^{(n+1)}) + z^r P(u^{(n)}; z) \exp[-(1-z^{-1})\mu(u_{n+1}+\beta u_n)]$$

for $|z| \leqq 1$, $z \neq 0$. Hence

$$
(3.3)\quad
\begin{aligned}
&P^*(s; z; n+1)\\
&= \sum_{i=0}^{\infty} (1-z^{-i})c_i^*(s; n+1) + z^r P^*\{(1-z^{-1})\mu\beta; z; n\}\psi(1-z^{-1}+s/\mu)
\end{aligned}
$$

for $|z| \leqq 1$, $z \neq 0$, $\operatorname{Re} s \geqq 0$, $\operatorname{Re}[(1-z^{-1})\mu\beta] \geqq 0$. The restrictions on z require it to lie in or on the unit circle and on or outside the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, with the point $z = 0$ deleted. We denote this domain of the z-plane by R.

Assuming (1.2), it can be shown by the methods of [2] that

$$(3.4)\qquad P(u; z) = \lim_{n\to\infty} EP(U_0, U_1, \cdots, U_{n-1}, u_n; z), \qquad u_n = u,$$

exists for $|z| \leqq 1$, $\operatorname{Re} s \geqq 0$ and is the generating function of a probability distribution. In equation (3.4) we have departed momentarily from our usual notation for n-tuples and the expectation is with respect to the random variables $U_0, U_1, \cdots, U_{n-1}$.

Using a natural notation we write

$$P^*(s; z) = E[P(U; z) \exp(-sU)], \qquad |z| \leqq 1, \ \operatorname{Re} s \geqq 0, \tag{3.5}$$

where U is a random variable with distribution function $A(x)$. Similarly we write

$$P^*(s; z) = \sum_{i=0}^{\infty} P_i^*(s) z^i$$

and

$$c_i^*(s) = \lim_{n\to\infty} c_i^*(s; n).$$

We note that (3.5) gives

$$P^*(s; 1) = \psi(s/\mu). \tag{3.6}$$

Letting $n \to \infty$ in (3.3) we obtain

$$P^*(s; z) = c(s; z) + z^r P^*[(1-z^{-1})\mu\beta; z]\psi(1-z^{-1})+s/\mu), \qquad z \in R, \ \operatorname{Re} s \geqq 0, \tag{3.7}$$

where

$$c(s; z) = \sum_{i=0}^{\infty} (1-z^{-i}) c_i^*(s), \qquad |z| \leqq 1, \ \operatorname{Re} s \geqq 0. \tag{3.8}$$

4. Evaluation of $P^*(s; z)$

Putting $s = (1-z^{-1})\mu\beta$ in (3.7), we obtain, for $z \in R$, $\operatorname{Re} s \geqq 0$,

$$P^*[(1-z^{-1})\mu\beta; z] = [c\{(1-z^{-1})\mu\beta; z\}]/[1-z^r\psi\{(1-z^{-1})(1+\beta)\}],$$

so that for $z \in R$, $\operatorname{Re} s \geqq 0$,

$$P^*(s; z) = c(s; z) + [c\{(1-z^{-1})\mu\beta; z\}]z^r\psi[1-z^{-1}+s/\mu] \cdot [1-z^r\psi\{(1-z^{-1})(1+\beta)\}]^{-1}. \tag{4.1}$$

Consider the function

$$F(s; z) = [\prod_{j=1}^{r} (1-T_j z)]P^*(s; z), \qquad |z| \leqq 1, \ \operatorname{Re} s \geqq 0.$$

As $P^*(s; z)$ is the generating function of a probability distribution, $P^*(s; z)$ and hence $F(s; z)$ must be a regular function of z for $|z| \leqq 1$, $\operatorname{Re} s \geqq 0$.

If we define

$$F(s; z) = [\prod_{j=1}^{r} (1-T_j z)]\{c(s; z) + [c\{(1-z^{-1})\mu\beta; z\}]z^r\psi(1-z^{-1}+s/\mu) \times (1-z^r\psi[(1-z^{-1})(1+\beta)])^{-1}\}$$

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for $|z| \geqq 1$, Re $s \geqq 0$, then as the zeros of $1-z^r\psi[(1-z^{-1})(1+\beta)]$ outside the unit circle are those of $\prod_{j=1}^r (1-T_j z)$, $F(s; z)$ must be a regular function of z for Re $s \geqq 0$. Hence, by analytic continuation, $F(s; z)$ is a regular function of z for all finite z for Re $s \geqq 0$.

Also, using (3.2) and (3.8), we can show by Abel's theorem that $\lim_{z\to\infty} c(s; z)$ exists and equals $\sum_{i=1}^\infty c_i^*(s)$, and it can also be shown that $c\{(1-z^{-1})\mu\beta; z\}$ and $c\{\mu\beta; z\}$ converge to the same limit $\sum_{i=1}^\infty c_i^*(\mu\beta)$ as $z \to \infty$.

Thus $\lim_{z\to\infty} F(s; z)/z$ exists, and by (4.1), is given by

$$\text{(4.2)} \quad \lim_{z\to\infty} \frac{F(s; z)}{z} = \left[\prod_{j=1}^r (-T_j)\right] \left\{\sum_{i=1}^\infty c_i^*(s) - \psi(1+s/\mu) \sum_{i=1}^\infty c_i^*(\mu\beta)(\psi[1+\beta])^{-1}\right\}.$$

Since a function which is analytic for all finite values of z and $O(|z|^k)$, k a non-negative integer, as $z \to \infty$ is a polynomial in z of degree less than or equal to k, it follows that

$$F(s; z) = A_r(s)z^r + A_{r-1}(s)z^{r-1} + \cdots + A_0(s), \qquad \text{Re } s \geqq 0,$$

where $A_r(s)$, $A_{r-1}(s), \cdots, A_0(s)$ are functions of s alone. Thus

$$\text{(4.3)} \quad P^*(s; z) + [\prod_{j=1}^r (1-T_j z)]^{-1}[A_r(s)z^r + A_{r-1}(s)z^{r-1} + \cdots + A_0(s)], \qquad |z| \leqq 1, \ \text{Re } s \geqq 0.$$

Hence if we define α_j, $j = 1, 2, \cdots, r$ by

$$\text{(4.4)} \qquad \prod_{j=1}^r (1-T_j z)^{-1} \equiv \sum_{j=1}^r \alpha_j (1-T_j z)^{-1},$$

legitimate since the T_j are all distinct, we obtain

$$\text{(4.5)} \quad \begin{aligned} P_0^*(s) &= A_0(s) \sum_{t=1}^r \alpha_t, \\ P_1^*(s) &= A_0(s) \sum_{t=1}^r \alpha_t T_t + A_1(s) \sum_{t=1}^r \alpha_t, \\ &\cdots \\ P_{r-1}^*(s) &= A_0(s) \sum_{t=1}^r \alpha_t T_t^{r-1} + A_1(s) \sum_{t=1}^r \alpha_t T_t^{r-2} + \cdots + A_{r-1}(s) \sum_{t=1}^r \alpha_t, \\ P_j^*(s) &= A_0(s) \sum_{t=1}^r \alpha_t T_t^j + A_1(s) \sum_{t=1}^r \alpha_t T_t^{j-1} + \cdots + A_r(s) \sum_{t=1}^r \alpha_t T_t^{j-r}, \quad j \geqq r. \end{aligned}$$

It follows readily from (4.4) that $\alpha_t \neq 0$, $t = 1, 2, \cdots, r$.

5. Determination of $A_j(s)$, $j = 0, 1, \cdots, r$

From (3.1),

$$P_j(u^{(n+1)}) = \sum_{i=0}^{\infty} P_{j+i-r}(u^{(n)})[\exp\{-\mu(u_{n+1}+\beta u_n)\}]\{\mu(u_{n+1}+\beta\mu_n)\}^i/i!, \quad j \geqq r$$

$$= \sum_{i=0}^{\infty}\sum_{l=0}^{i} P_{j+i-r}(u^{(n)}) \exp(-\mu\beta u_n)\frac{(\mu\beta u_n)^{i-l}}{(i-l)!}\exp(-\mu u_{n+1})\frac{(\mu n_{n+1})^l}{l!},$$

whence

$$P^*(s, n+1) = \sum_{i=0}^{\infty} \frac{\partial^i}{\partial\sigma^i}[P^*_{j+i-r}(\sigma\beta, n)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}, \qquad j \geqq r.$$

Letting $n \to \infty$ and using (4.5) we see that for $j \geqq 2r$

$$A_0(s)\sum_{t=1}^{r}\alpha_t T_t^j + A_1(s)\sum_{t=1}^{r}\alpha_t T_t^{j-1} + \cdots + A_r(s)\sum_{t=1}^{r}\alpha_t T_t^{j-r}$$

$$= \sum_{t=1}^{r}\alpha_t \sum_{i=0}^{\infty}\frac{(-\mu T_t)^i}{i!}\frac{\partial^i}{\partial\sigma^i}[\{A_0(\sigma\beta)T_t^{j-r}+\cdots+A_r(\sigma\beta)T_t^{j-2r}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$= \sum_{t=1}^{r}\alpha_t[\{A_0(\mu[1-T_t]\beta)T_t^{j-r}+\cdots+A_r(\mu[1-T_t]\beta)T_t^{j-2r}\}\psi\{1-T_t+s/\mu\}].$$

Hence

$$\sum_{t=1}^{r}\alpha_t T_t^k[A_0(s)T_t^{2r}+A_1(s)T_t^{2r-1}+\cdots+A_r(s)T_t^r-\psi\{1-T_t+s/\mu\}$$
$$\cdot\{A_0(\mu\beta[1-T_t])T_t^r+\cdots+A_r(\mu\beta[1-T_t])\}]$$
$$= 0 \text{ for } k \geqq 0.$$

Consider this result for $k = 0, 1, 2, \cdots, r-1$. Since $\alpha_j \neq 0$, $j = 1, 2, \cdots, r$, and the hypothesis that the T_j are distinct for $j = 1, 2, \cdots, r$ implies that the $r \times r$ matrix $(a_{ik}) = (T_i^{k-1})$ is non-singular, we have that

$$\text{(5.1)} \quad A_0(s)T_t^{2r}+A_1(s)T_t^{2r-1}+\cdots+A_r(s)T_t^r$$
$$= \psi\{1-T_t+s/\mu\}\{A_0(\mu\beta[1-T_t])T_t^r+\cdots+A_r(\mu\beta[1-T_t])\},$$
$$t = 1, 2, \cdots, r.$$

An argument similar to the above, starting from the expression for $P_r(u^{(n+1)})$ given by (3.1) yields

$$A_0(s)\sum_{t=1}^{r}\alpha_t T_t^r + A_1(s)\sum_{t=1}^{r}\alpha_t T_t^{r-1} + \cdots + A_r(s)\sum_{t=1}^{r}\alpha_t$$

$$= [\{A_0(\sigma\beta)\sum_{t=1}^{r}\alpha_t\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[\{A_0(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t+A_1(\sigma\beta)\sum_{t=1}^{r}\alpha_t\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$
$$+\cdots$$
$$+\frac{(-\mu)^{r-1}}{(r-1)!}\frac{\partial^{r-1}}{\partial\sigma^{r-1}}[\{A_0(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{r-1}+\cdots+A_{r-1}(\sigma\beta)\sum_{t=1}^{r}\alpha_t\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$
$$+\sum_{i=r}^{\infty}\frac{(-\mu)^i}{i!}\frac{\partial^i}{\partial\sigma^i}[A_0(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{i}+\cdots+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{i-r}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}.$$

Using (5.1), we deduce that

(5.2)
$$[\{A_1(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-1}+A_2(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-2}+\cdots+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$
$$+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[\{A_2(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-1}+\cdots+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r+1}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$
$$+\cdots$$
$$+\frac{(-\mu)^{r-1}}{(r-1)!}\frac{\partial^{r-1}}{\partial\sigma^{r-1}}[\{A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-1}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$
$$=0.$$

From (4.4),

$$z^k\prod_{t=1}^{r}(1-T_t z)^{-1}\equiv\sum_{t=1}^{r}\frac{\alpha_t}{T_t^k}(1-T_t z)^{-1},\qquad k=1,2,\cdots,r-1. \tag{5.3}$$

Substitution of $z=0$ gives

$$\sum_{t=1}^{r}\frac{\alpha_t}{T_t^k}=0,\qquad k=1,2,\cdots,r-1. \tag{5.4}$$

Multiplication of (5.3) by z, for $k=r-1$, yields, on letting $z\to\infty$,

$$\sum_{t=1}^{r}\frac{\alpha_t}{T_t^r}\neq(-1)^{r+1}\prod_{t=1}^{r}T_j^{-1},$$

so that

$$\sum_{t=1}^{r}\frac{\alpha_t}{T_t^r}\neq 0. \tag{5.5}$$

By virtue of (5.4), (5.2) reduces to

$$[A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}=0.$$

Similarly, use of the expressions for $P_i(u^{(n+1)})$, $i = 1, 2, \cdots, r-1$, given by (3.1) provides us with the equations

$$[\{A_{r-1}(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r-1}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$= 0,$$

$$\cdots$$

$$\cdots$$

$$[\{A_1(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}+\cdots+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-(2r-1)}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[\{A_2(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}+\cdots+A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-(2r-2)}\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$+\cdots$$

$$+\frac{(-\mu)^{r-1}}{(r-1)!}\frac{\partial^{r-1}}{\partial\sigma^{r-1}}[A_r(\sigma\beta)\sum_{t=1}^{r}\alpha_t T_t^{-r}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}$$

$$= 0,$$

which simplify, because of (5.5), to

$$[A_r(\sigma\beta)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu} = 0,$$

$$[A_{r-1}(\sigma\beta)\psi\{(\sigma+s\}]_{\sigma=\mu}+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[A_r(\sigma\beta)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu} = 0,$$

$$\cdots$$

$$[A_1(\sigma\beta)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}+\cdots+\frac{(-\mu)^{r-1}}{(r-1)!}\frac{\partial^{r-1}}{\partial\sigma^{r-1}}[A_r(\sigma\beta)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}=0,$$

or, since $\psi\{1+s/\mu\}$ will not vanish if s is real, to

$$\begin{aligned}
&[A_r(\sigma\beta)]_{\sigma=\mu} = 0,\\
(5.6)\qquad &[A_{r-1}(\sigma\beta)]_{\sigma=\mu}+\frac{(-\mu)}{1!}\frac{\partial}{\partial\sigma}[A_r(\sigma\beta)]_{\sigma=\mu} = 0,\\
&[A_1(\sigma\beta)]_{\sigma=\mu}+\cdots+\frac{(-\mu)^{r-1}}{(r-1)!}\frac{\partial^{r-1}}{\partial\sigma^{r-1}}[A_r(\sigma\beta)]_{\sigma=\mu} = 0.
\end{aligned}$$

Comparing (3.6), (4.3), we find that

$$(5.7)\qquad A_0(s)+A_1(s)+\cdots+A_r(s) = \prod_{t=1}^{r}(1-T_t)\psi(s/\mu).$$

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Since the T_t, $t = 1, 2, \cdots, r$ are distinct and lie inside the unit circle, (5.1), (5.7) form a set of $r+1$ independent linear equations in the $r+1$ variables $A_0(s)$, $A_1(s), \cdots, A_r(s)$. These can be solved, the r unknown constants $\sum_{i=0}^{r} A_i(\mu\beta)[1-T_t])$, $t = 1, 2, \cdots, r$, being obtained with the aid of the relations (5.6). Equations (4.5) then give the limiting distribution of the queueing system, the T_t, $t = 1, 2, \cdots, r$ being obtained from (2.1) and the α_t, $t = 1, 2, \cdots, r$ from (4.4).

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The Australian National University
Canberra