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VON NEUMANN OPERATORS IN $\mathscr{B}_1(\Omega)$

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ABSTRACT. For a connected open subset Ω of the plane and n a positive integer, let $\mathscr{B}_n(\Omega)$ be the space introduced by Cowen and Douglas in their paper, "Complex geometry and operator theory". Our main concern is the case $n=1$, in which case we show the existence of a functional calculus for von Neumann operators in $\mathscr{B}_1(\Omega)$ for which a spectral mapping theorem holds. In particular we prove that if the spectrum of $T\in\mathscr{B}_1(\Omega)$, $\sigma(T)$, is a spectral set for T, and if $\sigma(T)=\bar{\Omega}$, then $\sigma(f(T))=f(\Omega)^-$ for every bounded analytic function f on the interior of L, where L is compact, $\sigma(T)\subset L$, the interior of L is simply connected and L is minimal with respect to these properties. This functional calculus turns out to be nice in the sense that the general study of von Neumann operators in $\mathscr{B}_1(\Omega)$ is reduced to the special situation where Ω is an open connected subset of the unit disc $\mathbb{D}$ with $\partial\mathbb{D}\subset\partial\Omega$.

§1. **Introduction.** If K is a compact subset of the plane, then K is a *spectral set* for $T\in\mathscr{L}(\mathscr{H})$ if $\sigma(T)\subseteq K$ and $\|f(T)\|\le\max\{|f(z)|:z\in K\}$ for all rational functions f with poles off K. An operator T whose spectrum is a spectral set for T is called a *von Neumann operator.*

In dealing with von Neumann operators in $\mathscr{B}_1(\Omega)$ we will show that there exists a simply connected open set Ω_0 containing Ω such that the weak-star closure of the rational functions in T with poles off $\bar{\Omega}_0$ is isometrically isomorphic to the space $H^\infty(\Omega_0)$ of all bounded analytic functions in Ω_0. This result will furnish us with a functional calculus for von Neumann operators in $\mathscr{B}_1(\Omega)$ for which a spectral mapping theorem holds. That is, $\sigma(f(T))=f(\Omega)^-$ for all f in $H^\infty(\Omega_0)$. If T is a von Neumann operator in $\mathscr{B}_1(\Omega)$ and φ is the conformal mapping from Ω_0 onto $\mathbb{D}$, then $\varphi(T)$ is also a von Neumann operator, furthermore $\varphi(T)$ is in $\mathscr{B}_1(\varphi(\Omega))$. This result enables us to transfer the general study of von Neumann operators in $\mathscr{B}_1(\Omega)$ to the special case where

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Ω is an open connected subset of the unit disc $\mathbb{D}$, $\partial\mathbb{D} \subseteq \partial\Omega$, and the corresponding set $\Omega_0 = \mathbb{D}$.

§2. **Preliminaries.** Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$.

For a connected open subset Ω of the plane and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy:

(a) $\Omega \subseteq \sigma(T) = \{\omega \in \mathbb{C}: T - \omega \text{ is not invertible}\}$;

(b) $\operatorname{ran}(T - \omega) = \mathcal{H}$ for ω in Ω;

(c) $\bigvee_{\omega \in \Omega} \ker(T - \omega) = \mathcal{H}$; and

(d) $\dim \ker(T - \omega) = n$ for ω in Ω.

The space $\mathcal{B}_n(\Omega)$ has been introduced and investigated by Cowen and Douglas [5].

For $T \in \mathcal{L}(\mathcal{H})$, the approximate point spectrum of T and point spectrum of T will be denoted by $\sigma_{ap}(T)$ and $\sigma_p(T)$, respectively.

(2.1) LEMMA. *Let $T \in \mathcal{B}_n(\Omega)$ such that $\sigma(T) = \bar{\Omega}$. Then $\sigma(T) = \sigma_{ap}(T)$.*

Proof. By definition of $\mathcal{B}_n(\Omega)$, $\Omega \subseteq \sigma_p(T) \subseteq \sigma_{ap}(T)$. Because $\sigma_{ap}(T)$ is closed (Halmos [9], problem 62), $\bar{\Omega} \subseteq \sigma_{ap}(T)$. Hence $\sigma(T) = \sigma_{ap}(T)$.

Let K be a compact subset of the complex plane $\mathbb{C}$, and let $R(K)$ denote the algebra of all continuous complex-valued functions on K which can be approximated uniformly on K by rational functions whose poles all lie outside K. $R(K)$ is a *Dirichlet algebra* on ∂K if $\operatorname{Re} R(K)|_{\partial K}$ is dense in $C_{\mathbb{R}}(\partial K)$. That is, the real parts of the functions in $R(K)$ when restricted to ∂K are dense in the continuous real-valued functions on ∂K. $R(K)$ is a *hypodirichlet algebra* on ∂K if there exists invertible elements $f_1, f_2, \ldots, f_n$ in $R(K)$ such that the linear span of $\operatorname{Re} R(K)|_{\partial K}$, $\log|f_1|, \ldots,$ and $\log|f_n|$ is uniformly dense in $C_{\mathbb{R}}(\partial K)$.

If $R(K)$ is Dirichlet and z is in the interior of K (denoted by K^0), then there exists a unique measure m_z supported on ∂K such that $\int f\, dm_z = f(z)$ for all $f \in R(K)$. Let $\{G_n\}_{n=1}^{\infty}$ be the components of K^0 and fix $z_n \in G_n$ for all $n = 1, 2, \ldots$. Set $m = \sum_{n=1}^{\infty} 2^{-n} m_{z_n}$. This m will be referred to as the *harmonic measure* on ∂K. We let $H^\infty(\partial K)$ denote the weak-star closure of $R(K)$ in $L^\infty(m)$. The weak-star topology is denoted by w^*. It is a well known fact that the definitions of $L^\infty(m)$ and $H^\infty(\partial K)$ are independent of the sequence $\{z_n\}$ used to define the measure m.

For each function f in $H^\infty(\partial K)$ we define a function $\hat{f}$ on K^0 by $\hat{f}(z) = \int f\, dm_z$. It is a standard result that $\hat{f}$ is a bounded analytic function in K^0 (see Sarason [11], p. 5). It is also shown there that whenever $R(K)$ is a Dirichlet algebra the map $f \to \hat{f}$ is an isometric isomorphism of $H^\infty(\partial K)$ onto $H^\infty(K^0)$, the space of bounded analytic functions on K^0. It is customary not to distinguish between the two spaces $H^\infty(\partial K)$ and $H^\infty(K^0)$ and we will follow this custom too.

We will also use, either explicitly or implicitly, a well known fact from function theory which states that whenever $R(K)$ is a Dirichlet algebra, then the components of K^0 are simply connected. So it makes sense to talk about the conformal map from K^0 onto $\mathbb{D}$, in case K^0 is connected.

We will denote the Banach space of trace-class operators in $\mathcal{L}(\mathcal{H})$ with the trace norm $\| \|_1$ by $\mathcal{C}_1$. Recall from [6, Theorem 8, p. 105] that setting

$$\langle T, A\rangle = \operatorname{tr}(AT), \qquad A \in \mathcal{L}(\mathcal{H}), \qquad T \in \mathcal{C}_1,$$

defines a bilinear functional on $\mathcal{C}_1 X \mathcal{L}(\mathcal{H})$ that allows us to identify $\mathcal{C}_1^*$ with $\mathcal{L}(\mathcal{H})$. We refer to the weak-star topology $\mathcal{L}(\mathcal{H})$ inherits as a dual as the w^*-topology. Some authors choose to call this topology "ultra weak", though it is stronger than what is commonly referred to as the "weak operator topology". We denote the weak operator topology by *WOT*.

Let S^2 denote the extended complex plane. The *analytic capacity* of a planar set E is

$$\gamma(E) = \sup\{|f'(\infty)| : f \text{ is analytic on } S^2 \sim K \text{ for some compact subset } K \text{ of } E, |f| \leq 1\}.$$

If K is a compact set with $\sigma(T) \subseteq K$, then $\mathcal{R}_K(T)$ will denote the w^*-closure of the rational functions in T with poles off K.

(2.2) DEFINITION. If $T \in \mathcal{L}(\mathcal{H})$, then a compact subset $K \subseteq \mathbb{C}$ is *D-spectral for T* if K is a spectral set for T and $R(K)$ is a Dirichlet algebra.

If K is a spectral set for $T \in \mathcal{L}(\mathcal{H})$ and f is a rational function with poles off K, then clearly $f(T)$ is well defined. If Φ_K denotes the map that sends f to $f(T)$, then the fact that K is a spectral set says that Φ_K extends to a norm contraction $\Phi_K : R(K) \to \mathcal{R}_K(T)$.

An operator T in $\mathcal{L}(\mathcal{H})$ is *irreducible* if T has no nontrivial reducing subspaces.

(2.3) PROPOSITION. (Agler [1]). *Let T be a von Neumann operator such that T is irreducible and* $\sigma(T) = \sigma_{ap}(T)$. *If K is D-spectral for T, then* Φ_K *extends to a norm contractive algebra homomorphism* $\Phi_K : H^\infty(\partial K) \to \mathcal{R}_K(T)$. *Furthermore* Φ_K *is continuous when domain and range have their* w^* *topologies.*

§3. **Von Neumann Operators in $\mathcal{B}_1(\Omega)$.** In his paper [1] J. Agler proves the following theorem.

(3.1) THEOREM. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a von Neumann operator such that T is irreducible and* $\sigma(T) = \sigma_{ap}(T)$. *Then there exists a compact set K with the following properties:*

(1) *K is D-spectral for T,*

(2) K^0 *has one (simply connected) component,*

and

(3) *If φ is the conformal map from K^0 onto $\mathbb{D}$, then $\partial\mathbb{D} \subseteq \sigma(\varphi(T))$.*

Agler has also shown that the Φ of Proposition 2.3 associated with the K of Theorem 3.1 is an isometric, w^* homeomorphic algebra isomorphism from $H^\infty(\partial K)$ onto $\mathcal{R}_K(T)$. We give a simpler proof of this theorem and we use the idea of the proof to derive a few more interesting results. To do this we need a few technical lemmas.

(3.2) LEMMA. *If K is a compact subset of the plane then there is a countable ordinal α_0 such that for every $\alpha < \alpha_0$ there is a component V_α of $\mathbb{C} \sim K$ such that:*

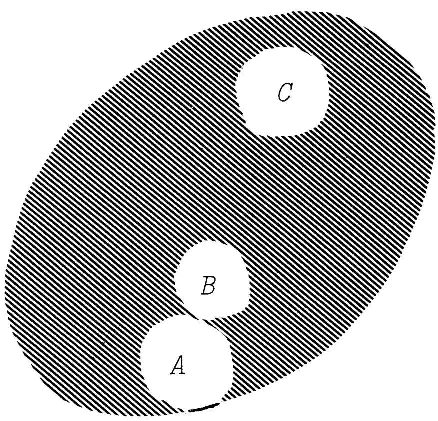
(a) *If 0 is the first ordinal, V_0 is the unbounded component of $\mathbb{C} \sim K$;*

(b) *For each ordinal α, $V_\alpha^- \cap [\bigcup_{\beta<\alpha} V_\beta]^- \neq \emptyset$;*

(c) *If V is a component of $\mathbb{C} \sim K$ and $V \neq V_\alpha$ for any α, then $V^- \cap [\bigcup_\alpha V_\alpha]^- = \emptyset$.*

Proof. The proof is an easy application of transfinite induction.

The enumeration of the components of $\mathbb{C} \sim K$ in the preceding lemma picks out those components that can be "chained" to the unbounded component. So if K is finitely connected, the unbounded component is enumerated but there may be no others. If K is the annulus, for example, only the unbounded component is selected. On the other hand if K is as in the figure below,



then V_0 is the unbounded component, $V_1 = A$, $V_2 = B$, and $\alpha_0 = 3$. If K is the

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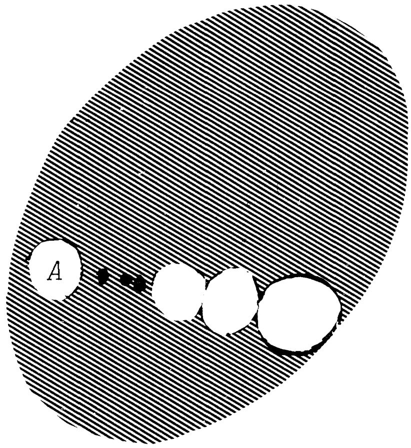
infinitely connected set pictured below,



then every component of $\mathbb{C} \sim K$ is enumerated and A corresponds to the first infinite ordinal.

If K is a compact set, $\{V_\alpha : \alpha < \alpha_0\}$ are those components of $\mathbb{C} \sim K$ that are picked out by the preceding lemma and $\hat{K}$ is the polynomially convex hull of K, then for each $\alpha, \alpha < \alpha_0$, let

$$L_\alpha = \hat{K} \sim \bigcup_{\beta < \alpha} V_\beta.$$

Also let

$$L_{\alpha_0} = \hat{K} \sim \bigcup_{\alpha < \alpha_0} V_\alpha = \bigcap_\alpha L_\alpha$$

and set

$$L = L_{\alpha_0}.$$

(3.3) LEMMA. *For each $\alpha \le \alpha_0$, $R(L_\alpha)$ is a Dirichlet algebra.*

Proof. See (Conway [3], p. 402).

Now let T be a von Neumann operator satisfying the hypothesis of Theorem 3.1. Set $K = \sigma(T)$ in the preceding inductive process and find L. Note that L contains $\sigma(T)$ and by Lemma 3.3, $R(L)$ is a Dirichlet algebra. Therefore L is a D-spectral set for T. Since T is irreducible, L^0 is connected ([1], Lemma E). Actually L is the smallest compact subset of $\mathbb{C}$ that contains K and has L^0 simply connected. In constructing L we did not use the full strngth of Lemma 3.2, this will be done in Theorem 3.6.

In the following lemma note that by Proposition 2.3 it makes sense to talk about the operator $g(T)$ whenever g is in $H^\infty(L^0)$.

(3.4) LEMMA. *Let T and L be as before. If $g \in H^\infty(L^0)$, then $g(\sigma(T) \cap L^0) \subseteq \sigma(g(T))$.*

Proof. Let $\lambda \in \sigma(T) \cap L^0$. Write $g(z) = g(\lambda) + (z-\lambda)h(z)$ where $h \in H^\infty(L^0)$. Then $g(T) - g(\lambda) = (T-\lambda)h(T) = h(T)(T-\lambda)$. Since $T-\lambda$ is not invertible, it follows that $g(T) - g(\lambda)$ is not invertible either.

Next we give a simpler proof of Agler's Theorem, though the idea is taken from Scott Brown's proof of Sarason's characterization of weak-star closure of polynomials (unpublished).

(3.5) THEOREM. *Let T and L be as before. Then there exists an isometric isomorphism $\Phi: H^\infty(\partial L) \to \mathcal{R}_L(T)$ which is a w^* homeomorphism.*

Proof. By Proposition 2.3 there is a norm contractive algebra homomorphism $\Phi_L: H^\infty(\partial L) \to \mathcal{R}_L(T)$ which is w^* continuous. We now show that $\Phi = \Phi_L$ is an isometry. Suppose there is a g in $H^\infty(L^0)$ such that $\|g(T)\| < \|g\|_{L^0}$. Put $G = \{a \in L^0 : |g(a)| > \|g(T)\|\}$ and let V be a component of G. Clearly G and V are open. We show that $V^- \cap \partial L \neq \emptyset$. Otherwise $V^- \subseteq L^0$. Now there exists $z_0 \in V^-$ such that $|g(z_0)| = \|g\|_{V^-} > \|g(T)\|$. There exists $B = B(z_0, r) \subseteq L^0$ such that $|g(z)| > \|g(T)\|$ for all z in B. Clearly $V \cup B \subseteq G$, so $V \cup B = V$ and $z_0 \in V$. Thus g attains its maximum on V^- at z_0 in V contradicting the maximum modulus principle. Next we show that $V \cap K = \emptyset$ ($K = \sigma(T)$). If this is true, then it is easy to see that Φ is an isometry. Indeed if $V \subseteq \mathbb{C} \sim K$, there is a component U of $\mathbb{C} \sim K$, $V \subseteq U$. But L was obtained in such a way that each component of $\mathbb{C} \sim K$ is included in either L or $\mathbb{C} \sim L$. Since $V \subseteq L$, $U \subseteq L$. Because $V^- \cap \partial L \neq \emptyset$ we have $U^- \cap \partial L \neq \emptyset$ contradicting condition (c) of Lemma 3.2. We denote the spectral radius of an operator A in $\mathcal{L}(\mathcal{H})$ by $r(A)$. By Lemma 3.4 $g(\sigma(T) \cap L^0) \subseteq \sigma(g(T))$, so $\|g(T)\| \geq r(g(T)) = \sup\{|\lambda| : \lambda \in \sigma(g(T))\} \geq \sup\{|\lambda| : \lambda \in g(\sigma(T) \cap L^0)\} = \sup\{|g(\lambda)| : \lambda \in \sigma(T) \cap L^0\}$. It is now clear that $V \cap \sigma(T) = \emptyset$. Therefore Φ is an isometry.

It is obvious that the range of Φ contains the rational functions in T with poles off L. The proof will now be completed by using the fact that any isometric, w^* continuous, linear map between the duals of two separable Banach spaces has w^* closed range and is actually a w^* homeomorphism onto its range.

Now let $T \in \mathcal{B}_1(\Omega)$ such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T. Then T is irreducible ([5], Corollary 1.19) and $\sigma(T) = \sigma_{ap}(T)$ by Lemma 2.1. Let φ be the conformal map from L^0 onto $\mathbb{D}$ and let $A = \varphi(T)$. Then $\|A\| = \|\varphi\|_\infty = 1$, and A is irreducible ([1], Lemma 2). If $h \in H^\infty$, then $\|h\|_{\sigma(A) \cap \mathbb{D}} = \|h\|_{\varphi(\sigma(T) \cap L^0)}$ ([1], Lemma 3). Hence $\|h\|_{\sigma(A) \cap \mathbb{D}} = \|h \circ \varphi\|_{\sigma(T) \cap L^0} = \|h \circ \varphi\|_{L^0} = \|h\|_\infty$ by the construction of L.

The following theorem allows us to transfer the general study of von

Neumann operators in $\mathcal{B}_1(\Omega)$ to the special case when Ω is an open connected subset of the unit disc $\mathbb{D}$ and $\partial\mathbb{D}\subseteq\partial\Omega$.

(3.6) THEOREM. *Let $T\in\mathcal{B}_1(\Omega)$ such that $\sigma(T)=\bar{\Omega}$ is a spectral set for T and let φ be the conformal map from L^0 onto $\mathbb{D}$. Then $A=\varphi(T)$ is a von Neumann operator.*

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be those components of $\mathbb{C}\sim K$ $(K=\sigma(T))$ that cannot be chained to the unbounded component of $\mathbb{C}\sim K$. Then $U_i\subseteq L$, $i\geq 1$. Now $\varphi: L^0\to\mathbb{D}$ and if r is a rational function with poles off $\sigma(A)$, then $r\circ\varphi\in H^\infty$ $(\text{int}(K\cup\bigcup_{j=N+1}^{\infty} U_j))$ for some N. Let $W_1=\bigcup_{0<\alpha<\alpha_0} V_\alpha$, $W_2=\bigcup_{j=1}^{N} U_j$, $W=W_1\cup W_2$ and set $X=\hat{K}\sim W$ and $Y=\hat{K}\sim W_2$ (see Fig. 1).

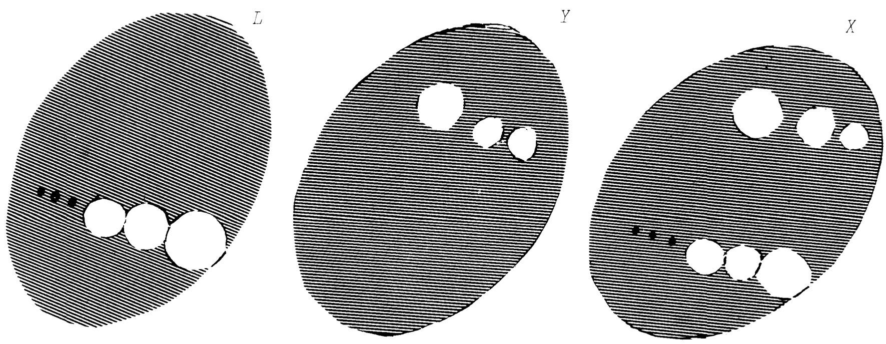


Figure 1

In order to show that $A=\varphi(T)$ is a von Neumann operator, it suffices to show that $\|r\circ\varphi(T)\|\leq\|r\circ\varphi\|_{\bar{\Omega}}$. So we need to approximate $r\circ\varphi$ by a suitable sequence in $R(\bar{\Omega})$. To this end we show that $R(X)$ is a hypodirichlet algebra and then use the approximation properties of these algebras. This will be our next goal.

If S^2 denotes the extended complex plane, then $S^2\sim Y^0$ and $S^2\sim L^0$ have only a finite number of components, $R(L)$ is a Dirichlet algebra (Lemma 3.3) and $R(Y)$ is a hypodirichlet algebra (Gamelin [7]). For a subset E of the plane, let $\gamma(E)$ denote the analytic capacity of E. Applying Theorem 7 of (Gamelin and Garnett [8]) to both $R(L)$ and $R(Y)$ we can find $\delta>0$ sufficiently small such that

$$\gamma(\overline{B(z,\delta)}\sim L)\geq\frac{\delta}{4},\qquad z\in\partial L,$$

and

$$\gamma(\overline{B(z,\delta)}\sim Y)\geq\frac{\delta}{4},\qquad z\in\partial Y.$$

Now it is easy to see that $\partial X = \partial L \cup \partial Y$, hence $\gamma(\overline{B(z,\delta)} \sim X) \geq \delta/4$, $z \in \partial X$. Since $S^2 \sim X^0$ has only a finite number of components again using the same theorem we conclude that $R(X)$ is a hypodirichlet algebra, it is pointwise boundedly dense in $H^\infty(X^0)$, and $R(\partial X) = C(\partial X)$.

Since $r \circ \varphi \in H^\infty(X^0)$ we can choose a sequence $\{f_n\} \subseteq R(X)$ such that $\|f_n\|_X \leq \|r \circ \varphi\|_X$ and $f_n(z) \to r \circ \varphi(z)$ for all $z \in X^0$. Now $\bar{\Omega} \subseteq X$, so $f_n \in R(\bar{\Omega})$ and $\|f_n\|_{\bar{\Omega}} \leq \|r \circ \varphi\|_X = \|r \circ \varphi\|_{\bar{\Omega}}$. Because T is von Neumann, $\|f_n(T)\| \leq \|f_n\|_{\bar{\Omega}}$ and since the unit ball of $\mathcal{L}(\mathcal{H})$ is *WOT* compact, by passing to a subsequence, if necessary, we may assume that $f_n(T) \to S$ (*WOT*) for some S in $\mathcal{L}(\mathcal{H})$.

For ω in Ω let $u(\omega)$ be a unit vector in $\ker(T - \omega)$. Then $f_n(T)u(\omega) \to Su(\omega)$ weakly. Also $f_n(T)u(\omega) = f_n(\omega)u(\omega)$ converges to $r \circ \varphi(\omega)u(\omega)$ in norm. Thus $Su(\omega) = r \circ \varphi(\omega)u(\omega)$. Since $\{u(\omega) : \omega \in \Omega\}$ spans a dense subspace of $\mathcal{H}$, $S = r \circ \varphi(T)$. It is clear that $\|r \circ \varphi(T)\| \leq \|r \circ \varphi\|_{\bar{\Omega}}$. We also know that $\Gamma_T : \{T\}' \to H^\infty(\Omega)$, where $\{T\}'$ is the commutant of T, is a contractive monomorphism (Cowen and Douglas [5], Proposition 1.21), from which we obtain $\|r \circ \varphi\| \leq \|r \circ \varphi(T)\|$. Therefore $\|r \circ \varphi\|_{\bar{\Omega}} = \|r \circ \varphi(T)\|$. It follows that $\varphi(T)$ is a von Neumann operator.

Before concluding this section we will record a few results that will be used in the sequel. In the following lemma, note that the fact that $R(K)$ is a Dirichlet algebra and K^0 has one component implies that K^0 is simply connected.

(3.7) LEMMA. *If K is a D-spectral set for T, K^0 has one component, and φ is the conformal map from K^0 onto $\mathbb{D}$, then the following are true for $\lambda \in K^0$:*

(i) $\ker(\lambda - T) = \ker(\varphi(\lambda) - \varphi(T))$;

(ii) $\operatorname{ran}(\lambda - T)$ *is closed if and only if* $\operatorname{ran}(\varphi(\lambda) - \varphi(T))$ *is closed.*

Proof. (i) If $x \in \ker(\lambda - T)$, then $Tx = \lambda x$. Choose $r_n \in R(K)$ with $r_n \to \varphi$ w^* in $H^\infty(\partial K)$ and $\|r_n\|_K \leq 1$ (Sarason [11], Lemma 4.3). Then $r_n(T)x = r_n(\lambda)x$, so $\varphi(T)x = \varphi(\lambda)x$. Hence $x \in \ker(\varphi(\lambda) - \varphi(T))$.

Conversely, let $y \in \ker(\varphi(\lambda) - \varphi(T))$. Then $\varphi(T)y = \varphi(\lambda)y$, so $p \circ \varphi(T)y = p \circ \varphi(\lambda)y$ for every polynomial p. Because $\varphi^{-1} \in H^\infty$, there is a uniformly bounded sequence $\{p_n\}$ of polynomials such that $p_n(z) \to \varphi^{-1}(z)$, $z \in \mathbb{D}$. Therefore, $p_n \circ \varphi(\lambda) \to \lambda$, $\lambda \in K^0$. Hence $p_n \circ \varphi \to z$ w^* in $H^\infty(\partial K)$. By Proposition 2.3, $p_n \circ \varphi(T) \to T$ w^* in $\mathcal{L}(\mathcal{H})$. Now $p_n \circ \varphi(T)y = p_n \circ \varphi(\lambda)y$. By passing to the limit, $Ty = \lambda y$. Therefore $y \in \ker(\lambda - T)$.

(ii) If $\lambda - T$ does not have closed range, then it is not bounded below on the ortho-complement of $\ker(\lambda - T) = \ker(\varphi(\lambda) - \varphi(T))$. Therefore there exists a sequence $\{x_p\}_{p=1}^\infty$ of unit vectors in the ortho-complement of $\ker(\varphi(\lambda) - \varphi(T))$ with $\|(\lambda - T)x_p\| = \varepsilon_p$ and $\lim_{p\to\infty} \varepsilon_p = 0$. Choose $r_n \in R(K)$ with $r_n \to \varphi$ w^* in $H^\infty(\partial K)$ and $\|r_n\|_k \leq 1$. Then $r_n(\lambda) \to \varphi(\lambda)$ and $r_n(T) \to \varphi(T)$ w^* in $\mathcal{L}(\mathcal{H})$. If $s_n(z) = (r_n(\lambda) - r_n(z))(\lambda - z)^{-1}$, then $s_n \in R(K)$. Because $\lambda \in K^0$ the maximum modulus principle implies there is a constant M such that $\|s_n\|_K \le M$ for all n. Thus $\|s_n(T)\| \le M$.

Now for arbitrary $y \in \mathcal{H}$ and $n, p \ge 1$

$$|((r_n(\lambda) - r_n(T))x_p, y)| = |(s_n(T)(\lambda - T)x_p, y)| \le \|s_n(T)\| \, \|(\lambda - T)x_p\| \, \|y\| \le M \, \|y\| \, \varepsilon_p.$$

Letting $n \to \infty$ we get,

$$|((\varphi(\lambda) - \varphi(T))x_p, y)| \le M \, \|y\| \, \varepsilon_p.$$

From this inequality we conclude that $\|(\varphi(\lambda) - \varphi(T))x_p\| \le M\varepsilon_p$. Therefore $\varphi(\lambda) - \varphi(T)$ is not bounded below on the ortho-complement of $\ker(\varphi(\lambda) - \varphi(T))$ and hence it does not have closed range. The converse follows similarly.

(3.8) Lemma. *Let $T \in \mathcal{B}_1(\Omega)$ such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T. Let L be as before and let φ be the conformal map from L^0 onto $\mathbb{D}$. Then $\varphi(T) \in \mathcal{B}_1(\varphi(\Omega))$, $\sigma(\varphi(T)) = \varphi(\Omega)^-$ and $\partial\mathbb{D} \subseteq \sigma(\varphi(T))$.*

Proof. In order to show that $\varphi(T) \in \mathcal{B}_1(\varphi(\Omega))$ we need to consider the following four conditions.

(1) Because $\Omega \subseteq \sigma(T) \cap L^0$ we get $\varphi(\Omega) \subseteq \varphi(\sigma(T) \cap L^0)$. By Lemma 3.4 we have $\varphi(\sigma(T) \cap L^0) \subseteq \sigma(\varphi(T))$. Therefore $\varphi(\Omega) \subseteq \sigma(\varphi(T))$.

(2) By Lemma 3 of [1] we have $\operatorname{ran}(\varphi(\lambda) - \varphi(T))^- = \mathcal{H}, \lambda \in L^0$. Since $T \in \mathcal{B}_1(\Omega)$ we conclude that $\operatorname{ran}(\lambda - T)$ is closed, $\lambda \in \Omega$. By Lemma 3.7, $\operatorname{ran}(\varphi(\lambda) - \varphi(T)) = \mathcal{H}$, $\lambda \in \Omega$.

(3) By Lemma 3.7, $\ker(\lambda - T) = \ker(\varphi(\lambda) - \varphi(T))$, $\lambda \in \Omega$.

(4) Since $T \in \mathcal{B}_1(\Omega)$, $\dim \ker(\lambda - T) = 1$, $\lambda \in \Omega$. From (3) we get $\dim \ker(\varphi(\lambda) - \varphi(T)) = 1$, $\lambda \in \Omega$.

To show that $\sigma(\varphi(T)) = \varphi(\Omega)^-$ note that $\varphi(\Omega)^- \subseteq \sigma(\varphi(T))$ by (1). To prove the reverse inequality consider the isometric isomorphism $\Phi: \mathcal{R}_L(T) \to H^\infty(L^0)$ of Theorem 3.5 and let $r: H^\infty(L^0) \to H^\infty(\Omega)$ be the restriction map. Then $r \circ \varphi: \mathcal{R}_L(T) \to H^\infty(\Omega)$ is an algebra monomorphism. By Theorem 10.18 of Rudin [10], $\partial\sigma_{\mathcal{R}_L(T)}(\varphi(T)) \subseteq \partial\sigma_{H^\infty(\Omega)}(\varphi)$. Because $\sigma_{\mathcal{R}_L(T)}(\varphi(T)) = \sigma_{H^\infty(L^0)}(\varphi) = \varphi(L^0)^- = \bar{\mathbb{D}}$, we have $\partial\mathbb{D} \subseteq \varphi(\Omega)^-$. Hence $\partial\mathbb{D} \subseteq \sigma(\varphi(T)) \subseteq \bar{\mathbb{D}}$. Therefore

$$\begin{aligned}\sigma(\varphi(T)) &= [\sigma(\varphi(T)) \cap \mathbb{D}] \cup \partial\mathbb{D} = \varphi(\sigma(T) \cap L^0) \cup \partial\mathbb{D} \\ &= \varphi(\bar{\Omega} \cap L^0) \cup \partial\mathbb{D} = \varphi((\Omega \cup \partial\Omega) \cap L^0) \cup \partial\mathbb{D} \subseteq \varphi(\Omega)^-.\end{aligned}$$

So

$$\varphi(\Omega)^- = \sigma(\varphi(T)).$$

§4. **Spectral mapping theorem.** Consider an operator $T \in \mathcal{B}_1(\Omega)$ such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T and let L be as in the preceding section. We want to determine the spectrum of $f(T)$ for $f \in H^\infty(L^0)$. The result is that a spectral mapping theorem holds:

$$\sigma(f(T)) = f(\Omega)^-, \qquad f \in H^\infty(L^0).$$

The idea of the proof of the next theorem is due to Conway and Olin [4, Lemma 8.9].

(4.1) THEOREM. *Let $T \in \mathcal{B}_1(\Omega)$ such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T. Then $\sigma(f(T)) = f(\Omega)^-$ for $f \in H^\infty(L^0)$.*

Proof. Let φ be the conformal map from L^0 onto $\mathbb{D}$. Then we have already shown that $\varphi(T) \in \mathcal{B}_1(\varphi(\Omega))$ (Lemma 3.8), $\sigma(\varphi(T)) = \varphi(\Omega)^-$ is a spectral set for $\varphi(T)$ (Theorem 3.6) and $\partial\mathbb{D} \subseteq \sigma(\varphi(T))$ (Lemma 3.8). Without loss of generality we may assume that Ω is an open connected subset of $\mathbb{D}$ such that $L^0 = \mathbb{D}$, $T \in \mathcal{B}_1(\Omega)$ is such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T, $\partial\mathbb{D} \subseteq \bar{\Omega}$ and we want to show that $\sigma(f(T)) = f(\Omega)^-$ for $f \in H^\infty$.

By Lemma 3.4, $f(\Omega)^- \subseteq \sigma(f(T))$, so we need only show $\sigma(f(T)) \subseteq f(\Omega)^-$, $f \in H^\infty$. To see this we assume $a \notin f(\Omega)^-$ and we show that $f(T) - a$ is an invertible operator in $\mathcal{L}(\mathcal{H})$.

Because $a \notin f(\Omega)^-$ there is $\delta > 0$ such that $\Omega \subseteq \{z \in \mathbb{D}: |f(z) - a| \geq \delta\}$. Put $J_1 = \{z \in \mathbb{D}: |f(z) - a| \geq \delta\}^-$ and $J_2 = \{z \in \mathbb{D}: |f(z) - a| \geq \delta/2\}^-$. It is clear that $\bar{\Omega} \subseteq J_1 \subseteq J_2$. Because $\partial\mathbb{D} \subseteq \bar{\Omega}$, it follows that $\partial\mathbb{D} \subseteq J_1 \subseteq J_2$.

Now $(f-a)^{-1} \in H^\infty(J_2^0)$. Because $\partial\mathbb{D} \subseteq J_2 \subseteq \bar{\mathbb{D}}$, $\mathbb{C} \sim \bar{\mathbb{D}}$ is the unbounded component of $\mathbb{C} \sim J_2$. By Theorem VIII. 10.7 of [7], there is a uniformly bounded sequence $\{f_n\}$ in $H^\infty(J_2^0)$ such that each f_n has an analytic continuation to a neighborhood of $\partial\mathbb{D}$, and $f_n(z) \to (f(z) - a)^{-1}$ for z in J_2^0. But $J_1 \subseteq J_2^0 \cup \partial\mathbb{D}$, so each f_n is analytic in a neighborhood of J_1. By Runge's Theorem ([2], p. 198) $f_n \in R(J_1)$. Moreover, f_n is uniformly bounded on J_1 and $f_n(z) \to (f(z) - a)^{-1}$ for all z in $J_1 \cap \mathbb{D}$. Because $\bar{\Omega} \subseteq J_1$, $f_n \in R(\bar{\Omega})$ and $\|f_n\|_{\bar{\Omega}} \leq M$ for some M.

Since $\bar{\Omega} = \sigma(T)$ is a spectral set for T, $\|f_n(T)\| \leq \|f_n\|_{\bar{\Omega}} \leq M$. But the unit ball of $\mathcal{L}(\mathcal{H})$ is *WOT* compact, so by passing to a subsequence we may assume $f_n(T) \to B$ (*WOT*) for some B in $\mathcal{L}(\mathcal{H})$. For $\omega \in \Omega$, let $u(\omega)$ be a unit vector in $\ker(T - \omega)$. Then $f_n(T)u(\omega) \to Bu(\omega)$ weakly. Also $f_n(T)u(\omega) = f_n(\omega)u(\omega)$ converges to $(f(\omega) - a)^{-1}u(\omega)$ in norm. Hence $Bu(\omega) = (f(\omega) - a)^{-1}u(\omega)$, $\omega \in \Omega$. But $B(f(T) - a)u(\omega) = (f(\omega) - a)(f(\omega) - a)^{-1}u(\omega)$, $\omega \in \Omega$. Since $\{u(\omega): \omega \in \Omega\}$ spans a dense subspace of $\mathcal{H}$, B is the inverse of $f(T) - a$.

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