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RELATION BETWEEN THE SHAPLEY VALUE AND NULL GROUP COHOMOLOGY

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The purpose of this paper is to give an application of homological methods to game theory. The main theorems give conditions of a homological nature for the existence and uniqueness of the Shapley value for games with continuum players.

Introduction

The Shapley value for n-person games was introduced in [3]. In [1] Aumann and Shapley have given the following definition of the Shapley value for games of continuum players: Let $I = [0, 1]$ be the set of players, let BV denote the space of all bounded variation set functions defined on Borel sets $\mathcal{B}$ of I, and let FA denote the set of all bounded finitely additive set functions on $(I, \mathcal{B})$. Finally, let G be the group of all Borel automorphisms of the interval I. Then a Shapley value is a map $\varphi : Q \mapsto FA$, where Q is a G-invariant subspace of BV, and φ is linear, positive and it satisfies the following conditions:

(I) $\varphi g_* = g_* \varphi$ for all $g \in G$;

(II) $(\varphi v)(I) = v(I)$ for all $v \in Q$.

(Here $(g_*\varphi)(S) := \varphi(g(S))$ for $g \in G$ and $S \in \mathcal{B}$.)

In [1] there were investigated the subspaces Q of BV for which

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there exists the Shapley value φ and also those for which φ is uniquely determined. A natural question arises: how many Shapley values exist for a given linear space and how it depends on the group G. In this paper we give a partial answer to this question using the homological methods.

Since axioms (I) and (II) state a functor property in the category of G-modules, it apparently leads to the algebraic approach to the problem of existence and uniqueness of the Shapley value.

In Section 1 we define the action of a group G on the Banach algebra BV, where G is a group of the Borel automorphisms of a Polish space; and we introduce special cohomological spaces. Section 3 contains the main result of this paper (Theorem 3.1). If the null cohomological space, $H^0(G, FA)$ is nontrivial, then there exist G-Shapley value on every subspace $Q \supseteq FA$ with $Q \subseteq BV$.

From the proof of Theorem 3.1 it also follows how many distinct Shapley values are available. A necessary condition for the existence exactly one G-Shapley value is established in Theorem 3.2.

For basic notions of homological algebra we refer to [2].

1. Preliminaries and notations

Let $(X, \mathcal{B})$ be any measurable space, that is, $\mathcal{B}$ is a σ-algebra of subsets of X. By BV we denote the Banach space of all real valued set functions of bounded variation defined on $(X, \mathcal{B})$. By FA (respectively FA^+) we denote the subspace of all bounded finitely additive (respectively nonnegative) set functions in BV (see [1]).

By $\mathrm{Aut}(X, \mathcal{B})$ we denote the group of all automorphisms of the space $(X, \mathcal{B})$, where the multiplication is defined as follows:

$$(1.1) \quad \left(g_1 \cdot g_2\right)(S) := g_2\left(g_1(S)\right) \quad \text{for all } g_1, g_2 \in \mathrm{Aut}(X, \mathcal{B}), \text{ and } S \in \mathcal{B} .$$

Let G be any subgroup of $\mathrm{Aut}(X, \mathcal{B})$. The action of the group G on the space BV, that is, the map

$$(1.2) \qquad G \times BV \ni (g, v) \mapsto g \cdot v \in BV$$

is defined by

$$(1.3) \quad (gv)(S) = v\left(g(S)\right) \quad \text{for all } g \in G, \ v \in BV \text{ and } S \in \mathcal{B} .$$

It is easy to verify that the following hold:

(1.4) (i) $g(\alpha v_1+\beta v_2) = \alpha g v_1 + \beta g v_2$,

(ii) $(g_1 g_2)v = g_1(g_2 v)$,

(iii) $ev = v$, where e is the neutral element of G ,

for each $g, g_1, g_2 \in G$, $v, v_1, v_2 \in BV$ and $\alpha, \beta \in \mathbb{R}$.

Let $\mathbb{R}[G]$ be the group algebra of the group G over real numbers, that is, the elements of $\mathbb{R}[G]$ have the form

(1.5) $$\sum_g r_g g , \quad r_g \in \mathbb{R} , \quad g \in G \text{ and } r_g = 0$$

for all but finitely many $g \in G$, and the multiplication is defined by

(1.6) $$\left(\sum_g r_g \cdot g\right)\left(\sum_h r_h \cdot h\right) = \sum_{g,h} (r_g \cdot r_h) \cdot (g \cdot h) .$$

Let us define the action of $\mathbb{R}[G]$ on BV as follows:

(1.7) $$\mathbb{R}[G] \times BV \ni \left(\sum_g r_g \cdot g, v\right) \mapsto \sum_g r_g(gv) \in BV .$$

By (1.4), (1.7) and the definition of $\mathbb{R}[G]$, it follows that BV may be regarded as $\mathbb{R}[G]$-module. Moreover, the additive group $\mathbb{R}$ of real numbers also is a $\mathbb{R}[G]$-module (it is called trivial $\mathbb{R}[G]$-module), when the action of $\mathbb{R}[G]$ in $\mathbb{R}$ is given by

(1.8) $$h \cdot r = r \text{ for each } r \in \mathbb{R} \text{ and } h \in \mathbb{R}[G] .$$

For each subgroup G of $\mathrm{Aut}(X, \mathcal{B})$ we define the sequence $\{H^q(G, \cdot)\}$, $q = 0, 1, 2, 3, \ldots$, of q-dimensional cohomology groups by setting

(1.9) $$H^q(G, A) := \mathrm{Ext}^q_{\mathbb{R}[G]}(\mathbb{R}, A)$$

for any $\mathbb{R}[G]$-module A (see [2]). Clearly, each $H^q(G, A)$ is a linear space over $\mathbb{R}$. From the definition it follows that $H^q(G, \cdot)$ is the covariant functor from the category $\mathrm{Mod}_{\mathbb{R}[G]}$ of all $\mathbb{R}[G]$-modules to the category $\mathrm{Vect}_{\mathbb{R}}$ of all vector spaces over $\mathbb{R}$.

We note that $\{H^q(G, \cdot)\}_q$ has the following properties:

(1.10) $H^0(G, A) = \mathrm{Hom}_{\mathbb{R}[G]}(\mathbb{R}, A)$ for each $\mathbb{R}[G]$-module A,

(1.11) every short exact sequence $0 \mapsto A \mapsto B \mapsto C \mapsto 0$ of $\mathbb{R}[G]$-modules induces the long exact sequence

$$0 \mapsto H^0(G, A) \mapsto H^0(G, B) \mapsto H^0(G, C) \mapsto H^1(G, A) \mapsto H^1(G, B) \mapsto \ldots$$

of the vector spaces.

LEMMA 1.1. *$H^0(G, A) \doteq A^G$, where*

$$A^G = \{a \in A : ga = a, \text{ for every } g \in G\} .$$

Proof. If $\varphi \in H^0(G, A) = \mathrm{Hom}_{\mathbb{R}[G]}(\mathbb{R}, A)$, then φ is a function from $\mathbb{R}$ into A such that $\varphi(r) = r\varphi(1)$ for all $r \in \mathbb{R}$. Moreover, for each $g \in G$ we have

$$\varphi(1) = \varphi(g \cdot 1) = g \cdot \varphi(1) \text{ , hence } \varphi(1) \in A^G .$$

Finally, we observe that the map

$$\mathrm{Hom}_{\mathbb{R}[G]}(\mathbb{R}, A) \ni \varphi \mapsto \varphi(1) \in A^g$$

is the isomorphism of the vector spaces and this completes the proof.

2. Definition of the G-Shapley value in the framework of cohomology

Let $|\ | : BV \mapsto BV$ denote the set function defined by

(2.1) $|v|(S) = |v(S)|$ for every $S \in \mathcal{B}$, $v \in BV$.

For every $g \in G$ we denote by g_* the linear map from BV to BV defined by formula

(2.2) $g_*(v)(S) = v\big(g(S)\big)$ for all $v \in BV$, $g \in G$, $S \in \mathcal{B}$.

Following [1], a G-Shapley value φ on Q is the $\mathbb{R}$-linear map $\varphi : Q \mapsto FA$ satisfying the following conditions:

(2.3) (i) for every $g \in G$, $g_*\varphi = \varphi g_*$ (the symmetry);

(ii) $(\varphi v)(X) = v(X)$ for each $v \in Q$ (the Pareto optimality);

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(iii) $\varphi(|v|) = |\varphi(v)|$ for each $v \in Q$.

The condition (2.1) (i) and linearity of φ imply that $\varphi \in \mathrm{Hom}_{\mathbb{R}[G]}(Q, FA)$. Now we define the $\mathbb{R}[G]$-homomorphism $I : BV \mapsto \mathbb{R}$ (where $\mathbb{R}$ is regarded as the trivial $\mathbb{R}[G]$-module), by the formula

(2.4) $$I(v) = v(X) \quad \text{for each} \quad v \in BV ,$$

where $(X, \mathcal{B})$ is a given measurable space.

For every $Q \subseteq BV$ we denote by I_Q the restriction of I to Q . Now we are ready for the cohomological definition of the Shapley value.

DEFINITION 1. We shall say that $\varphi \in \mathrm{Hom}_{\mathbb{R}[G]}(Q, FA)$ is the G-Shapley value on the linear G-invariant subspace $Q \subseteq BV$ if φ satisfies two conditions:

(SVI) $I_{FA} \circ \varphi = I_Q$;

(SVII) $\varphi \circ |\ | = |\ | \circ \varphi$.

I_Q, I_{FA} and $|\ |$ are mappings defined above, and $\circ$ denotes the composition of the maps.

3. Cohomological conditions for the existence of the G-Shapley value when $H^0(G, FA)$ is nontrivial.

Necessary condition for the uniqueness of the G-Shapley value

Let φ be a G-Shapley value on Q . For any $q \geq 0$, we denote by $\varphi^{*,q}$, $I_Q^{*,q}$ and $I_{FA}^{*,q}$ the homomorphisms induced by φ, I_Q and I_{FA} , respectively. Then we have the following:

LEMMA 3.1. *If φ is the G-Shapley value on Q then for each $q \geq 0$ the following diagram*

$(\mathrm{SVI})_q$ $$\begin{array}{ccc} H^q(G, Q) & \xrightarrow{\ \varphi^{*,q}\ } & H^q(G, FA) \\ & {\scriptstyle I_Q^{*,q}} \searrow \quad \swarrow {\scriptstyle I_{FA}^{*,q}} & \\ & H^q(G, \mathbb{R}) & \end{array}$$

commutes.

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The lemma follows immediately from the cohomological definition of the G-Shapley value, and from the functional properties of the cohomology theory.

COROLLARY 3.1. *If for some* $q \geq 0$ *and for each* $\mathbb{R}$*-linear map* $h : H^q(G, Q) \mapsto H^q(G, FA)$ *the diagram* $(\mathrm{SVI})_q$ *does not commute, then there is no* G*-Shapley value on* Q .

Since we always assume that Q is a fixed G-invariant subspace of BV , $FA \subseteq Q$, and G is a group such that $H^0(G, FA) \neq \{0\}$, it also follows that $H^0(G, Q) \neq \{0\}$.

Let A be any subset of BV , and let $v \in BV-A$. Then by A_v we denote the $\mathbb{R}$-linear subspace of BV containing A and $\{gv : g \in G\}$, that is,

$$A_v := \mathrm{lin}_{\mathbb{R}}(\{gv : g \in G\} \cup A) . \tag{3.1}$$

It is easy to see that if A is the G-invariant subset of BV then the sets $\{gv : g \in G\}$ and A are disjoint, and the space A_v is a G-invariant subspace of BV .

LEMMA 3.2. *Each* G*-Shapley value* φ *on* Q *can be extended to a* G*-Shapley value* φ_v *defined on the* G*-invariant space* Q_v *, when* $v \in BV-Q$.

Proof. Let $v \in BV-Q$. Then, for each $g \in G$, $gv \notin Q$. Indeed, if for some $g_1 \in G$, $u = g_1 v \in Q$, then $v = g_1^{-1} u \in Q$, because Q is G-invariant, but it yields a contradiction.

Let $v = v_1 - v_2$ be the Jordan decomposition of v . By G_v we denote the subgroup of G defined by the formula

$$G_v = \{g \in G : gv = v\} . \tag{3.2}$$

Now we extend φ to the $\mathbb{R}$-linear map $\varphi_v : Q_v \mapsto FA$ by setting

$$(3.3) \qquad \varphi_v(u) = \begin{cases} \varphi(u) & \text{for } u \in Q \;, \\ \\ g\left(\mu_v^1 - \mu_v^2\right) & \text{for } \mu = gv \;, \; g \in G \;, \end{cases}$$

where μ_v^1 and μ_v^2 are any elements of FA^+ satisfying two conditions:

$$(3.4) \qquad \mu_v^1, \; \mu_v^2 \in H^0\left(G_v, \; FA^+\right) \;,$$

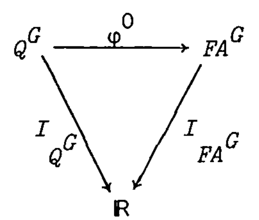
$$(3.5) \qquad \mu_v^1(X) = v_1(X) \quad \text{and} \quad \mu_v^2(X) = v_2(X) \;;$$

such elements exist because $H^0\left(G_v, \; FA^+\right) \supset H^0(G, \; FA^+) \neq \{0\}$.

It is easy to verify that the mapping φ_v defined above satisfies the condition (SVI) and (SVII) on Q_v . Thus φ_v is a G-Shapley value on the G-invariant space $Q_v \supseteq Q$.

LEMMA 3.3. *A G-Shapley value φ on the subspace Q exists if and only if there exists a linear map $\varphi_0 : Q^G \mapsto FA^G$ such that the following diagram*

(SVI)$_0$



commutes and

$$(\text{SVII})_0 \qquad |\;| \circ \varphi_0 = \varphi_0 \circ |\;|$$

(φ_0 *is the G-Shapley value on* Q^G).

Proof. If φ is the G-Shapley value on Q , then $\varphi\left(Q^G\right) \subset FA^G$, and φ_0 defined by $\varphi_0 = \varphi | Q^G = \varphi_0^*$ satisfies (SVI)$_0$. Indeed, it follows from Lemma 3.1, because $H^0(G, \mathbb{R}) = \text{Hom}_{\mathbb{R}[G]}(\mathbb{R}, \mathbb{R}) = \mathbb{R} = \mathbb{R}$ (G acts on $\mathbb{R}$ trivially). The condition (SVII)$_0$ is obviously satisfied. Conversely, let

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$\varphi_0 : Q^G \mapsto FA^G$ be a $\mathbb{R}$-linear map satisfying $(\mathrm{SVI})_0$ and $(\mathrm{SVII})_0$. We shall construct the G-Shapley value φ on Q. If $Q^G = Q$, then we put $\varphi = \varphi_0$. Now let us assume that $Q \neq Q^G$. Let $v \in Q-Q^G$. Applying Lemma 3.2 for φ_0, Q^G and v we obtain the G-Shapley value $(\varphi_0)_v$ on the G-invariant space $Q_v^G \dotplus Q^G$. Now we apply Lemma 3.2 for $(\varphi_0)_v$, Q_v^G and $v' \in Q-Q_v^G$ we obtain a G-Shapley value $((\varphi_0)_v)_{v'}$ on the space $\left(Q_v^G\right)_{v'}$, and so on. Proceeding by the transfinite induction finally we obtain a G-Shapley value on Q.

THEOREM 3.1. *Let Q be G-invariant subspace of BV. Assume further that $Q \supseteq FA$ and $H^0(G, FA)$ is nontrivial. Then there exists at least* dim $H^0(G, FA)$*-distinct G-Shapley values on Q.*

Proof. If $Q \supseteq FA$ and $FA^G \neq \{0\}$, then $Q^G \neq \{0\}$ and FA^G contains the $\mathbb{R}$-linear subspace isomorphic with $(R, +)$. Indeed, FA^G is the $\mathbb{R}$-linear vector space and if $v \in FA^G$ then v^+ and v^- (in the Jordan decomposition of v) both are in $(FA^+)^G$. Since

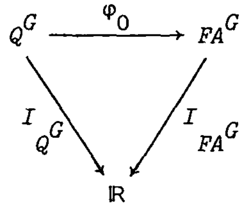
$$v^+(S) = \text{supremum} \sum_{i=1}^{k} \max\left(v(S_i)-v(S_{i-1}), 0\right),$$

$\emptyset = S_0 \subset \ldots \subset S_k = S$ and $v^- = v^+ - v$.

Thus there exists $\lambda \in (FA^+)^G$ such that $\mathbb{R} \cdot \lambda \subset FA^G$ and $\lambda(X) = 1$. If we put

$$\varphi_0(v) = v(X) \cdot \lambda \quad \text{for all} \quad v \in Q^G, \tag{3.6}$$

then the diagram



commutes.

From (3.6) we have $\varphi_0 \circ |\ | = |\ | \circ \varphi_0$. Hence by Lemma 3.3 there is a G-Shapley value on Q . Finally there are at least $\dim H^0(G, FA)$ many different G-Shapley values. For it suffices to put for every element u of the basis of $H^0(G, FA)$ over $\mathbb{R}$,

$$\varphi_u(v) = v(X) \frac{u}{u(X)} . \tag{3.7}$$

PROPOSITION 3.1. *If* $H^0(G, FA) = \{0\}$ *and* $H^0(G, Q) \neq \{0\}$ *then there is no* G*-Shapley value on* Q .

Proof. Suppose there is a G-Shapley value φ on Q . Then there is $v \in H^0(G, Q)$, $v(X) \neq 0$, so that, for every $g \in G$, $g\varphi(v) = \varphi(gv) = \varphi(v)$. Hence $\varphi(v) \in H^0(G, FA)$, that is, $\varphi(v) = 0$ which is a contradiction with $\varphi(v)(X) = v(X) \neq 0$.

THEOREM 3.2. *There exists a unique* G*-Shapley value on the space* Q *if and only if for each* $v \in Q^+$ *the space* $H^0(G_v, FA)$ *is one-dimensional.*

REMARK. If $v \in Q^G$ then $G_v = G$ and $H^0(G_v, FA) = H^0(G, FA)$.

Proof. Necessity. Suppose $\dim H^0(G, FA) > 1$. Let $\{u_i\}_{i\in I}$, $|I| > 1$, be an algebraic base of the linear space $H^0(G, FA)$. Without loss of generality we can assume that $u_i(X) = 1$ for every $i \in I$. Put $\varphi_i^0(v) = v(X)u_i$ for every $i \in I$ and $v \in Q^G$. Obviously $\varphi_i^0 \neq \varphi_j^0$ for $i \neq j$, $i, j \in I$.

By virtue of Lemma 3.3 we may extend each $\varphi_{u_i}^0$ $(i \in I)$ to distinct

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Shapley values.

Similarly, suppose that $\dim H^0(G_v, FA) > 1$ for some $v \in Q^+ - Q^G$. Then there are $\mu_1, \mu_2 \in H^0(G_v, FA^+)$ such that $\mu_1(X) = \mu_2(X) = v(X)$ and $\mu_1 \neq \mu_2$. We have already proved in Theorem 3.1 that there exists the G-Shapley value φ_0 over Q^G if $H^0(G, FA)$ is nontrivial.

Denote by φ_1, φ_2 two distinct extensions of φ_0 over Q_v, where $\varphi_i(v) = v(X)\mu_i$ for $i = 1, 2$. They exist by Lemma 3.2.

Sufficiency. Let $H^0(G_v, FA)$ be one-dimensional vector space for every $v \in Q$. Let φ and ψ be two G-Shapley values on Q. We claim that $\varphi = \psi$. Indeed, let $v \in Q^+$ be arbitrary. It follows immediately from the definition of the G-Shapley value, that $\varphi(v), \psi(v) \in H^0(G_v, FA^+)$. Since $\dim H^0(G_v, FA) = 1$, we have $\varphi(v) = c_1\mu$, $\psi(v) = c_2\mu$, for some $c_1, c_2 \in \mathbb{R}$. Here μ is a base of the space $H^0(G_v, FA)$. But $c_1\mu(X) = \varphi(v)(X) = v(X) = \psi(v)(X) = c_2\mu(X)$. Hence $c_1 = c_2$ and $\varphi(v) = \psi(v)$. Since v was arbitrary then $\varphi = \psi$.

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