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Solutions for Semilinear Elliptic Systems with Critical Sobolev Exponent and Hardy Potential

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Abstract. In this paper we consider an elliptic system with an inverse square potential and critical Sobolev exponent in a bounded domain of $\mathbb{R}^N$. By variational methods we study the existence results.

1 Introduction

In this paper we study the existence of nontrivial solutions of the following system

$$(S_{A,\mu})\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2}u = au + bv + (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1} & \text{in } \Omega,\\ -\Delta v - \dfrac{\mu}{|x|^2}v = bu + cv + (\beta+1)|u|^{\alpha+1}v|v|^{\beta-1} & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing 0 in its interior; a, b, c are real parameters; $\alpha,\ \beta > 0$ such that $\alpha + \beta \leq \frac{4}{N-2}$; and $0 \leq \mu < \overline{\mu} := (\frac{N-2}{2})^2$.

We start by giving a brief history for the scalar case. The problem

$$(P_{\lambda,\mu})\begin{cases} L_\mu u := -\triangle u - \mu\dfrac{u}{|x|^2} = u|u|^{\frac{4}{N-2}} + \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has been considered by many authors (see [6, 9, 11, 12] and the references cited therein). The quasilinear case was treated for example by Ghoussoub and Yuan in [10].

Problem $(P_{\lambda,0})$ has been the object of the famous paper of Brézis and Nirenberg in [4]. Jannelli in [11] generalized the results of [4] to problem $(P_{\lambda,\mu})$ for $\mu \geq 0$. He proved the following:

- If $0 \leq \mu \leq \overline{\mu} - 1$, then problem $(P_{\lambda,\mu})$ has at least one positive solution in $H_0^1(\Omega)$ for all $0 < \lambda < \mu_1$, where $\mu_1 = \mu_1(\mu)$ is the first eigenvalue of L_μ in $H_0^1(\Omega)$.

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- If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then problem $(P_{\lambda,\mu})$ has at least one positive solution for $\lambda_*(\mu) < \lambda < \mu_1$, where

$$\lambda_*(\mu) = \min_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{|\nabla\varphi(x)|^2}{|x|^{2\sigma}}\,dx}{\int_\Omega \frac{\varphi^2(x)}{|x|^{2\sigma}}\,dx},$$

  with $\sigma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$.
- If $\overline{\mu} - 1 < \mu < \overline{\mu}$ and $\Omega = B(0, R)$ (*i.e.,* the ball centred at 0 with radius R), then problem $(P_{\lambda,\mu})$ has no nontrivial solution for $\lambda \le \lambda_*(\mu)$.
- If $\lambda \le 0$ and Ω is a smooth starshaped domain, then by a Pohozaev type identity problem $(P_{\lambda,\mu})$ has no positive solution.
- The case $\lambda \ge \mu_1$ has been discussed in several papers; we quote [5–7, 9].

Cappozi and Gazzola [5] proved the following results:

- If $N = 4$, $\lambda > 0$, and $\lambda \notin \sigma_0$, where σ_0 denotes the spectrum of $-\triangle$ with zero Dirichlet boundary problem, then problem $(P_{\lambda,0})$ has at least one nontrivial solution.
- If $N \ge 5$, then problem $(P_{\lambda,0})$ has at least one nontrivial solution for all $\lambda > 0$.

Ferrero and Gazzola [9] developed some technical asymptotic estimates in their proof for $\mu \ge 0$. Chen [7] gave a partial positive answer to an open problem proposed in [9] by using the linking theorem and delicate energy estimates. Recently Cao and Han [6] solved completely the open problem proposed in [9]; they proved that if $N \ge 5$ and $0 \le \mu < \overline{\mu} - (\frac{N+2}{N})^2$, then problem $(P_{\lambda,\mu})$ admits a nontrivial solution for all $\lambda > 0$. They established an asymptotic behavior of the eigenfunction, which is crucial in their proof.

In this work we deal with the case of elliptic systems, we refer to de Figueiredo (see [8]) for a general view about the theory of elliptic systems. The results of [4] have been also generalized to system $(S_{A,0})$ by Alves *et al.* [1].

Our system $(S_{A,\mu})$ can be written as follows:

$$\begin{cases} -\overrightarrow{\Delta} U - \mu \frac{U}{|x|^2} = AU + \nabla H & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\overrightarrow{\triangle} := \binom{\triangle}{\triangle}$, $U = \binom{u}{v}$, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $H(u, v) = |u|^{\alpha+1}|v|^{\beta+1}$.

Borrowing ideas of Alves *et al.* [1] and Cao and Han [6], we prove some existence and nonexistence results for $(S_{A,\mu})$ with $\mu > 0$. By establishing a Pohozaev type identity adapted for systems, we give a nonexistence result. We distinguish three main cases, depending on the position of the eigenvalues of the matrix A for the existence results.

The paper is organized as follows. In Section 2 we recall some preliminaries and main results, Section 3 contains the case where the eigenvalues of the matrix A are negative. Section 4 is devoted to the case where the eigenvalues of the matrix A are between 0 and μ_1. In Section 5, we consider the case where the eigenvalues belong to $[\mu_1, +\infty[$.

2 Preliminaries and Main Results

Notations We make use of the following notation:

- $L^p(\Omega)$, $1 \le p \le \infty$, denote Lebesgue spaces, the norm L^p is denoted by $|\cdot|_p$ for $1 \le p \le \infty$.
- $E\colon = H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the norm $\|(u,v)\|_\mu = (\|u\|_\mu^2 + \|v\|_\mu^2)^{\frac{1}{2}}$, where

$$\|u\|_\mu = \Big(\int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2}\, dx\Big)^{\frac{1}{2}};$$

  this norm is equivalent to the standard norm in E by Hardy's inequality.
- E' is the dual of E.
- $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.
- B_R is the ball centered at 0 with radius R.
- supp φ denotes the support of the function φ.
- $\langle \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.
- $\circ(1)$ denotes $\circ_n(1) \to 0$ as $n \to +\infty$.
- $C_1, C_2, C_3, \dots$ denote (possibly different) positive constants.

Let $\mathcal{M} = \big\{ \big(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\big) : a > 0, c > 0, b^2 < ac \big\}$. If $A \in \mathcal{M}$, then there exist two eigenvalues λ_1, λ_2 such that $0 < \lambda_1 \le \lambda_2$.

We have

$$\lambda_1(|u|^2 + |v|^2) \le \langle AU, U\rangle \le \lambda_2(|u|^2 + |v|^2) \quad \text{for all } (u,v) \in \mathbb{R}^2. \tag{2.1}$$

As a consequence of the Hardy inequality, the operator L_μ with zero Dirichlet boundary condition is positive and has a discrete spectrum σ_μ in $H_0^1(\Omega)$ if $0 \le \mu < \overline{\mu}$. The smallest eigenvalue μ_1 is simple and $\mu_i \to +\infty$ as $i \to +\infty$.

Moreover, each L^2 normalized eigenfunction e_i corresponding to $\mu_i \in \sigma_\mu$ belongs to the space $H_0^1(\Omega)$ and is not in $L^\infty(\Omega)$, however for the case when $\mu = 0$, $e_i \in L^\infty(\Omega)$.

Lemma 2.1 *Let Ω be a domain (not necessarily bounded), $0 \le \mu < \overline{\mu}$ and $\alpha + \beta \le \frac{4}{N-2}$. We define*

$$S_\mu = S_\mu(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2})\, dx}{(\int_\Omega |u|^{\alpha+\beta+2}\, dx)^{\frac{2}{\alpha+\beta+2}}} \tag{2.2}$$

and

$$S_{\mu,\alpha,\beta} = S_{\mu,\alpha,\beta}(\Omega) := \inf_{(u,v) \in E\setminus\{(0,0)\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2 - \mu \frac{u^2+v^2}{|x|^2})\, dx}{(\int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\, dx)^{\frac{2}{\alpha+\beta+2}}}. \tag{2.3}$$

Then we have

$$S_{\mu,\alpha,\beta} = \Big[\Big(\frac{\alpha+1}{\beta+1}\Big)^{\frac{\beta+1}{\alpha+\beta+2}} + \Big(\frac{\alpha+1}{\beta+1}\Big)^{\frac{-\alpha-1}{\alpha+\beta+2}} \Big] S_\mu.$$

Moreover, if ω_0 realizes S_μ, then $(u_0, v_0) = (B\omega_0, C\omega_0)$ realizes $S_{\mu,\alpha,\beta}$ for any positive constants B and C such that $\frac{B}{C} = (\frac{\alpha+1}{\beta+1})^{\frac{1}{2}}$.

Proof The proof of the lemma is essentially given in [1] with minor modifications. ■

As in [11] we consider the family of functions

$$\omega_\varepsilon^*(x) = \frac{C_\varepsilon^{\frac{N-2}{4}}}{(\varepsilon^2|x|^{\frac{\sigma'}{\sqrt{\overline{\mu}}}} + |x|^{\frac{\sigma}{\sqrt{\overline{\mu}}}})^{\sqrt{\overline{\mu}}}} \quad \text{for } \varepsilon > 0$$

where $C_\varepsilon = 4\varepsilon^2 N(\overline{\mu} - \mu)/(N-2)$, $\sigma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$ and $\sigma' = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$.

For $\varepsilon > 0$, the function ω_ε^* solves the equation

$$-\triangle u - \mu \frac{u}{|x|^2} = u|u|^{\frac{4}{N-2}} \quad \text{in} \quad \mathbb{R}^N \backslash \{0\}.$$

From Lemma 2.1 we conclude that the problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1} & \text{in } \mathbb{R}^N \backslash \{0\}, \\ -\Delta v - \mu \frac{v}{|x|^2} = (\beta+1)|u|^{\alpha+1}v|v|^{\beta-1} & \text{in } \mathbb{R}^N \backslash \{0\}, \\ u(x) = v(x) = 0 & \text{as } |x| \to \infty \end{cases}$$

has a solution in the form $(B\omega_\varepsilon^*, C\omega_\varepsilon^*)$, where B and C are positive constants satisfying

$$\frac{B}{C} = \Big(\frac{\alpha+1}{\beta+1}\Big)^{\frac{1}{2}}.$$

Let $0 \le \phi(x) \le 1$ be a function in $C_0^\infty(\Omega)$ defined as

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le R, \\ 0 & \text{if } |x| \ge 2R, \end{cases}$$

where $B_{2R} \subset \Omega$.

Taking

$$\widetilde{\omega}_\varepsilon = \frac{\omega_\varepsilon}{\|\omega_\varepsilon\|_{2^*}} \quad \text{with } \omega_\varepsilon = \phi(x)\omega_\varepsilon^*.$$

Let us introduce the corresponding functional energy of system $(S_{A,\mu})$

$$J_\mu(u,v) = \frac{1}{2}\|(u,v)\|_\mu^2 - \frac{1}{2}\int_\Omega \langle AU, U\rangle\, dx - \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\, dx.$$

It is well known that a weak solution $(u,v) \in E$ (in our case $u \ne 0$ and $v \ne 0$) of $(S_{A,\mu})$ is precisely a critical point of J_μ. That is,

$$\int_\Omega \Big(\nabla u \nabla\varphi + \nabla v \nabla\psi - \frac{\mu}{|x|^2}(u\varphi + v\psi) - (au\varphi + bv\varphi + bu\psi + cv\psi)\Big)\, dx$$
$$- \int_\Omega \big((\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1}\varphi + (\beta+1)|u|^{\alpha+1}v|v|^{\beta-1}\psi\big)\, dx = 0$$

for all $(\varphi, \psi) \in E$.

Definition 2.2 Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$,

(i) (u_n, v_n) is a $(\mathrm{PS})_c$ sequence in E for I at level c if $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$ strongly in E' as $n \to +\infty$.

(ii) We say that I satisfies the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence in E for I has a convergent subsequence.

In this paper we obtain the following results.

Theorem 2.3 *Let $A \in M_{2\times 2}$ symmetric matrix such that $\lambda_1 \le \lambda_2 \le 0$ and $\alpha + \beta = \frac{4}{N-2}$. If Ω is a smooth starshaped domain with respect to the origin then system $(S_{A,\mu})$ has no nontrivial solution.*

Theorem 2.4 *Suppose $\alpha + \beta = \frac{4}{N-2}$ and $A \in \mathcal{M}$. If $0 \le \mu \le \overline{\mu} - 1$, then system $(S_{A,\mu})$ has a solution for all $\lambda_2 < \mu_1$. If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then system $(S_{A,\mu})$ has a solution for all $\mu^* < \lambda_1 \le \lambda_2 < \mu_1$ where*

$$\mu^* = \min_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{|\nabla \varphi(x)|^2}{|x|^{2\sigma}}\, dx}{\int_\Omega \frac{\varphi^2(x)}{|x|^{2\sigma}}\, dx}$$

and $\sigma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$.

Corollary 2.5 *Suppose $0 \le \mu < \overline{\mu}$, $\alpha + \beta < \frac{4}{N-2}$, and $A \in \mathcal{M}$. Then system $(S_{A,\mu})$ has a solution for all $\lambda_2 < \mu_1$.*

Theorem 2.6 *Suppose $N \ge 5$, $\alpha + \beta = \frac{4}{N-2}$, $0 \le \mu < \overline{\mu} - (\frac{N+2}{N})^2$, and $A \in \mathcal{M}$. Assume one of the following conditions holds:*

- *There exists $k \in \mathbb{N}^*$ such that $\mu_k \le \lambda_1 \le \lambda_2 < \mu_{k+1}$.*
- *There exist $k, k' \in \mathbb{N}^*$, $k \ne k'$ such that*

$$\mu_k \le a - |b| \le \lambda_1 \le a + |b| < \mu_{k+1} \le \mu_{k'} \le c - |b| \le \lambda_2 \le c + |b| < \mu_{k'+1}.$$

Then system $(S_{A,\mu})$ has at least one solution.

3 Eigenvalues of A are Nonpositive

In this section, we give an nonexistence result which is based on a Pohozaev type identity adapted for systems.

Proof of Theorem 2.3 We will use a Pohozaev type identity. The idea consists of multiplying each equation by $\langle x, \nabla u\rangle$ and $\langle x, \nabla v\rangle$, respectively, and integrating by parts. We obtain

$$\int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)\langle x, \nu\rangle\, d\sigma + \Big(\frac{N-2}{2}\Big) \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\, dx \tag{3.1}$$
$$= \mu\Big(\frac{N-2}{2}\Big) \int_\Omega \frac{u^2 + v^2}{|x|^2}\, dx + \frac{N}{2} \int_\Omega \langle AU, U\rangle\, dx + N \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\, dx,$$

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where ν is the outwards normal to $\partial\Omega$. On the other hand, multiplying each equation by u, v respectively and integrating over Ω we obtain

$$\int_\Omega \Big(|\nabla u|^2 - \frac{\mu}{|x|^2} u^2 \Big)\, dx = \int_\Omega \big(a|u|^2 + bvu + (\alpha+1)|u|^{\alpha+1}\,|v|^{\beta+1} \big)\, dx \tag{3.2}$$

$$\int_\Omega \Big(|\nabla v|^2 - \frac{\mu}{|x|^2} v^2 \Big)\, dx = \int_\Omega \big(buv + c|v|^2 + (\beta+1)|u|^{\alpha+1}\,|v|^{\beta+1} \big)\, dx. \tag{3.3}$$

Replacing (3.2) and (3.3) in (3.1), we obtain

$$\int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)\langle x, \nu\rangle\, d\sigma = \int_\Omega \langle AU, U\rangle\, dx.$$

Using (2.1) with $\lambda_2 \le 0$, we get

$$\int_{\partial\Omega} (|\nabla u|^2 + |\nabla v|^2)\langle x, \nu\rangle\, d\sigma \le 0,$$

which is in contradiction with the fact that Ω is starshaped, *i.e.*, $\langle x, \nu\rangle > 0$ a.e. on $\partial\Omega$. ■

4 Eigenvalues of A Belong to $[0, \mu_1[$

For proving Theorem 2.4, we need some auxiliary results.

Lemma 4.1 *Let $0 < \lambda_1 \le \lambda_2 < \mu_1$, $0 \le \mu < \overline{\mu}$, and $\alpha + \beta \le \frac{4}{N-2}$.*

(i) *There exist $\rho > 0$ and $R > 0$ such that $J_\mu(u, v) \ge \rho$ for all $(u, v) \in E$ with $\|(u, v)\|_\mu = R$.*

(ii) *There exists $(u_0, v_0) \in E$ with $\|(u_0, v_0)\|_\mu > R$ such that $J_\mu(u_0, v_0) \le 0$.*

Proof From (2.1) and (2.3) we get

$$J_\mu(u, v) \ge \frac{1}{2}\Big(1 - \frac{\lambda_2}{\mu_1}\Big)\, \|(u, v)\|_\mu^2 - C\, \|(u, v)\|_\mu^{\alpha+\beta+2} \ge \rho$$

for $\|(u, v)\|_\mu = R$ small enough.

We have

$$J_\mu(tu, tv) = \frac{t^2}{2}\, \|(u, v)\|_\mu^2 - \frac{t^2}{2} \int_\Omega \langle AU, U\rangle\, dx - t^{\alpha+\beta+2} \int_\Omega |u|^{\alpha+1}\,|v|^{\beta+1}\, dx \longrightarrow +\infty$$

as $t \to +\infty$, thus there exists (u_0, v_0) with $\|(u_0, v_0)\| > R$ such that $J_\mu(u_0, v_0) \le 0$. ■

Let

$$c := \inf_{g\in\Gamma}\big(\max_{t\in[0,1]} J_\mu[g(t)]\big),$$

where

$$\Gamma := \big\{ g \in \mathcal{C}([0, 1], E) : g(0) = (0, 0),\ g(1) = (u_0, v_0) \big\}.$$

Now we will prove that J_μ satisfies $(\mathrm{PS})_c$ below some critical threshold.

Lemma 4.2 *If $c < \frac{2}{N-2}(\frac{1}{2^*}S_{\mu,\alpha,\beta})^{\frac{N}{2}}$, then J_μ satisfies $(\mathrm{PS})_c$.*

Proof Let (u_n, v_n) be a $(\mathrm{PS})_c$ sequence in E. We obtain, for large n,

$$(4.1)\quad 2J_\mu(u_n,v_n)-\big\langle J'_\mu(u_n,v_n),(u_n,v_n)\big\rangle = (\alpha+\beta)\int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx \le 2c+o(1).$$

On the other hand, there exists $C_\lambda := C(N,\alpha,\beta,\lambda) > 0$ such that

$$(4.2)\qquad |u|^{\alpha+1}|v|^{\beta+1} - \lambda(|u|^2+|v|^2) \ge -C_\lambda$$

for all $(u,v) \in \mathbb{R}\times\mathbb{R}\backslash\{\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}\}$, where λ is a positive constant.

Indeed, consider the function

$$H_\lambda(u,v) = |u|^{\alpha+1}|v|^{\beta+1} - \lambda(|u|^2+|v|^2).$$

Then (u,v) is an extremum point of H_λ if

$$(4.3)\qquad (\alpha+1)u|u|^{\alpha-1}|v|^{\beta+1} - 2\lambda u = 0$$

and

$$(4.4)\qquad (\beta+1)|u|^{\alpha+1}v|v|^{\beta-1} - 2\lambda v = 0.$$

Multiplying (4.3) and (4.4) by $(\beta+1)u$ and $(\alpha+1)v$ respectively and subtracting them, we get

$$\Big|\frac{u}{v}\Big| = \Big(\frac{\alpha+1}{\beta+1}\Big)^{1/2} \quad \textit{i.e.,}\ |v| = k|u| \text{ with } k := \Big(\frac{\alpha+1}{\beta+1}\Big)^{-1/2}.$$

Put

$$g(u) := H_\lambda(|u|,k|u|) = k^{\beta+1}|u|^{2^*} - \lambda(1+k^2)|u|^2,$$

$g(u)$ attains its minimum at

$$u_0 = \Big(\frac{2\lambda(1+k^2)}{2^*k^{\beta+1}}\Big)^{\frac{1}{\alpha+\beta}}, \quad \text{with} \quad g(u_0) = -C_\lambda := -\frac{1}{N}\frac{\big(2\lambda(1+k^2)\big)^{\frac{2^*}{\alpha+\beta}}}{(2^*k^{\beta+1})^{\frac{2}{\alpha+\beta}}}.$$

Thus, we have $H_\lambda(u,v) \ge -C_\lambda$ for all $(u,v)\in\mathbb{R}\times\mathbb{R}\backslash\{\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}\}$. Finally, (4.1), and (4.2), with $\lambda := \lambda_2$, yield

$$\begin{aligned}\|(u_n,v_n)\|_\mu^2 &= 2J_\mu(u_n,v_n) + \int_\Omega \langle AU_n,U_n\rangle\,dx + 2\int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx\\ &\le 2J_\mu(u_n,v_n) + \lambda_2\int_\Omega(|u_n|^2+|v_n|^2)\,dx + 2\int_\Omega |u_n|^{\alpha+1}|v_n|^{\beta+1}\,dx\\ &\le C\big(1+\|(u_n,v_n)\|_\mu\big);\end{aligned}$$

consequently (u_n, v_n) is bounded in E.

Thus there exists a subsequence, again denoted by (u_n, v_n), such that

$$
(4.5)\qquad
\begin{aligned}
&(u_n, v_n) \to (u, v) \text{ weakly in } E,\\
&\Big(\frac{u_n}{x}, \frac{v_n}{x}\Big) \to \Big(\frac{u}{x}, \frac{v}{x}\Big) \text{ weakly in } [L^2(\Omega)]^2,\\
&(u_n, v_n) \to (u, v) \text{ strongly in } L^r \times L^s \text{ for all } 1 \le r, s < 2^*,\\
&(u_n, v_n) \to (u, v) \text{ a.e. on } \Omega,
\end{aligned}
$$

it follows that (u, v) is a weak solution of system $(S_{A,\mu})$, *i.e.*,

$$
(4.6)\qquad \langle J'_\mu(u, v), (\varphi, \psi)\rangle = 0 \quad \text{for all } (\varphi, \psi) \in E.
$$

We put $\varphi_n = u_n - u$ and $\psi_n = v_n - v$. From the Brézis–Lieb Lemma [3], we obtain the following relations

$$
(4.7)\qquad |\nabla u_n|_2^2 = |\nabla u|_2^2 + |\nabla \varphi_n|_2^2 + o(1),
$$

$$
(4.8)\qquad \Big|\frac{u_n}{x}\Big|_2^2 = \Big|\frac{u}{x}\Big|_2^2 + \Big|\frac{\varphi_n}{x}\Big|_2^2 + o(1),
$$

$$
(4.9)\qquad |\nabla v_n|_2^2 = |\nabla v|_2^2 + |\nabla \psi_n|_2^2 + o(1),
$$

$$
(4.10)\qquad \Big|\frac{v_n}{x}\Big|_2^2 = \Big|\frac{v}{x}\Big|_2^2 + \Big|\frac{\psi_n}{x}\Big|_2^2 + o(1),
$$

and

$$
(4.11)\qquad \int_\Omega |u_n|^{\alpha+1} |v_n|^{\beta+1}\, dx = \int_\Omega |u|^{\alpha+1} |v|^{\beta+1}\, dx + \int_\Omega |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1}\, dx + o(1).
$$

Using (4.5) to (4.11) we get

$$
(4.12)\qquad J_\mu(u, v) + \frac{1}{2}\|(\varphi_n, \psi_n)\|_\mu^2 - \int_\Omega |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1}\, dx = c + o(1)
$$

and

$$
\begin{aligned}
\|(u, v)\|_\mu^2 + \|(\varphi_n, \psi_n)\|_\mu^2 &= \int_\Omega \langle AU, U\rangle\, dx\\
&\quad + 2^* \int_\Omega (|u|^{\alpha+1} |v|^{\beta+1} + |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1})\, dx + o(1).
\end{aligned}
$$

Since $\langle J'(u, v), (u, v)\rangle = 0$,

$$
\|(\varphi_n, \psi_n)\|_\mu^2 = 2^* \int_\Omega |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1}\, dx + o(1).
$$

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Therefore, along a subsequence we may assume that, as $n \to +\infty$,

$$\|(\varphi_n, \psi_n)\|_\mu^2 \to k \quad \text{and} \quad 2^* \int_\Omega |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1} \, dx \to k.$$

By (2.3) we get

$$\|(\varphi_n, \psi_n)\|_\mu^2 \geq S_{\mu,\alpha,\beta} \Big(\int_\Omega |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1} \, dx \Big)^{2/2^*}.$$

At the limit we have $k \geq S_{\mu,\alpha,\beta}(\frac{k}{2^*})^{\frac{2}{2^*}}$. It follows that either $k = 0$ or $k \geq 2^*(\frac{S_{\mu,\alpha,\beta}}{2^*})^{\frac{N}{2}}$.

The case $k = 0$ is trivial.

If $k > 0$, then $k \geq 2^*(S_{\mu,\alpha,\beta}/2^*)^{N/2}$. Passing to the limit in (4.12), we obtain

$$J_\mu(u, v) + \frac{k}{N} = c < \frac{2}{N-2} \Big(\frac{1}{2^*} S_{\mu,\alpha,\beta} \Big)^{N/2}.$$

From this, we conclude that $J_\mu(u, v) < 0$ for all $(u, v) \in E$.

Taking $(\phi, \psi) = (u, v)$ in (4.6) we have

$$J_\mu(u, v) = \Big(\frac{2^*}{2} - 1 \Big) \int_\Omega |u|^{\alpha+1} |v|^{\beta+1} \, dx \geq 0;$$

thus we get a contradiction. Hence (u_n, v_n) converges strongly to (u, v) in E. ∎

Remark Lemma 4.2 is true for $\lambda_2 \geq \mu_1$.

Lemma 4.3 *Suppose that* $0 \leq \mu < \overline{\mu}$ *and* $0 < \lambda_1 \leq \lambda_2 < \mu_1$ *then we have*

$$\sup_{t \geq 0} J_\mu(tB\widetilde{\omega}_\varepsilon, tC\widetilde{\omega}_\varepsilon) < \frac{2}{N-2} \Big(\frac{1}{2^*} S_{\mu,\alpha,\beta} \Big)^{N/2} \quad \text{for } \varepsilon > 0 \text{ small.}$$

Proof Let $B, C > 0$ such that $\frac{B}{C} = (\frac{\alpha+1}{\beta+1})^{\frac{1}{2}}$. We have

$$J_\mu(tB\widetilde{\omega}_\varepsilon, tC\widetilde{\omega}_\varepsilon) \leq t^2 \Big(\frac{B^2 + C^2}{2} \Big) Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) - t^{2^*} B^{\alpha+1} C^{\beta+1} := g(t),$$

where $Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) := \|\widetilde{\omega}_\varepsilon\|_\mu^2 - \lambda_1 |\widetilde{\omega}_\varepsilon|_2^2$. Observe that the function g attains its maximum at $(t_0, g(t_0))$, where

$$t_0 = \Big(\frac{B^2 + C^2}{2^* B^{\alpha+1} C^{\beta+1}} Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) \Big)^{\frac{1}{\alpha+\beta}}$$

and

$$g(t_0) = \frac{2}{N-2} \Big(\frac{B^2 + C^2}{2^* B^{\alpha+1} C^{\beta+1}} Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) \Big)^{\frac{N}{2}}.$$

Thanks to Jannelli [11] we have

$$Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) < S_\mu \quad \text{if } \mu \le \overline{\mu} - 1, \qquad \text{for all } 0 < \lambda_1 \le \lambda_2 < \mu_1,$$
$$Q_{\lambda_1}(\widetilde{\omega}_\varepsilon) < S_\mu \quad \text{if } \overline{\mu} - 1 < \mu < \overline{\mu}, \qquad \text{for all } \mu^* < \lambda_1 \le \lambda_2 < \mu_1,$$

with ε sufficiently small.

By Lemma 2.1 we deduce that

$$\sup_{t\ge 0} J_\mu(tB\widetilde{\omega}_\varepsilon, tC\widetilde{\omega}_\varepsilon) < \frac{2}{N-2}\Big(\frac{1}{2^*} S_{\mu,\alpha,\beta}(\Omega)\Big)^{\frac{N}{2}} \quad \text{for } \varepsilon > 0 \text{ small.}$$

■

Proof of Theorem 2.4 From Lemmas 4.1, 4.2, and 4.3, J_μ satisfies the conditions of the mountain pass theorem [2]. Then there exists $(u, v) \in E$ such that $J'_\mu(u, v) = 0$ and $J_\mu(u, v) = c > 0$. ■

5 Eigenvalues of A Are Higher Than or Equal to μ_1

In this section, we consider two subcases:

(a) There exists $k \in \mathbb{N}^*$ such that $\mu_k \le \lambda_1 \le \lambda_2 < \mu_{k+1}$.
(b) There exist $k, k' \in \mathbb{N}^*$, $k \ne k'$ such that $\mu_k \le \lambda_1 < \mu_{k+1} \le \mu_{k'} \le \lambda_2 < \mu_{k'+1}$.

Consider the technique introduced by Ferrero and Gazzola in [9]. Fix $k \in \mathbb{N}^*$, and for each $i \in \mathbb{N}^*$ denote by e_i an L^2 normalized eigenfunction relative to $\mu_i \in \sigma_\mu$. Let X_k denote the space spanned by the eigenfunctions corresponding to the eigenvalues $\mu_1, \mu_2, \dots, \mu_k$, $Y_k = (X_k)^\perp$ and let $P_k\colon H_0^1(\Omega) \to X_k$ denote the orthogonal projection.

Always take $m \in \mathbb{N}$ large enough so that $B_{1/m} \subset \Omega$, and consider the function $\zeta_m\colon \Omega \to \mathbb{R}$ defined by

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x \in B_{\frac{1}{m}}, \\ m|x| - 1 & \text{if } x \in B_{\frac{2}{m}} \backslash B_{\frac{1}{m}}, \\ 1 & \text{if } x \in \Omega \backslash B_{\frac{2}{m}}, \end{cases}$$

the approximate eigenfunctions $e_i^m := e_i \zeta_m$, and the space $X_k^m := \operatorname{span}\{e_i^m : i = 1, \dots, k\}$. For all $\varepsilon > 0$, consider the shifted functions

$$\omega_\varepsilon^m(x) = \begin{cases} \omega_\varepsilon^*(x) - \omega_\varepsilon^*\big(\frac{1}{m}\big) & \text{if } x \in B_{\frac{1}{m}} \backslash \{0\} \\ 0 & \text{if } x \in \Omega \backslash B_{\frac{1}{m}}. \end{cases}$$

We shall need the following lemma.

Lemma 5.1 [6]
(i) $\|e_i^m - e_i\|_\mu \to 0$ *as* $m \to \infty$.
(ii) $\max_{\{u \in X_k^m, |u|_2 = 1\}} \|u\|_\mu^2 \le \mu_k + C_1 m^{-2\sqrt{\overline{\mu}-\mu}}$.

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From [9] with $\varepsilon = m^{-(\frac{N+2}{N-2})\sqrt{\bar\mu-\mu}}$, we obtain the estimates

$$
\begin{aligned}
&\|\omega_\varepsilon^m\|_\mu^2 \le S_\mu^{\frac{N}{2}} + C_1 m^{-N\sqrt{\bar\mu-\mu}},\\
&|\omega_\varepsilon^m|_{2^*}^{2^*} \ge S_\mu^{\frac{N}{2}} - C_2 m^{-\frac{2N}{N-2}\sqrt{\bar\mu-\mu}},\\
&|\omega_\varepsilon^m|_2^2 \ge C_3 m^{-(N+2)}.
\end{aligned} \tag{5.1}
$$

5.1 Eigenvalues of A belong to $[\mu_k, \mu_{k+1}[$ with $k \in \mathbb{N}^*$

Now we verify that the functional J_μ has linking geometry conditions.

Proposition 5.2 *Assume that $\lambda_1, \lambda_2 \in [\mu_k, \mu_{k+1}[$ for some $k \in \mathbb{N}^*$.*

(i) *There exist $\rho, \delta > 0$ such that $J_\mu(u, v) \ge \delta$ for all $(u, v) \in (\partial B_\rho \cap Y_k)^2$.*

(ii) *There exists $R > \rho$ such that $J_\mu|_{\partial Q_\varepsilon^m} \le p(m)$ with $p(m) \to 0$ as $m \to +\infty$, where*

$$Q_\varepsilon^m = \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m / 0 \le r < R\}\big) \times \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m / 0 \le r < R\}\big).$$

Proof For any $(u, v) \in (Y_k)^2$, we have

$$\|(u, v)\|_\mu^2 \ge \mu_{k+1} \int_\Omega (|u|^2 + |v|^2)\, dx. \tag{5.2}$$

Using (5.2) and (2.3), we get

$$J_\mu(u, v) \ge \frac{1}{2}\Big(1 - \frac{\lambda_2}{\mu_{k+1}}\Big) \|(u, v)\|_\mu^2 - C_1 \|(u, v)\|_\mu^{2^*}.$$

Thus we can choose $\rho = \|(u, v)\|_\mu$ sufficiently small enough and $\delta > 0$ such that $J_\mu|_{(\partial B_\rho \cap Y_k)^2} \ge \delta$. For $(u, v) \in (X_k^m)^2$, we have

$$J_\mu(u, v) \le \frac{1}{2}\|(u, v)\|_\mu^2 - \frac{\lambda_1}{2} \int_\Omega (|u|^2 + |v|^2)\, dx - \int_\Omega |u|^{\alpha+1} |v|^{\beta+1}\, dx.$$

From (4.2) and Lemma 5.1, we obtain

$$
\begin{aligned}
J_\mu(u, v) &\le \frac{1}{2}(\mu_k - \lambda_1 + C_2 m^{-2\sqrt{\bar\mu-\mu}}) \int_\Omega (|u|^2 + |v|^2)\, dx - \int_\Omega |u|^{\alpha+1} |v|^{\beta+1}\, dx\\
&\le -H_\lambda(u, v) \quad \text{with } \lambda := C_2 m^{-2\sqrt{\bar\mu-\mu}}.
\end{aligned}
$$

Then

$$J_\mu(u, v) \le C_3 m^{-N\sqrt{\bar\mu-\mu}} \quad \text{where } C_3 := \frac{1}{N} \frac{\big(2C_2(1+k^2)\big)^{\frac{2^*}{\alpha+\beta}}}{(2^* k^{\beta+1})^{\frac{2}{\alpha+\beta}}}.$$

Consequently, we have

$$\lim_{m\to\infty} \max_{(u,v)\in(X_k^m)^2} J_\mu(u, v) = 0.$$

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On the other hand, we have

$$J_\mu(\mathrm{Br}\,\omega_\varepsilon^m, \mathrm{Cr}\,\omega_\varepsilon^m) \le r^2\Big(\frac{B^2+C^2}{2}\Big)\,\|\omega_\varepsilon^m\|_\mu^2 - r^{2^*}B^{\alpha+1}C^{\beta+1}|\omega_\varepsilon^m|_{2^*}^{2^*},$$

so $J_\mu(\mathrm{Br}\,\omega_\varepsilon^m, \mathrm{Cr}\,\omega_\varepsilon^m)$ becomes negative if $r = R$ with R large enough. Therefore,

$$J_\mu(u,v) \le C_3 m^{-N\sqrt{\bar\mu-\mu}}$$
$$\text{for all } (u,v) \in \big(X_k^m \cup (X_k^m \oplus R\{B\omega_\varepsilon^m\})\big) \times \big(X_k^m \cup (X_k^m \oplus R\{C\omega_\varepsilon^m\})\big).$$

Since

$$\max_{0\le r\le R} J_\mu(\mathrm{Br}\,\omega_\varepsilon^m, \mathrm{Cr}\,\omega_\varepsilon^m) < +\infty$$
$$\text{for } (u,v) \in \big((\overline{B}_R \cap X_k^m) \oplus R\{B\omega_\varepsilon^m\}\big) \times \big((\overline{B}_R \cap X_k^m) \oplus R\{C\omega_\varepsilon^m\}\big),$$

as $(u,v) \in (X_k^m \oplus \mathbb{R}^+\{B\omega_\varepsilon^m\}) \times (X_k^m \oplus \mathbb{R}^+\{C\omega_\varepsilon^m\})$, we may write $u = w_1 + tB\omega_\varepsilon^m$ and $v = w_2 + tC\omega_\varepsilon^m$. Hence $\mathrm{meas}(\mathrm{supp}(\omega_\varepsilon^m) \cap \mathrm{supp}(w_i)) = 0$. Then $J_\mu\,|_{\partial Q_\varepsilon^m} \le 0$ for R large enough with

$$Q_\varepsilon^m = \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m / 0 \le r < R\}\big) \times \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m / 0 \le r < R\}\big). \blacksquare$$

5.2 Eigenvalues of A Belong to $[\mu_k, \mu_{k+1}[\, \times\, [\mu_{k'}, \mu_{k'+1}[$ with $k < k'$, $k, k' \in \mathbb{N}^*$

Proposition 5.3 *Suppose $A \in \mathcal{M}$ and*

$$\mu_k \le a - |b| \le \lambda_1 \le a + |b| < \mu_{k+1} \le \mu_{k'} \le c - |b| \le \lambda_2 \le c + |b| < \mu_{k'+1}$$

for some $k, k' \in \mathbb{N}^*$.

(i) *There exist $\rho, \delta > 0$ such that $J_\mu(u,v) \ge \delta$ for all $(u,v) \in (\partial B_\rho \cap Y_k) \times (\partial B_\rho \cap Y_{k'})$.*

(ii) *There exists $R > \rho$ such that $J_\mu\,|_{\partial Q_\varepsilon^m} \le p(m)$, with $p(m) \to 0$ as $m \to +\infty$ and*

$$Q_\varepsilon^m = \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m / 0 \le r < R\}\big) \times \big((\overline{B}_R \cap X_{k'}^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m / 0 \le r < R\}\big).$$

Proof For any $(u,v) \in Y_k \times Y_{k'}$, we have

$$\|u\|_\mu^2 \ge \mu_{k+1}\int_\Omega |u|^2\,dx \quad\text{and}\quad \|v\|_\mu^2 \ge \mu_{k'+1}\int_\Omega |v|^2\,dx. \tag{5.3}$$

Then (2.3), (5.3) and Young's inequality imply that

$$J_\mu(u,v) \ge \frac{1}{2}\Big(1 - \frac{a+|b|}{\mu_{k+1}}\Big)\|u\|_\mu^2 - C_1\|u\|_\mu^{2^*} + \frac{1}{2}\Big(1 - \frac{c+|b|}{\mu_{k'+1}}\Big)\|v\|_\mu^2 - C_2\|v\|_\mu^{2^*} \ge \delta$$

for $\rho = \|(u,v)\|_\mu$ sufficiently small.

For any $(u,v) \in X_k^m \times X_{k'}^m$, we obtain from (2.1), (4.2) and Lemma 5.1 that

$$\begin{aligned} J_\mu(u,v) &\le \frac{1}{2}\int_\Omega \Big[\big(\mu_k - (a-|b|)\big)|u|^2 + \big(\mu_{k'} - (c-|b|)\big)|v|^2 \\ &\qquad + C_3 m^{-2\sqrt{\overline{\mu}-\mu}}(|u|^2+|v|^2)\Big]\,dx - \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx \\ &\le -H_\lambda(u,v) \quad \text{with } \lambda := C_3 m^{-2\sqrt{\overline{\mu}-\mu}}, \end{aligned}$$

so $J_\mu(u,v) \le C_4 m^{-N\sqrt{\overline{\mu}-\mu}}$. Then $\underline{\lim}_{m\to\infty} \max_{(u,v)\in X_k^m\times X_{k'}^m} J_\mu(u,v) = 0$. With similar arguments as in Proposition 5.2, we get $J_\mu|_{\partial Q_\varepsilon^m} \le 0$, where

$$Q_\varepsilon^m = \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m / 0 \le r < R\}\big) \times \big((\overline{B}_R \cap X_{k'}^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m / 0 \le r < R\}\big).$$

■

Set $c_\varepsilon = \inf_{h\in\Gamma_{\varepsilon,m}} \max_{U\in Q_\varepsilon^m} J_\mu(h(U))$ with

$$\Gamma_{\varepsilon,m} = \{h \in C(Q_\varepsilon^m, E)/h(U) = U, \forall U \in Q_\varepsilon^m\}$$

and

$$\begin{aligned} Q_\varepsilon^m = &\big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m/0 \le r < R\}\big) \\ &\times \big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m/0 \le r < R\}\big) \quad \text{if } \mu_k \le \lambda_1 \le \lambda_2 < \mu_{k+1} \end{aligned}$$

or

$$\begin{aligned} Q_\varepsilon^m = &\big((\overline{B}_R \cap X_k^m) \oplus \{\mathrm{Br}\,\omega_\varepsilon^m/0 \le r < R\}\big) \times \big((\overline{B}_R \cap X_{k'}^m) \oplus \{\mathrm{Cr}\,\omega_\varepsilon^m/0 \le r < R\}\big) \\ &\text{if } \mu_k \le \lambda_1 < \mu_{k+1} \le \mu_{k'} \le \lambda_2 < \mu_{k'+1}. \end{aligned}$$

Lemma 5.4 *Let $\mu \in \big[0, \overline{\mu} - (\frac{N+2}{N})^2\big)$ and $A \in \mathcal{M}$. Assume one of the following conditions holds:*

(i) *There exists $k \in \mathbb{N}^*$ such that $\mu_k \le \lambda_1 \le \lambda_2 < \mu_{k+1}$.*

(ii) *There exist $k, k' \in \mathbb{N}^*$, $k \ne k'$ such that $\mu_k \le a - |b| \le \lambda_1 \le a + |b| < \mu_{k+1} \le \mu_{k'} \le c - |b| \le \lambda_2 \le c + |b| < \mu_{k'+1}$.*

Then

$$c_\varepsilon < \frac{2}{N-2}\Big(\frac{S_{\mu,\alpha,\beta}}{2^*}\Big)^{\frac{N}{2}}.$$

Proof Let

$$\max_{(u,v)\in Q_\varepsilon^m} J_\mu(u,v) = J_\mu(y_m + t_\varepsilon^m B\omega_\varepsilon^m, z_m + t_\varepsilon^m C\omega_\varepsilon^m)$$

where $B, C > 0$ such that $\frac{B}{C} = (\frac{\alpha+1}{\beta+1})^{\frac{1}{2}}$ and

$$(y_m, z_m) \in \begin{cases} (X_k^m)^2 & \text{if } \mu_k \le \lambda_1 \le \lambda_2 < \mu_{k+1}, \\ X_k^m \times X_{k'}^m & \text{if } \mu_k \le \lambda_1 < \mu_{k+1} \le \mu_{k'} \le \lambda_2 < \mu_{k'+1}. \end{cases}$$

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From Propositions 5.2 and 5.3 we have

$$J_\mu(y_m, z_m) \le C_1 m^{-N\sqrt{\overline{\mu}-\mu}}.$$

Since $\operatorname{meas}(\operatorname{supp}(\omega_\varepsilon^m) \cap \operatorname{supp}(y_m)) = 0$ and $\operatorname{meas}(\operatorname{supp}(\omega_\varepsilon^m) \cap \operatorname{supp}(z_m)) = 0$, we conclude that

$$\begin{aligned} c_\varepsilon &\le \max_{(u,v)\in Q_\varepsilon^m} J_\mu(u,v) = J_\mu(y_m, z_m) + J_\mu(t_\varepsilon^m B\omega_\varepsilon^m, t_\varepsilon^m C\omega_\varepsilon^m) \\ &\le C_1 m^{-N\sqrt{\overline{\mu}-\mu}} + (t_\varepsilon^m)^2 \frac{(B^2+C^2)}{2}(\|\omega_\varepsilon^m\|_\mu^2 - \lambda_1|\omega_\varepsilon^m|_2^2) \\ &\quad - B^{\alpha+1}C^{\beta+1}(t_\varepsilon^m)^{2^*}|\omega_\varepsilon^m|_{2^*}^{2^*}. \end{aligned}$$

Using (5.1), we obtain

$$\begin{aligned} c_\varepsilon \le C_1 m^{-N\sqrt{\overline{\mu}-\mu}} + (t_\varepsilon^m)^2 \frac{(B^2+C^2)}{2}(S_\mu^{\frac{N}{2}} + C_2 m^{-N\sqrt{\overline{\mu}-\mu}} - \lambda_1 C_3 m^{-(N+2)}) \\ - B^{\alpha+1}C^{\beta+1}(t_\varepsilon^m)^{2^*}(S_\mu^{\frac{N}{2}} - C_4 m^{-\frac{2N}{N-2}\sqrt{\overline{\mu}-\mu}}). \end{aligned}$$

Put

$$\begin{aligned} h(t_\varepsilon^m) := \frac{(t_\varepsilon^m)^2}{2}(B^2+C^2)(S_\mu^{\frac{N}{2}} + C_2 m^{-N\sqrt{\overline{\mu}-\mu}} - \lambda_1 C_3 m^{-(N+2)}) \\ - B^{\alpha+1}C^{\beta+1}(t_\varepsilon^m)^{2^*}(S_\mu^{\frac{N}{2}} - C_4 m^{-\frac{2N}{N-2}\sqrt{\overline{\mu}-\mu}}). \end{aligned}$$

Then

$$\max_{t_\varepsilon^m>0} h(t_\varepsilon^m) \le \frac{2}{N-2}\Big(\frac{S_{\mu,\alpha,\beta}}{2^*}\Big)^{\frac{N}{2}} + C_5 m^{-N\sqrt{\overline{\mu}-\mu}} - \lambda_1 C_6 m^{-(N+2)}.$$

Thus

$$\begin{aligned} c_\varepsilon &\le \max_{(u,v)\in Q_\varepsilon^m} J_\mu(u,v) \\ &\le C_1 m^{-N\sqrt{\overline{\mu}-\mu}} + \frac{2}{N-2}\Big(\frac{S_{\mu,\alpha,\beta}}{2^*}\Big)^{\frac{N}{2}} + C_5 m^{-N\sqrt{\overline{\mu}-\mu}} - \lambda_1 C_6 m^{-(N+2)}. \end{aligned}$$

Then we have

$$c_\varepsilon < \frac{2}{N-2}\Big(\frac{S_{\mu,\alpha,\beta}}{2^*}\Big)^{\frac{N}{2}} \quad \text{for } \mu \in \Big[0, \overline{\mu} - \Big(\frac{N+2}{N}\Big)^2\Big) \text{ and } m \text{ large enough.} \quad \blacksquare$$

Proof of Theorem 2.6 From Lemma 4.2 and Propositions 5.2 and 5.3, J_μ satisfies all assumptions of the linking theorem [2]. Then J_μ has a critical point whose critical value belongs to $\big(0, \frac{2}{N-2}(\frac{S_{\mu,\alpha,\beta}}{2^*})^{\frac{N}{2}}\big)$. ■

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