J. Austral. Math. Soc. (*Series A*) **69** (2000), 116–126

DIAGRAMMATICALLY REDUCIBLE COMPLEXES AND HAKEN MANIFOLDS

J. M. CORSON and B. TRACE

(Received 16 April 1999; revised 28 February 2000)

Communicated by W. D. Neumann

Abstract

We show that diagrammatically reducible two-complexes are characterized by the property: every finite subcomplex of the universal cover collapses to a one-complex. We use this to show that a compact orientable three-manifold with nonempty boundary is Haken if and only if it has a diagrammatically reducible spine. We also formulate an analogue of diagrammatic reducibility for higher dimensional complexes. Like Haken three-manifolds, we observe that if $n \geq 4$ and M is a compact connected n-dimensional manifold with a triangulation, or a spine, satisfying this property, then the interior of the universal cover of M is homeomorphic to Euclidean n-space.

2000 *Mathematics subject classification*: primary 57M20, 57N10, 20F06.
Keywords and phrases: diagrammatically reducible, manifold, Haken manifold, spine, collapse, covering space, two-complex.

1. Introduction

In this paper we establish a connection between Haken 3-manifolds and diagrammatically reducible 2-complexes. More precisely, we show that a compact orientable 3-manifold M with nonempty boundary is Haken if and only if it has a diagrammatically reducible spine K (Theorem 4.4).

To carry out this construction, we first give a characterization of diagrammatically reducible 2-complexes, which is a result of independent significance (Theorem 2.4): A 2-complex X is diagrammatically reducible if and only if every finite subcomplex of the universal cover of X collapses to a 1-complex. In the case of finite 2-complexes, this was conjectured by Brick [Bk].

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

Diagrammatically reducible 2-complexes were introduced by Sieradski [Si] and were subsequently studied by Gersten [Ge1, Ge2] and others. They are an interesting class of aspherical 2-complexes, with applications in equations over groups. Haken manifolds are an important, well-behaved, class of compact 3-manifolds; see, for example, [He] as a general reference. Knot complements are examples of orientable Haken 3-manifolds. Gersten has previously shown [Ge2] that orientable Haken 3-manifolds have the homotopy type of a diagrammatically reducible 2-complex. And earlier Chiswell, Collins, and Huebschmann [CCH] had shown that bounded Haken 3-manifolds have the homotopy type of a Diagrammatically Aspherical 2-complex (a weaker property).

Our equivalent formulation of diagrammatic reducibility makes sense, with a minor modification, for higher dimensional complexes. With this in mind, we say that a simplicial complex K satisfies the property P_1 if: every finite subcomplex of the universal cover of K is contained in a finite subcomplex that collapses to a 1-complex. In dimension two it is not necessary to go to a larger subcomplex since subcomplexes of a finite 2-complex that collapses to a 1-complex also collapse to 1-complexes. In fact it is easy to see that a finite 2-complex collapses to a 1-complex if and only if every 2-dimensional subcomplex contains a 2-cell with a free face; see Section 2 for this terminology. Thus, for 2-complexes the P_1 condition is equivalent to diagrammatic reducibility, by Theorem 2.4.

Using this notion we extend a well-known result about Haken manifolds in dimension three. Namely, if M^n is a compact, connected, n-dimensional manifold with a triangulation, or a spine, with the property P_1 ($n \geq 4$), then the interior of M^n is covered by $\mathbb{R}^n$ (Theorem 3.2). By a *spine* of a PL manifold M we mean a simplicial complex K such that some triangulation of M simplicially collapses to a subcomplex isomorphic to K. For a general reference on piecewise linear topology, we refer the reader to [RS].

It should be noted that there is also an interesting characterization, due to Gersten [Ge2], of diagrammatic reducibility in terms of branched coverings. It may be worth investigating what this condition means in higher dimensions, and possibly comparing with the P_1 condition above.

2. Diagrammatically reducible complexes

In this section we work in the category of combinatorial 2-complexes. Thus, for our purposes every 2-cell of a 2-complex is attached along a (finite) edge-circuit, and by a map of 2-complexes we mean a combinatorial map (that is, a map in which each open cell in the domain is mapped homeomorphically onto an open cell in the target).

Let X be a 2-complex. We say that an open $(n-1)$-cell t is a *free face* of an

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

open n-cell e if it occurs exactly once in the boundary of e and it does not occur in the boundary of any other n-cell. Recall that under these circumstances, the passage from X to the subcomplex $X \setminus (e \cup t)$ is called an elementary collapse. We say that X collapses to a subcomplex A if there is a finite sequence of elementary collapses passing from X to A. (In this case, of course, X and A have the same homotopy type.)

For convenience, we say that a 2-complex is *closed* if it is finite and none of its cells has a free face. Notice that every finite 2-complex collapses to a closed subcomplex.

Given a closed surface F, we say that a map $f : F \to X$ is a *near immersion* if F supports a combinatorial cell structure for which f is a combinatorial map and $f|_{F \setminus F^0}$ is an immersion. Here F^0 denotes the 0-skeleton of the cell structure of F, and by an immersion we mean a local embedding. Then we have:

DEFINITION. A 2-complex X is *diagrammatically reducible* (abbreviated DR) if there is no near immersion of S^2 into X.

The next lemma is used in the proof of the main result in this section. For use in the proof we make a definition: A *complete set of cutting curves* on a closed orientable surface F is a collection of disjoint simple closed curves such that cutting the surface along these curves yields a genus zero surface.

LEMMA 2.1. *Suppose $f : F \to X$ is a near immersion, where F is a closed surface and X is a 1-connected 2-complex. Then there exists a near immersion $S^2 \to X$ (that is, X is not* DR).

PROOF. By first subdividing X, and F correspondingly, we may assume that X is a simplicial complex. We may also assume, by taking an orientable double cover, that F is orientable.

Choose a complete set of cutting curves $\gamma_1, \dots, \gamma_k$ for F such that each curve avoids the finitely many points at which f is not a local embedding. Then each $f(\gamma_i)$ is an immersed curve in X. By appropriately subdividing X (and pulling back the subdivision to F), we can arrange that the γ_i lie in the 1-skeleton and thus are embedded edge-circuits. Since X is simply connected, each $f(\gamma_i)$ is null-homotopic and hence bounds a van Kampen diagram (D_i, ϕ_i) in X. Recall that a van Kampen diagram (D, ϕ) in X is a finite 1-connected planar 2-complex D and a combinatorial map $\phi : D \to X$; see, for example, [LS] for more details.

Form a 2-complex L by 'attaching' the diagram D_i to F along γ_i, for each $i = 1, \dots, k$. It should be noted that under this 'attaching' some identifications of F along γ_i may be performed. Define a combinatorial map $\phi : L \to X$ by $\phi|_F = f$ and $\phi|_{D_i} = \phi_i$ $(1 \le i \le k)$. Note that L is a closed, 1-connected 2-complex and that L embeds in S^3 (as shown in Figure 1) such that $S^3 \setminus L$ is a disjoint union of open 3-cells

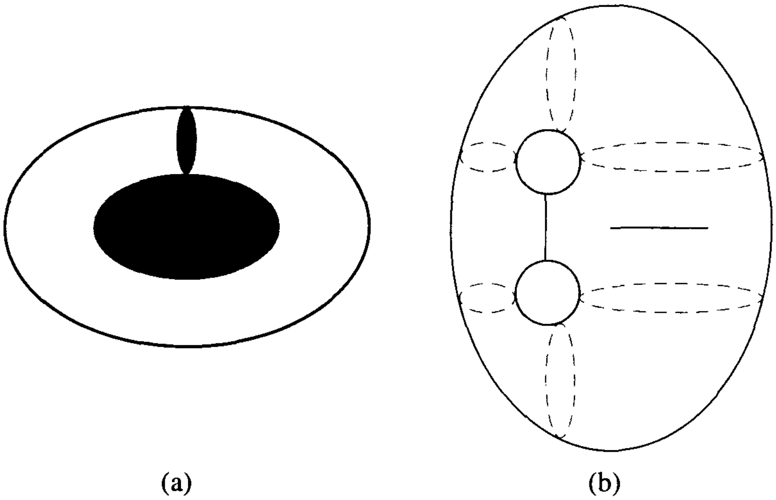

FIGURE 1. Embedding of L in the 3-sphere in the case where F is the torus. In (a) the attached diagrams are disks however in general, as in (b), they are 'pinched' disks.

(two of them in this case). Furthermore, the map f factors through L; $f = \phi \circ \iota$, where $\iota : F \to L$ is the natural map into the adjunction space L.

Hence, there exists a 2-complex K with the following four properties:

(1) K is a closed 1-connected 2-complex embedded in S^3.

(2) $S^3 \setminus K$ is a disjoint union of open 3-cells, each of which is attached to K by an immersion $S^2 \to K$.

(3) The map f factors through K; that is, there exist combinatorial maps $g : F \to K$ and $h : K \to X$ such that $f = h \circ g$.

(4) Amongst all 2-complexes satisfying 1–3, K has a minimal number of 2-cells.

Now let $j : S^2 \to K$ be the attaching map (immersion) of one of the 3-cells in $S^3 \setminus K$. We claim that the map $h \circ j : S^2 \to X$ is a near immersion.

To see this, suppose $h \circ j$ is not a near immersion. Then there exists a pair of distinct closed 2-cells σ and τ, in the pull-back cell structure of S^2, such that $\sigma \cap \tau$ contains a 1-cell e and $h(j(\sigma)) = h(j(\tau))$. Thus, for each point $x \in \sigma$, there is a unique point $x' \in \tau$ such that $h(j(x)) = h(j(x'))$, and $x = x'$ if $x \in e$. Let K' denote the 2-complex obtained by identifying $j(x)$ and $j(x')$, for each $x \in \sigma$.

There are four ways in which the 2-cells σ and τ can meet: in one edge, the union of two edges, one edge and a disjoint vertex, or the three edges making up the entire (common) boundary of the 2-cells. In any case, observe that the embedding of K in S^3 can be continuously deformed to an embedding of K', folding $j(\sigma) \cup j(\tau)$ at e in the the direction of the 3-cell bounded by the immersion j. In the first two cases, the number of 3-cells in $S^3 \setminus K'$ is the same as in $S^3 \setminus K$. In the third case, the number of

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

3-cells is increased by one. And in the last case, the number of 3-cells is decreased by one.

The complement of this embedding of K' is again a disjoint union of open 3-cells, and after collapsing any 2-cells of K' with a free edge that may have been introduced, we see that (1) and (2) hold. Also (3) holds for K', since h factors through K'. But K' has one less 2-cell than K, contradicting (4). Our claim therefore follows, and hence X is not DR. □

REMARK. As an alternative to viewing K as embedded in S^3 in the above proof, the conditions (1) and (2) can be replaced by simply requiring the existence of a collection of immersions $S^2 \to K$ such that each open 2-cell of K is hit exactly twice.

We note two easy consequences before turning to the theorem.

COROLLARY 2.2. *Suppose $f : F \to X$ is a near immersion, where F is a closed surface and X is* DR. *Then the image of $f_* : \pi_1(F) \to \pi_1(X)$ is nontrivial.*

PROOF. If f_* is the trivial homomorphism, then f lifts to the universal cover $\widetilde{X}$. But $\widetilde{X}$ is DR, contradicting Lemma 2.1. □

COROLLARY 2.3. *Let X be a closed 2-complex. If X is* DR, *then $\pi_1(X, x_0)$ is infinite (and torsion-free).*

PROOF. By [CT1, Theorem 2.1] there is a near immersion $f : F \to X$, where F is some closed surface. Thus, $\pi_1(X, x_0) \neq 1$ by Corollary 2.2. The result now follows since X is an aspherical 2-complex; see [Ge1]. □

THEOREM 2.4. *A 2-complex X is* DR *if and only if every finite subcomplex of the universal cover $\widetilde{X}$ collapses to a 1-complex.*

PROOF. First assume that X is DR and let L be a finite subcomplex of $\widetilde{X}$. Then L collapses to a closed subcomplex L_0, which we claim is a 1-complex. For if L_0 were 2-dimensional, then by [CT1, Theorem 2.1] there would be a near immersion $f : F \to L_0$, for some closed surface F. But that would imply, by Lemma 2.1, that $\widetilde{X}$ is not DR, a contradiction. (Clearly a 2-complex is DR if and only if its universal cover is DR.)

Conversely, suppose $f : S^2 \to X$ is a near immersion. Then f lifts to a near immersion $f' : S^2 \to \widetilde{X}$ in the universal cover. But the image of a near immersion of a closed surface is a closed 2-dimensional subcomplex. Thus the image of f' is a finite subcomplex of $\widetilde{X}$ that does not collapse to a 1-complex. □

In the case of a finite 2-complex X (or any 2-complex whose universal cover has only a countable number of cells), note that Theorem 2.4 can be stated as was conjectured by Brick [Bk]: X is DR if and only if the universal cover of X is the union of an ascending sequence of finite subcomplexes, each of which collapses to a 1-complex.

3. Generalization of diagrammatic reducibility

Henceforth, we consider only simplicial complexes. Thus by a k-complex we now mean a simplicial complex of dimension $\leq k$. To indicate that a simplicial complex L (simplicially) collapses to a subcomplex K, we write $L \searrow K$. See the book by Rourke and Sanderson [RS] for a general reference on piecewise linear topology.

DEFINITION. For each nonnegative integer k, we say that a simplicial complex K *satisfies the property* P_k provided: every finite subcomplex of $\widetilde{K}$ is contained in a finite subcomplex that collapses to a k-complex.

We are only interested here in the cases $k = 0$ and $k = 1$. As we noted in the introduction, a 2-complex X satisfies P_1 if and only if it is DR. Thus the condition P_1 can be viewed as a generalization of diagrammatic reducibility, for simplicial complexes of arbitrary dimension.

The next lemma is true for any nonnegative integer k.

LEMMA 3.1. *Suppose K is a subcomplex of a finite simplicial complex L and that $L \searrow K$. If K satisfies property P_k, then L also satisfies P_k.*

PROOF. We may assume that $L = K \cup \{s^n, s^{n-1}\}$, where s^n and s^{n-1} are open simplices that are not contained in K and s^{n-1} is a face of s^n. Let X be a finite subcomplex of $\widetilde{L}$. Observe that $\widetilde{L}$ is obtained from $\widetilde{K}$ by attaching lifts of s^n, each of which has a free face projecting to s^{n-1}. So $X \searrow A$ where A is the subcomplex of X obtained by deleting all the lifts of s^n and s^{n-1}. Since $A \subset \widetilde{K}$, there is a finite subcomplex B of $\widetilde{K}$, containing A, such that B collapses to a k-complex. Put $Y = B \cup X$, a finite subcomplex of $\widetilde{L}$ containing X. Then $Y \searrow B$ (by collapsing away each lift of s^n) which then collapses to a k-complex. □

In the next section we show that every Haken 3-manifold has a triangulation satisfying property P_1, and it is well known that the interior of every Haken 3-manifold is covered by $\mathbb{R}^3$. We observe next that the same is true in higher dimensions.

THEOREM 3.2. *Let M^n be a compact, connected, n-dimensional manifold ($n \geq 4$) that has a triangulation or spine with the property P_1. Then the universal cover of* Int M^n *is (topologically) homeomorphic to $\mathbb{R}^n$.*

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

PROOF. If M^n has a spine satisfying P_1, then by Lemma 3.1 it also has a triangulation with this property. So let M^n be triangulated in this fashion.

Let C be a compact subset of $\widetilde{M}$. We show that C is contained in a PL n-cell. By property P_1, there is a finite connected subcomplex X of $\widetilde{M}$, that collapses to a 1-complex, such that $C \subset X$. Let V be a regular neighbourhood of X in $\widetilde{M}$. Then V is an n-dimensional handlebody (a 0-handle with 1-handles attached).

Since $\widetilde{M}$ is simply connected, there exists a finite connected subcomplex Y of $\widetilde{M}$, containing X, such that $\pi_1(X) \to \pi_1(Y)$ is the trivial homomorphism. Let W be a regular neighbourhood of Y, so that W is an n-dimensional handlebody and $V \subset W$ induces a trivial homomorphism of fundamental groups.

Now, since $n \geq 4$, it follows by a general position argument that V is ambient isotopic in W to a subset of the 0-handle of W. This is a special case of the Zeeman Engulfing Theorem; see for example [Ru, Theorem 4.6.1]. Therefore, V is contained in an n-cell, and hence this n-cell contains C.

Thus, every compact subset of $\widetilde{M}$ is contained in an n-cell. It follows that $\operatorname{Int} \widetilde{M}$ is the union of an ascending sequence of open n-cells. The proof is completed by appealing to Brown's Theorem [Bn]. □

As a consequence we have the following (the case $n = 3$ is handled in the next section): Let K be a finite, connected, diagrammatically reducible 2-complex. If M is any n-dimensional thickening of K, that is, triangulated n-manifold that collapses to K, then $\operatorname{Int} M$ is covered by $\mathbb{R}^n$. Of course, not every finite 2-complex has a 3-dimensional thickening, but they all have n-dimensional thickenings, for every $n \geq 4$.

4. Haken three-manifolds

Turning to 3-dimensional manifolds we next show that an orientable Haken 3-manifold with nonempty boundary has a spine which is DR, in a strong sense.

THEOREM 4.1. *Let M be an orientable Haken 3-manifold with nonempty boundary. Then M has a 2-dimensional spine K satisfying the property P_0 (in particular, K is* DR).

We first establish two preliminary results. Here, and elsewhere, we say that an embedding $j : A \to X$, or its image $j(A)$, is *incompressible* if $j_* : \pi_1(A) \to \pi_1(X)$ is injective for any choice of base point in $j(A)$.

LEMMA 4.2. *Suppose K and Σ are finite simplicial complexes and $g : \Sigma \times \{-1, 1\} \to K$ is a simplicial map such that $g|_{\Sigma\times\{-1\}}$ and $g|_{\Sigma\times\{1\}}$ are incompressible embeddings. If K and Σ both have the property P_0, then $L = K \cup_g (\Sigma \times [-1, 1])$ also satisfies P_0.*

PROOF. We may assume that L and Σ are connected, and that K has one or two components. Then $\pi_1(L)$ is either an HNN extension of $\pi_1(K)$ or an amalgamated free product of the fundamental groups of the distinct components of K; in each case the splitting is over a subgroup isomorphic to $\pi_1(\Sigma)$. The universal cover of L therefore consists of copies of the universal covers of the components of K connected by copies of the universal cover of $\Sigma \times [-1, 1]$ in a 'tree-like' fashion.

Denote by $p : \widetilde{L} \to L$ the universal covering map, and let X be a finite connected subcomplex of $\widetilde{L}$. Then X meets only finitely many closures of components of $p^{-1}(\Sigma \times (-1, 1))$, each of which is a copy of $\widetilde{\Sigma} \times [-1, 1]$. Denote these components $(\widetilde{\Sigma} \times [-1, 1])_1, \ldots, (\widetilde{\Sigma} \times [-1, 1])_m$. By hypothesis, we can choose a subcomplex of the form $A_i = T_i \times [-1, 1]$ of $(\widetilde{\Sigma} \times [-1, 1])_i$ where T_i is a finite collapsible subcomplex of $\widetilde{\Sigma}$, large enough that $X \cap (\widetilde{\Sigma} \times [-1, 1])_i \subset A_i$ (for $i = 1, \ldots, m$). Put $Y = X \cup A_1 \cup A_2 \cdots \cup A_m$, a finite subcomplex of $\widetilde{L}$.

Then Y meets only finitely many components of $p^{-1}(K)$, say $\widetilde{K}_1, \ldots, \widetilde{K}_n$, each of which is a copy of the universal cover of a component of K. Choose, as we may by the hypothesis on K, a collapsible subcomplex B_j of $\widetilde{K}_j$ such that $Y \cap \widetilde{K}_j \subset B_j$ for each $j = 1, \ldots, n$. Set $Z = B_1 \cup \cdots \cup B_n \cup A_1 \cup \cdots \cup A_m$, a finite subcomplex of $\widetilde{L}$ containing X. Note that Z consists of the complexes B_i joined by 'generalized 1-handles' A_j in a 'tree-like' manner.

We complete the proof by observing that Z is collapsible. Initially collapse each $A_i = T_i \times [-1, 1]$ onto the subcomplex $(T_i \times \{-1, 1\}) \cup (*_i \times [-1, 1])$ where $*_i$ is some vertex of T_i. In this way we collapse Z onto a subcomplex consisting of the parts B_j joined together by arcs (in a 'tree-like' fashion). Then we can collapse each part B_j onto a spanning tree in its 1-skeleton, thus collapsing Z onto a tree in its 1-skeleton. Finally we collapse this tree to a vertex, as required. □

It is obvious that 1-dimensional simplicial complexes have property P_0. We next observe that the same is true for triangulations of compact aspherical surfaces.

LEMMA 4.3. *If Σ is a 2-dimensional simplicial complex homeomorphic to a compact aspherical surface, then Σ satisfies property P_0.*

PROOF. If Σ has nonempty boundary, then Σ has a 1-dimensional spine and the result follows from Lemma 3.1. So assume that Σ is a closed surface. Then each component of $\widetilde{\Sigma}$ is a triangulation of the plane, and it is easy to see that every finite subcomplex of a triangulation of the plane is contained in a collapsible one. □

PROOF OF THEOREM 4.1. We assume, without loss of generality, that M is connected.

It is well known (see [He, Theorem 13.3]) that M admits a hierarchy of the following

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

form:

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = B^3,$$

where M_i is obtained from M_{i-1} by cutting along a properly embedded surface $F_i \subset M_{i-1}$ which satisfies:

(1) F_i is incompressible in M_{i-1};
(2) F_i is compact, connected, and orientable;
(3) $\partial F_i \neq \emptyset$;
(4) (implicit from $M_n = B^3$) F_i does not separate M_{i-1}.

We now associate to such a hierarchy of M a 2-dimensional spine—which satisfies property P_0.

To begin, let K_i denote a 1-dimensional spine for each F_i, $i = 1, \dots, n$. For simplicity, we assume that K_i is collapsed as much as possible. In particular, K_i is a point if $F_i = D^2$. Recall that 'M_i is obtained from M_{i-1} by cutting along F_i' means that we view $F_i \times [-1, 1] \subset M_{i-1}$ such that

$$(F_i \times [-1, 1]) \cap \partial M_{i-1} = \partial(F_i \times [-1, 1]) \cap \partial M_{i-1} = \partial F_i \times [-1, 1]$$

and $M_i = M_{i-1} - \left[F_i \times (-1, 1)\right]$. Evidently, there are two copies of F_i in ∂M_i : $F_i^+ = F_i \times \{1\}$ and $F_i^- = F_i \times \{-1\}$. Let K_i^+ and K_i^- denote the copies of K_i in F_i^+ and F_i^-, respectively.

We next construct certain 1-complexes $C_i \subset \partial M_i$ for $i = 1, \dots, n$. Initially, set $C_1 = K_1^- \cup K_1^+$. We assume (without loss) that F_2 meets C_1 transversely in a finite number of points, say $\{p_1, \dots, p_k\}$. For $i = 1, \dots, k$, let A_i denote an embedded arc in F_2 such that A_i joins p_i to K_2, $\operatorname{Int} A_i$ misses $K_2 \cup \partial F_2$, and $A_i \cap A_j = \emptyset$ if $i \neq j$. Set $S_2 = K_2 \cup (\bigcup_{i=1}^k A_i)$.

Since F_2 meets C_1 transversely, we may assume that $C_1 \cap (F_2 \times [-1, 1]) = \{p_1, \dots, p_k\} \times [-1, 1]$; that is, that C_1 meets $F_2 \times [-1, 1]$ in $[-1, 1]$-fibers. Now C_2 is defined by cutting C_1 along $\{p_1, \dots, p_k\}$ and gluing S_2^- and S_2^+ to this cut 1-complex, where $S_2^\pm$ are the copies of S_2 in $F_2^\pm$, respectively. In other words, $C_2 = \left[C_1 - (\{p_1, \dots, p_k\} \times [-1, 1])\right] \cup S_2^- \cup S_2^+$.

The process of passing from C_1 to C_2 is now repeated in obtaining C_{i+1} from C_i for $i = 1, \dots, n-1$.

We now describe the spine K for M by stating the intersection of K with the 'generalized handles' of M associated to its hierarchy: K is defined by the property that $K \cap M_n$ is the cone on C_n, and $K \cap (F_i \times [-1, 1])$ is $S_i \times [-1, 1]$ if $i > 1$, and $K \cap (F_1 \times [-1, 1])$ is $K_1 \times [-1, 1]$.

It is relatively straightforward to see that K is a spine for M. First of all, $F_1 \times [-1, 1]$ collapses to $(K_1 \times [-1, 1]) \cup (F_1 \times \{-1, 1\})$. Note that S_2 is a spine of F_2 and $F_2 \times [-1, 1]$ collapses to $(S_2 \times [-1, 1]) \cup (F_2 \times \{-1, 1\})$. Proceeding sequentially in

https://doi.org/10.1017/S1446788700001865 Published online by Cambridge University Press

this manner we obtain $M \searrow (K \cup M_n)$ and finally $M \searrow K$ since $K \cap M_n$ is the cone over C_n.

We show that K satisfies property P_0 inductively. Note that the preceding paragraph actually shows more, namely that $K \cap M_i$ is a spine of M_i for $i = 1, \dots, n$. The induction starts at $K \cap M_n$, which is collapsible and hence satisfies P_0. Then observe that $S_i^+, S_i^- \hookrightarrow K \cap M_i$ are incompressible embeddings and $K \cap M_{i-1} = (K \cap M_i) \cup (S_i \times [-1, 1])$. The inductive step, and hence the proof, is thus completed by Lemma 4.2.

We next observe that the converse of Theorem 4.1 holds, thus giving a characterization of orientable Haken 3-manifolds with boundary.

THEOREM 4.4. *A compact orientable 3-manifold M with nonempty boundary is Haken if and only if it has a diagrammatically reducible 2-dimensional spine K.*

PROOF. Suppose M is a compact 3-manifold with a DR spine K, and choose a triangulation of M that collapses to K. Then, by Lemma 3.1, the triangulation of M satisfies the condition P_1 (as K satisfies P_1 by Theorem 2.4). Recall that K, and hence M, is aspherical. It is well known that an irreducible, compact, aspherical 3-manifold with boundary is Haken. Thus, the proof is completed by the claim (which also holds for closed manifolds): Every compact 3-manifold with a triangulation satisfying P_1 is irreducible.

To see this, let S be a PL 2-sphere in M. Then S lifts to a 2-sphere $\widetilde{S}$ in $\widetilde{M}$ which, by property P_1, is contained in some finite connected subcomplex X of $\widetilde{M}$ that collapses to a 1-complex. Then a regular neighbourhood of X in $\widetilde{M}$ must be a 3-dimensional handlebody (consisting of a 0-handle and 1-handles). Since such handlebodies are irreducible, we conclude that $\widetilde{S}$ bounds a 3-cell which projects to a 3-cell in M bounded by S, as required. □

REMARK 4.5. For closed 3-manifolds the situation is more complicated. On the one hand, a construction similar to that of the spine for Theorem 4.1, using induction on the length of a hierarchy, shows that every closed Haken 3-manifold has a triangulation satisfying P_0 (and thus P_1). However, the converse is false for the following reason. There are closed 3-manifolds which are not Haken, but for which some finite sheeted cover is Haken (virtually Haken manifolds). Let M be such a 3-manifold and let M' be a finite cover of M which is Haken. Then M' supports a triangulation satisfying the property P_0. By a standard fact from PL topology, there is a subdivision of the triangulation of M' and a triangulation of M for which the covering projection is a simplicial map. This subdivided triangulation of M' also satisfies P_0, which follows from the fact that every subdivision of a 3-dimensional collapsible simplicial complex remains collapsible [Ch]. Since M and M' have the same universal cover, it follows that the triangulation of M also satisfies P_0.

We do not know whether every closed 3-manifold with a triangulation satisfying P_0 is virtually Haken.

References

[Bk] S. G. Brick, 'A note on coverings and Kervaire complexes', *Bull. Austral Math. Soc.* **46** (1992), 1–21.

[Bn] M. Brown, 'The monotone union of open n-cells is an open n-cell', *Proc. Amer. Math. Soc.* **12** (1961), 812–814.

[Ch] D. R. J. Chillingworth, 'Collapsing three-dimensional convex polyhedra', *Proc. Camb. Phil. Soc.* **63** (1967), 353–357. Correction: *Proc. Camb. Phil. Soc.* **88** (1980), 307–310.

[CCH] I. M. Chiswell, D. J. Collins and J. Huebschmann, 'Aspherical group presentations', *Math. Z.* **178** (1981), 1–36.

[CT1] J. M. Corson and B. Trace, 'Geometry and algebra of nonspherical 2-complexes', *J. London Math. Soc.* **54** (1996), 180–198.

[CT2] ———, 'The 6-property for simplicial complexes and a combinatorial Cartan-Hadamard Theorem for manifolds', *Proc. Amer. Math. Soc.* **126** (1998), 917–924.

[Ge1] S. M. Gersten, 'Reducible diagrams and equations over groups', in: *Essays in group theory* (ed. S. M. Gersten), *Publ. Math. Sci. Res. Inst. 8* (Springer, New York, 1987) pp. 15–73.

[Ge2] ———, 'Branched coverings of 2-complexes and diagrammatic reducibility', *Trans. Amer. Math. Soc.* **303** (1987), 689–706.

[He] J. Hempel, *3-manifolds*, Annals of Mathematics Studies 86 (Princeton University Press, Princeton, 1976).

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergeb. Math. 89 (Springer, New York, 1977).

[RS] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Ergeb. Math. 69 (Springer, New York, 1972).

[Ru] T. B. Rushing, *Topological embeddings*, Pure Appl. Math. 52 (Academic Press, New York and London, 1973).

[Si] A. J. Sieradski, 'A coloring test for asphericity', *Quart. J. Math. Oxford* **34** (1983), 97–106.

Department of Mathematics
University of Alabama
Box 870350
Tuscaloosa, AL 35487-0350
USA
e-mail: jcorson@mathdept.as.ua.edu
e-mail: btrace@mathdept.as.ua.edu