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TOEPLITZ DETERMINANTS WHOSE ELEMENTS ARE THE COEFFICIENTS OF ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract

Let $\mathcal{S}$ denote the class of analytic and univalent functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which are of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. We determine sharp estimates for the Toeplitz determinants whose elements are the Taylor coefficients of functions in $\mathcal{S}$ and certain of its subclasses. We also discuss similar problems for typically real functions.

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1. Introduction and preliminaries

Let $\mathcal{H}$ denote the space of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of functions f in $\mathcal{H}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of univalent (that is, one-to-one) functions has attracted much interest for over a century and is a central area of research in complex analysis. A function $f \in \mathcal{A}$ is called starlike if $f(\mathbb{D})$ is starlike with respect to the origin, that is, $tf(z) \in f(\mathbb{D})$ for every t with $0 \le t \le 1$. Let $\mathcal{S}^*$ denote the class of starlike functions in $\mathcal{S}$. It is well known that a function $f \in \mathcal{A}$ is starlike if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

An important member of the class $\mathcal{S}^*$ as well as of the class $\mathcal{S}$ is the Koebe function k defined by $k(z) = z/(1-z)^2$. This function plays the role of extremal function in most of the problems for the classes $\mathcal{S}^*$ and $\mathcal{S}$.

A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{D})$ is a convex domain. Let C denote the class of convex functions in $\mathcal{S}$. It is well known that a function $f \in \mathcal{A}$ is in C if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad \text{for } z\in\mathbb{D}.$$

From the above it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a starlike function $g \in \mathcal{S}^*$ and a real number $\alpha \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right)>0 \quad \text{for } z\in\mathbb{D}. \tag{1.2}$$

Let $\mathcal{K}$ denote the class of all close-to-convex functions. It is well known that every close-to-convex function is univalent in $\mathbb{D}$ (see [3]). Geometrically, $f \in \mathcal{K}$ means that the complement of the image-domain $f(\mathbb{D})$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

Let $\mathcal{R}$ denote the class of functions f in $\mathcal{A}$ satisfying $\operatorname{Re} f'(z) > 0$ in $\mathbb{D}$. It is well known that functions in $\mathcal{R}$ are close-to-convex and hence univalent. Functions in $\mathcal{R}$ are sometimes called functions of bounded rotation.

A function f satisfying the condition $(\operatorname{Im} z)(\operatorname{Im} f(z)) \geq 0$ for $z \in \mathbb{D}$ is called typically real. Let $\mathcal{T}$ denote the class of all typically real functions. Robertson [9] proved that $f \in \mathcal{T}$ if and only if there exists a probability measure μ on $[-1, 1]$ such that

$$f(z)=\int_{-1}^{1} k(z,t)\,d\mu(t),$$

where

$$k(z,t)=\frac{z}{1-2tz+z^2} \quad \text{for } z\in\mathbb{D} \text{ and } t\in[-1,1].$$

Recently, Aleman and Constantin [1] provided a nice connection between univalent function theory and fluid dynamics. They seek explicit solutions to the incompressible two-dimensional Euler equations by means of a univalent harmonic map. More precisely, the problem of finding all solutions describing the particle paths of the flow in Lagrangian variables is reduced to finding harmonic functions satisfying an explicit nonlinear differential system in $\mathbb{C}^n$ with $n = 3$ or $n = 4$ (see also [2]).

Hankel matrices and determinants play an important role in several branches of mathematics and have many applications [12]. Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. For a summary of applications of Toeplitz matrices to a wide range of areas of pure and applied mathematics, we refer to [12]. Recently, Thomas and Halim [11] introduced the symmetric Toeplitz determinant $T_q(n)$ for analytic functions f of the form (1.1), defined by

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

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where $n, q = 1, 2, 3, \ldots$, and $a_1 = 1$. In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

For small values of n and q, estimates of the Toeplitz determinant $|T_q(n)|$ for functions in $\mathcal{S}^*$ and $\mathcal{K}$ have been studied in [11]. Similarly, estimates of the Toeplitz determinant $|T_q(n)|$ for functions in $\mathcal{R}$ have been studied in [8], when n and q are small. Apart from [8] and [11], there appears to be little in the literature concerning estimates of Toeplitz determinants. In [8, 11], we observe an invalid assumption in the proofs. It is the purpose of this paper to give estimates for Toeplitz determinants $T_q(n)$ for functions in $\mathcal{S}, \mathcal{S}^*, \mathcal{C}, \mathcal{K}, \mathcal{R}$ and $\mathcal{T}$, when n and q are small.

Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.3}$$

such that $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Functions in $\mathcal{P}$ are sometimes called Carathéodory functions. To prove our main results, we need some preliminary results for functions in $\mathcal{P}$.

Lemma 1.1 [3, page 41]. *For a function $p \in \mathcal{P}$ of the form (1.3), the sharp inequality $|c_n| \le 2$ holds for each $n \ge 1$. Equality holds for the function $p(z) = (1+z)/(1-z)$.*

Lemma 1.2 [4, Theorem 1]. *Let $p \in \mathcal{P}$ be of the form (1.3) and $\mu \in \mathbb{C}$. Then*

$$|c_n - \mu c_k c_{n-k}| \le 2 \max\{1, |2\mu - 1|\} \quad \text{for } 1 \le k \le n-1.$$

If $|2\mu - 1| \ge 1$ then the inequality is sharp for the function $p(z) = (1+z)/(1-z)$ or its rotations. If $|2\mu - 1| < 1$ then the inequality is sharp for $p(z) = (1+z^n)/(1-z^n)$ or its rotations.

2. Main results

Theorem 2.1. *Let $f \in \mathcal{S}$ be of the form (1.1). Then*

(i) $|T_2(n)| = |a_n^2 - a_{n+1}^2| \le 2n^2 + 2n + 1$ *for* $n \ge 2$;
(ii) $|T_3(1)| \le 24$.

Both inequalities are sharp.

Proof. Let $f \in \mathcal{S}$ be of the form (1.1). Then clearly

$$|T_2(n)| = |a_n^2 - a_{n+1}^2| \le |a_n^2| + |a_{n+1}^2| \le n^2 + (n+1)^2 = 2n^2 + 2n + 1. \tag{2.1}$$

Equality holds in (2.1) for the function f defined by

$$f(z) := \frac{z}{(1-iz)^2} = z + 2iz^2 - 3z^3 - 4iz^4 + 5z^5 + \cdots. \tag{2.2}$$

Again, if $f \in \mathcal{S}$ is of the form (1.1) then by the Fekete–Szegö inequality for functions in $\mathcal{S}$,

$$\begin{aligned} |T_3(1)| &= |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \\ &\le 1 + 2|a_2^2| + |a_3|\,|a_3 - 2a_2^2| \le 1 + 8 + (3)(5) = 24. \end{aligned} \tag{2.3}$$

Equality holds in (2.3) for the function f defined by (2.2). □

Remark 2.2. Since the function f defined by (2.2) belongs to $\mathcal{S}^*$ and $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$, the sharp inequalities in Theorem 2.1 also hold for functions in $\mathcal{S}^*$ and $\mathcal{K}$. In particular, the sharp inequalities $|T_2(2)| \le 13$ and $|T_2(3)| \le 25$ hold for functions in $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{S}$.

Theorem 2.3. *Let $f \in \mathcal{S}^*$ be of the form (1.1). Then $|T_3(2)| \le 84$. The inequality is sharp.*

Proof. Let $f \in \mathcal{S}^*$ be of the form (1.1). Then there exists a function $p \in \mathcal{P}$ of the form (1.3) such that $zf'(z) = f(z)p(z)$. Equating coefficients,

$$a_2 = c_1, \quad a_3 = \tfrac{1}{2}(c_2 + c_1^2) \quad \text{and} \quad a_4 = \tfrac{1}{6}c_1^3 + \tfrac{1}{2}c_1c_2 + \tfrac{1}{3}c_3. \tag{2.4}$$

By a simple computation $T_3(2)$ can be written as $T_3(2) = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)$. If $f \in \mathcal{S}^*$ then clearly, $|a_2 - a_4| \le |a_2| + |a_4| \le 6$. Therefore, we only need to maximise $|a_2^2 - 2a_3^2 + a_2a_4|$ for functions in $\mathcal{S}^*$. Writing a_2, a_3 and a_4 in terms of c_1, c_2 and c_3 with the help of (2.4),

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| &= |c_1^2 - \tfrac{1}{3}c_1^4 - \tfrac{1}{2}c_1^2c_2 - \tfrac{1}{2}c_2^2 + \tfrac{1}{3}c_1c_3| \\ &\le |c_1|^2 + \tfrac{1}{3}|c_1|^4 + \tfrac{1}{2}|c_2|^2 + \tfrac{1}{3}|c_1|\,|c_3 - \tfrac{3}{2}c_1c_2|. \end{aligned}$$

From Lemmas 1.1 and 1.2, it easily follows that

$$|a_2^2 - 2a_3^2 + a_2a_4| \le 4 + \tfrac{16}{3} + \tfrac{4}{2} + \tfrac{2}{3}(4) = 14.$$

Therefore, $|T_3(2)| \le 84$ and the inequality is sharp for the function f defined by (2.2). □

Remark 2.4. In [11], it was claimed that $|T_2(2)| \le 5$, $|T_2(3)| \le 7$, $|T_3(1)| \le 8$ and $|T_3(2)| \le 12$ hold for functions in $\mathcal{S}^*$ and these estimates are sharp. Similar results were also obtained for certain close-to-convex functions. For the function f defined by (2.2), a simple computation gives $|T_2(2)| = 13$, $|T_2(3)| = 25$, $|T_3(1)| = 24$ and $|T_3(2)| = 84$, which shows that these estimates are not correct. In proving these estimates the authors assumed that $c_1 > 0$ which is not justified, since the functional $|T_q(n)|$ $(n \ge 1, q \ge 2)$ is not rotationally invariant.

To prove our next result we need the following results for functions in $\mathcal{S}^*$.

Lemma 2.5 [5, Theorem 3.1]. *Let $g \in \mathcal{S}^*$ be of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then $|b_2b_4 - b_3^2| \le 1$, and the inequality is sharp for the Koebe function $k(z) = z/(1 - z)^2$ and its rotations.*

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Lemma 2.6 [6, Theorem 1]. *Let $g \in \mathcal{S}^*$ be of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then, for any $\lambda \in \mathbb{C}$,*

$$|b_3 - \lambda b_2^2| \leq \max\{1, |3 - 4\lambda|\}.$$

The inequality is sharp for $k(z) = z/(1-z)^2$ and its rotations if $|3 - 4\lambda| \geq 1$, and for $(k(z^2))^{1/2}$ and its rotations if $|3 - 4\lambda| < 1$.

Lemma 2.7 [7, Theorem 2.2]. *Let $g \in \mathcal{S}^*$ be of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then*

$$|\lambda b_n b_m - b_{n+m-1}| \leq \lambda nm - (n+m-1) \quad \text{for } \lambda \geq \frac{2(n+m-1)}{nm},$$

where $n, m = 2, 3, \ldots$. The inequality is sharp for the Koebe function $k(z) = z/(1-z)^2$ and its rotations.

Lemma 2.8. *Let $f \in \mathcal{K}$ be of the form (1.1). Then $|a_2 a_4 - 2a_3^2| \leq 21/2$.*

Proof. Let $f \in \mathcal{K}$ be of the form (1.1). Then there exists a starlike function g of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and a real number $\alpha \in (-\pi/2, \pi/2)$ such that (1.2) holds. This implies that there exists a Carathéodory function $p \in \mathcal{P}$ of the form (1.3) such that

$$e^{i\alpha} \frac{zf'(z)}{g'(z)} = p(z) \cos\alpha + i \sin\alpha.$$

Comparing coefficients,

$$\begin{aligned}
2a_2 &= b_2 + c_1 e^{-i\alpha} \cos\alpha \\
3a_3 &= b_3 + b_2 c_1 e^{-i\alpha} \cos\alpha + c_2 e^{-i\alpha} \cos\alpha \\
4a_4 &= b_4 + b_3 c_1 e^{-i\alpha} \cos\alpha + b_2 c_2 e^{-i\alpha} \cos\alpha + c_3 e^{-i\alpha} \cos\alpha,
\end{aligned}$$

and a simple computation gives

$$\begin{aligned}
72(a_2 a_4 - 2a_3^2) = (9b_2 b_4 - 16b_3^2) &+ (9b_4 - 23b_2 b_3) c_1 e^{-i\alpha} \cos\alpha \\
&+ (9b_3 - 16b_2^2) c_1^2 e^{-2i\alpha} \cos^2\alpha + (9b_2^2 - 32b_3) c_2 e^{-i\alpha} \cos\alpha \\
&+ (9c_3 - 23c_1 c_2 e^{-i\alpha} \cos\alpha) b_2 e^{-i\alpha} \cos\alpha \\
&+ (9c_1 c_3 - 16c_2^2) e^{-2i\alpha} \cos^2\alpha.
\end{aligned}$$

Consequently, by the triangle inequality,

$$\begin{aligned}
72|a_2 a_4 - 2a_3^2| \leq {}& |9b_2 b_4 - 16b_3^2| + |9b_4 - 23b_2 b_3|\,|c_1| + |9b_3 - 16b_2^2|\,|c_1^2| \\
&+ |9b_2^2 - 32b_3|\,|c_2| + |9c_3 - 23c_1 c_2 e^{-i\alpha} \cos\alpha|\,|b_2| + |9c_1 c_3 - 16c_2^2|.
\end{aligned} \tag{2.5}$$

By Lemmas 2.5, 2.6 and 2.7, it easily follows that

$$|9b_2 b_4 - 16b_3^2| \leq 9|b_2 b_4 - b_3^2| + 7|b_3|^2 \leq 9 + 63 = 72, \tag{2.6}$$

$$|9b_4 - 23b_2 b_3| = 9|b_4 - \tfrac{23}{9} b_2 b_3| \leq 9(\tfrac{46}{3} - 4) = 102, \tag{2.7}$$

$$|9b_3 - 16b_2^2| = 9|b_3 - \tfrac{16}{9} b_2^2| \leq 9(\tfrac{64}{9} - 3) = 37, \tag{2.8}$$

$$|9b_2^2 - 32b_3| = 32|b_3 - \tfrac{9}{32} b_2^2| \leq 32(3 - \tfrac{9}{8}) = 60. \tag{2.9}$$

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Again by Lemma 1.2,

$$|9c_3 - 23c_1c_2e^{-i\alpha}\cos\alpha| = 9|c_3 - \mu c_1c_2| \le 18\max\{1, |2\mu - 1|\}$$

where $\mu = (23/9)e^{-i\alpha}\cos\alpha$. Now note that

$$\begin{aligned}|2\mu - 1|^2 &= (\tfrac{23}{9}\cos 2\alpha + \tfrac{14}{9})^2 + (\tfrac{23}{9}\sin 2\alpha)^2 \\ &= (\tfrac{23}{9})^2 + (\tfrac{14}{9})^2 + 2(\tfrac{23}{9})(\tfrac{14}{9})\cos 2\alpha,\end{aligned}$$

and so

$$1 \le |2\mu - 1| \le \tfrac{37}{9}.$$

Therefore,

$$|9c_3 - 23c_1c_2e^{-i\alpha}\cos\alpha| \le 74. \tag{2.10}$$

Again by Lemma 1.2,

$$|9c_1c_3 - 16c_2^2| \le 9|c_1c_3 - c_4| + 9|c_4 - \tfrac{16}{9}c_2^2| \le 18 + 46 = 64. \tag{2.11}$$

By Lemma 1.1, and using inequalities (2.6)–(2.11) in (2.5),

$$|a_2a_4 - 2a_3^2| \le \tfrac{1}{72}(72 + 204 + 148 + 120 + 148 + 64) = \tfrac{21}{2}.$$

This concludes the proof. □

Theorem 2.9. *Let $f \in \mathcal{K}$ be of the form (1.1). Then $|T_3(2)| \le 86$.*

Proof. Let $f \in \mathcal{K}$ be of the form (1.1). Then by Lemma 2.8,

$$\begin{aligned}|T_3(2)| &= |a_2^3 - 2a_2a_3^2 - a_2a_4^2 + 2a_3^2a_4| \\ &\le |a_2|^3 + 2|a_2||a_3^2| + |a_4||a_2a_4 - 2a_3^2| \le 8 + 36 + 42 = 86.\end{aligned}$$

This concludes the proof. □

Remark 2.10. In Theorem 2.3, we have proved that $|T_3(2)| \le 84$ for functions in $\mathcal{S}^*$, and the inequality is sharp for the function f defined by (2.2). It is natural to conjecture that $|T_3(2)| \le 84$ holds for functions in $\mathcal{K}$ and that equality holds for the function f defined by (2.2).

Theorem 2.11. *Let $f \in C$ be of the form (1.1). Then*

(i) $|T_2(n)| \le 2$ *for* $n \ge 2$.
(ii) $|T_3(1)| \le 4$.
(iii) $|T_3(2)| \le 4$.

All inequalities are sharp.

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Proof. Let $f \in \mathcal{C}$ be of the form (1.1). Then there exists a function $p \in \mathcal{P}$ of the form (1.3) such that $f'(z) + zf''(z) = f'(z)p(z)$. Equating coefficients,

$$2a_2 = c_1, \quad 3a_3 = \tfrac{1}{2}(c_2 + c_1^2) \quad \text{and} \quad 4a_4 = \tfrac{1}{6}c_1^3 + \tfrac{1}{2}c_1c_2 + \tfrac{1}{3}c_3. \tag{2.12}$$

Clearly,

$$|T_2(n)| = |a_n^2 - a_{n+1}^2| \le |a_n^2| + |a_{n+1}^2| \le 1 + 1 = 2. \tag{2.13}$$

Equality holds in (2.13) for the function f defined by

$$f(z) := \frac{z}{1 - iz} = z + iz^2 - z^3 - iz^4 + z^5 + \cdots. \tag{2.14}$$

Again if $f \in \mathcal{C}$ is of the form (1.1) then from Lemma 1.2 and (2.12),

$$\begin{aligned}|T_3(1)| &= |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \\ &\le 1 + 2|a_2^2| + |a_3|\,|a_3 - 2a_2^2| \le 1 + 2 + \tfrac{1}{6}|c_2 - 2c_1^2| \le 4.\end{aligned} \tag{2.15}$$

It is easy to see that equality holds in (2.15) for the function f defined by (2.14).

For the third part, note that $T_3(2) = (a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)$. If $f \in \mathcal{C}$ then clearly $|a_2 - a_4| \le |a_2| + |a_4| \le 2$. Thus, we need to maximise $|a_2^2 - 2a_3^2 + a_2a_4|$ for functions in $\mathcal{C}$. Writing a_2, a_3 and a_4 in terms of c_1, c_2 and c_3 with the help of (2.12),

$$\begin{aligned}|a_2^2 - 2a_3^2 + a_2a_4| &= \tfrac{1}{144}|5c_1^4 - 36c_1^2 + 7c_1^2c_2 + 8c_2^2 - 6c_1c_3| \\ &\le \tfrac{1}{144}(5|c_1|^4 + 36|c_1|^2 + 8|c_2|^2 + 6|c_1|\,|c_3 - \tfrac{7}{6}c_1c_2|).\end{aligned}$$

From Lemmas 1.1 and 1.2, it easily follows that

$$|a_2^2 - 2a_3^2 + a_2a_4| \le \tfrac{1}{144}(80 + 144 + 32 + 32) = 2.$$

Therefore, $|T_3(2)| \le 4$, and the inequality is sharp for the function f defined by (2.14). □

Theorem 2.12. *Let $f \in \mathcal{R}$ be of the form (1.1). Then*

(i) $|T_2(n)| \le 4/n^2 + 4/(n+1)^2$ *for* $n \ge 2$.
(ii) $|T_3(1)| \le \frac{35}{9}$.
(iii) $|T_3(2)| \le \frac{7}{3}$.

The inequalities in (i) and (ii) are sharp.

Proof. Let $f \in \mathcal{R}$ be of the form (1.1). Then there exists a function $p \in \mathcal{P}$ of the form (1.3) such that $f'(z) = p(z)$. Equating coefficients gives $na_n = c_{n-1}$ and so

$$|a_n| = \frac{1}{n}|c_{n-1}| \le \frac{2}{n}, \quad n \ge 2.$$

The inequality is sharp for the function f defined by $f'(z) = (1 + z)/(1 - z)$ and its rotations. Thus,

$$|T_2(n)| = |a_n^2 - a_{n+1}^2| \le |a_n^2| + |a_{n+1}^2| \le \frac{4}{n^2} + \frac{4}{(n+1)^2}. \tag{2.16}$$

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Equality holds in (2.16) for the function f defined by

$$f'(z) := \frac{1+iz}{1-iz}. \tag{2.17}$$

Next, if $f \in \mathcal{R}$ is of the form (1.1), then

$$\begin{aligned}|T_3(1)| &= |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \le 1 + 2|a_2^2| + |a_3|\,|a_3 - 2a_2^2| \\ &\le 1 + 2 + \tfrac{2}{3}|\tfrac{1}{3}c_2 - \tfrac{1}{2}c_1^2| \le 3 + \tfrac{2}{9}|c_2 - \tfrac{3}{2}c_1^2| \le 3 + \tfrac{8}{9} = \tfrac{35}{9}.\end{aligned} \tag{2.18}$$

It is easy to see that equality in (2.18) holds for the function f defined by (2.17).

Again, if $f \in \mathcal{R}$ is of the form (1.1) then

$$\begin{aligned}|T_3(2)| &= |a_2^3 - 2a_2a_3^2 - a_2a_4^2 + 2a_3^2a_4| \le |a_2|^3 + 2|a_2|\,|a_3^2| + |a_4|\,|a_2a_4 - 2a_3^2| \\ &\le 1 + \tfrac{8}{9} + \tfrac{1}{2}|a_2a_4 - 2a_3^2| \le \tfrac{17}{9} + \tfrac{1}{2}|a_2a_4 - 2a_3^2|.\end{aligned}$$

Thus, we need to find the maximum value of $|a_2a_4 - 2a_3^2|$ for functions in $\mathcal{R}$. By (2.11), it easily follows that

$$|a_2a_4 - 2a_3^2| = \tfrac{1}{72}|9c_1c_3 - 16c_2^2| \le \tfrac{64}{72} = \tfrac{8}{9}.$$

Therefore,

$$|T_3(2)| \le \tfrac{17}{9} + \tfrac{4}{9} = \tfrac{7}{3}.$$

This concludes the proof. □

REMARK 2.13. Theorem 2.12 shows that for $f \in \mathcal{R}$, the sharp inequalities $|T_2(2)| \le 13/9$ and $|T_2(3)| \le 17/36$ hold. In [8], it was claimed that $|T_2(2)| \le 5/9$, $|T_2(3)| \le 4/9$, $|T_3(1)| \le 13/9$ and $|T_3(2)| \le 4/9$ hold for functions in $\mathcal{R}$ and these estimates are sharp. For the function f defined by (2.17), a simple computation gives $|T_2(2)| = 13/9$, $|T_2(3)| = 17/36$, $|T_3(1)| = 35/9$ and $|T_3(2)| \le 25/12$, showing that these estimates are not correct. As explained above, the authors assumed that $c_1 > 0$, which is not justified, since the functional $|T_q(n)|$ $(n \ge 1, q \ge 2)$ is not rotationally invariant.

If $f \in \mathcal{T}$ is given by (1.1), then the coefficients of f can be expressed by

$$a_n = \int_{-1}^{1} \frac{\sin(n \arccos t)}{\sin(\arccos t)}\, d\mu(t) = \int_{-1}^{1} U_{n-1}(t)\, d\mu(t), \quad n \ge 1,$$

where $U_n(t)$ are Chebyshev polynomials of degree n of the second kind.

Let $A_{n,m}$ denote the region of variability of the point (a_n, a_m), where a_n and a_m are coefficients of a given function $f \in \mathcal{T}$ with the series expansion (1.1), that is, $A_{n,m} := \{(a_n(f), a_m(f)) : f \in \mathcal{T}\}$. Thus, $A_{n,m}$ is the closed convex hull of the curve

$$\gamma_{n,m} : [-1, 1] \ni t \to (U_{n-1}(t), U_{m-1}(t)).$$

By the Carathéodory theorem we conclude that it is sufficient to discuss only functions

$$F(z, \alpha, t_1, t_2) := \alpha k(z, t_1) + (1 - \alpha)k(z, t_2), \tag{2.19}$$

where $0 \le \alpha \le 1$ and $-1 \le t_1 \le t_2 \le 1$.

Let X be a compact Hausdorff space and $J_\mu = \int_X J(t)\, d\mu(t)$. Szapiel [10, Theorem 1.49, page 37] proved the following theorem.

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Theorem 2.14. *Let $J : [\alpha,\beta] \to \mathbb{R}^n$ be continuous. Suppose that there exists a positive integer k, such that for each nonzero $\overrightarrow{p}$ in $\mathbb{R}^n$ the number of solutions of any equation $\langle \overrightarrow{J(t)}, \overrightarrow{p} \rangle$ = constant, $\alpha \le t \le \beta$, is not greater than k. Then, for every $\mu \in P_{[\alpha,\beta]}$ such that J_μ belongs to the boundary of the convex hull of $J([\alpha,\beta])$, the following statements are true:*

(1) *if $k = 2m$, then*

   (a) $|\mathrm{supp}(\mu)| \le m$, *or*
   (b) $|\mathrm{supp}(\mu)| = m + 1$ *and* $\{\alpha,\beta\} \subset \mathrm{supp}(\mu)$;

(2) *if $k = 2m + 1$, then*

   (a) $|\mathrm{supp}(\mu)| \le m$, *or*
   (b) $|\mathrm{supp}(\mu)| = m + 1$ *and one of the points α and β belongs to* $\mathrm{supp}(\mu)$.

In the above, the symbol $\langle \overrightarrow{u}, \overrightarrow{v} \rangle$ means the scalar product of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, whereas the symbols P_X and $|\mathrm{supp}(\mu)|$ describe the set of probability measures on X, and the cardinality of the support of μ, respectively.

Putting $J(t) = (U_1(t), U_2(t))$, $t \in [-1, 1]$ and $\overrightarrow{p} = (p_1, p_2)$, we can see that any equation of the form $p_1U_1(t) + p_2U_2(t) =$ constant, $t \in [-1, 1]$, has at most two solutions. According to Theorem 2.14, the boundary of the convex hull of $J([-1, 1])$ is determined by atomic measures μ whose support consists of at most two points. Thus, we have the following result.

Lemma 2.15. *The boundary of $A_{2,3}$ consists of points (a_2, a_3) that correspond to the functions $F(z, 1, t, 0) = k(z, t)$, or $F(z, \alpha, 1, -1)$ with $0 \le \alpha \le 1$ and $-1 \le t \le 1$, where $F(z, \alpha, t_1, t_2)$ is defined by (2.19).*

In a similar way, one can also obtain the following result.

Lemma 2.16. *The boundary of $A_{3,4}$ consists of points (a_3, a_4) that correspond to the functions $F(z, \alpha, t, -1)$, or $F(z, \alpha, t, 1)$ with $0 \le \alpha \le 1$ and $-1 \le t \le 1$, where $F(z, \alpha, t_1, t_2)$ is defined by (2.19).*

Before we proceed further, we give some examples of typically real functions.

Example 2.17. For each $t \in [-1, 1]$, the function $k(z, t) = z/(1 - 2tz + z^2)$ is a typically real function. For $k(z, 1) = z/(1 - z)^2$, we have $T_2(n) = n^2 - (n + 1)^2 = -(2n + 1)$ and $T_3(n) = a_n^3 - 2a_{n+1}^2 a_n - a_{n+2}^2 a_n + 2a_{n+1}^2 a_{n+2} = 4(n + 1)$.

Example 2.18. The function $f(z) = -\log(1 - z) = z + \sum_{n=2}^{\infty}(1/n)z^n$ is typically real. For this function,

$$T_2(n) = \frac{1}{n^2} - \frac{1}{(n+1)^2} \quad \text{and} \quad T_3(n) = \frac{4(n^2 + 3n + 1)}{n^3(n+1)^2(n+2)^2}.$$

Lemma 2.19. *If $f \in \mathcal{T}$ then $T_2(n)$ attains its extreme values on the boundary of $A_{n,n+1}$.*

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Proof. Let $\phi(x, y) = x^2 - y^2$, where $x = a_n$ and $y = a_{n+1}$. The only critical point of ϕ is $(0, 0)$ and $\phi(0, 0) = 0$. Since ϕ may be positive as well as negative for $(x, y) \in A_{n,n+1}$ (see Examples 2.17 and 2.18), the extreme values of ϕ are attained on the boundary of $A_{n,n+1}$. □

In a similar way, we can prove the following result.

Lemma 2.20. *If $f \in \mathcal{T}$ then $T_3(1)$ attains its extreme values on the boundary of $A_{2,3}$.*

Since all coefficients of $f \in \mathcal{T}$ are real, we look for the lower and the upper bounds of $T_q(n)$ instead of the bound of $|T_q(n)|$. The proof of the following theorem is obvious.

Theorem 2.21. *For $f \in \mathcal{T}$ of the form (1.1), we have $-(n + 1)^2 \le T_2(n) \le n^2$. Moreover,*

(i) *If n is odd then $\max\{T_2(n) : f \in \mathcal{T}\} = n^2$ and equality is attained for the function $F(z, 1/2, 1, -1)$.*

(ii) *If n is even then $\min\{T_2(n) : f \in \mathcal{T}\} = -(n + 1)^2$ and equality is attained for the function $F(z, 1/2, 1, -1)$.*

Theorem 2.22. *For $f \in \mathcal{T}$, $\max\{T_2(2) : f \in \mathcal{T}\} = 5/4$.*

Proof. By Lemma 2.15, it is enough to consider the functions $F(z, 1, t, 0) = k(z, t)$ and $F(z, \alpha, 1, -1)$ with $0 \le \alpha \le 1$ and $-1 \le t \le 1$.

Case 1. For $F(z, 1, t, 0) = k(z, t) = z + 2tz^2 + (4t^2 - 1)z^3 + (8t^3 - 4t)z^4 + \cdots$,

$$a_2^2 - a_3^2 = -16t^4 + 12t^2 - 1 \le \tfrac{5}{4}.$$

Case 2. For $F(z, \alpha, 1, -1) = z + (4\alpha - 2)z^2 + 3z^3 + (8\alpha - 4)z^4 + \cdots$,

$$a_2^2 - a_3^2 = (2 - 4\alpha)^2 - 9 \le -5.$$

The conclusion follows from Cases 1 and 2, with the maximum attained for $F(z, 1, t, 0) = k(z, t)$ with $t = \sqrt{3}/(2\sqrt{2})$. □

Corollary 2.23. *For $f \in \mathcal{T}$, the sharp inequality $-9 \le T_2(2) \le 5/4$ holds.*

Theorem 2.24. *For $f \in \mathcal{T}$, we have $\min\{T_2(3) : f \in \mathcal{T}\} = -7$.*

Proof. By Lemma 2.16, it is enough to consider the functions $F(z, \alpha, t, -1)$ and $F(z, \alpha, t, -1)$ with $0 \le \alpha \le 1$ and $-1 \le t \le 1$.

Case 1. For the function

$$F(z, \alpha, t, -1) = z + 2(\alpha + \alpha t - 1)z^2 + (4\alpha t^2 - 4\alpha + 3)z^3 + (4\alpha + 8\alpha t^3 - 4\alpha t - 4)z^4 + \cdots,$$

we have $T_2(3) = a_3^2 - a_4^2 = (4\alpha t^2 - 4\alpha + 3)^2 - (4\alpha + 8\alpha t^3 - 4\alpha t - 4)^2 := \phi(\alpha, t)$. By elementary calculus, one can verify that

$$\min_{0 \le \alpha \le 1, -1 \le t \le 1} \phi(\alpha, t) = \phi(0, 0) = -7.$$

Case 2. For the function

$$F(z,\alpha,t,1) = z + 2(1-\alpha+\alpha t)z^2 + (3-4\alpha+4\alpha t^2)z^3 + (4-4\alpha-4\alpha t+8\alpha t^3)z^4 + \cdots,$$

we have $a_2^2 - a_3^2 = \phi(\alpha,-t)$, and so

$$\min_{0\le\alpha\le 1,-1\le t\le 1} \phi(\alpha,-t) = \phi(0,0) = -7.$$

The conclusion follows from Cases 1 and 2 and the maximum is attained for the function $F(z,0,0,1)$ or $F(z,0,0,-1)$. □

Corollary 2.25. *For $f \in \mathcal{T}$, the sharp inequality $-7 \le T_2(3) \le 9$ holds.*

Theorem 2.26. *For $f \in \mathcal{T}$, we have $\max\{T_3(1) : f \in \mathcal{T}\} = 8$ and $\min\{T_3(1) : f \in \mathcal{T}\} = -8$.*

Proof. By Lemma 2.15, it is enough to consider the functions $F(z,1,t,0) = k(z,t)$ and $F(z,\alpha,1,-1)$ with $0 \le \alpha \le 1$ and $-1 \le t \le 1$.

Case 1. For $F(z,1,t,0) = k(z,t) = z + 2tz^2 + (4t^2-1)z^3 + (8t^3-4t)z^4 + \cdots$, we have $T_3(1) = 1 - 2a_2^2 + 2a_2^2a_3 - a_3^2 = 8t^2(2t^2-1) := \phi_1(t)$ and it is easy to verify that

$$\max_{-1\le t\le 1} \phi_1(t) = \phi_1(-1) = 8 \quad \text{and} \quad \min_{-1\le t\le 1} \phi_1(t) = \phi_1(-1/2) = -1.$$

Case 2. For the function $F(z,\alpha,1,-1) = z + (4\alpha-2)z^2 + 3z^3 + (8\alpha-4)z^4 + \cdots$, we have $T_3(1) = 8(8\alpha^2 - 8\alpha + 1) := \psi_1(\alpha)$ and it is again easy to verify that

$$\max_{0\le\alpha\le 1} \psi_1(\alpha) = \psi_1(0) = 8 \quad \text{and} \quad \min_{0\le\alpha\le 1} \psi_1(\alpha) = \psi_1(1/2) = -8.$$

The conclusion follows from Cases 1 and 2; the maximum is attained for the function $F(z,1,-1,0) = k(z,-1)$ and the minimum is attained for the function $F(z,1/2,1,-1)$. □

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