The Apolar Locus of Two Tetrads of Points on a Conic.

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1. In Vol. XXXV. (Session 1916–17), Part I., of the *Proceedings of the Edinburgh Mathematical Society*, I discussed in considerable detail the properties of the Apolar Locus of two tetrads of points. I shewed there that, subject to certain defined conditions, a unique quartic curve would be obtained, which would be the Apolar Locus of the two given tetrads. I mentioned, however, in § 7 of the paper, that in the case when the two tetrads lie on the same conic, the above-mentioned conditions are not independent, and that, in fact, not a unique quartic but a pencil of quartics is obtained.

The discussion of the case when the two tetrads lie on a conic will be given in this paper. In addition to the interest afforded by the investigation in relation to "Apolarity," many important results are obtained with reference to the geometrical interpretation of the invariants and covariants of two binary tetrads on a conic.

2. We choose the conic on which the two tetrads lie as our Norm-Conic $x = t^2$, $y = 1$, $z = 2t$. Adopting the notation of the paper above referred to, we take our ϕ-tetrad to be $t^4 + 6kt^2 + 1 = 0$, and our ψ-tetrad as $a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4 = 0$. Hence the tangential equations of these two tetrads are given by

$$\phi_t^4 \equiv l^4 + m^4 + 16n^4 + 24km^2 n^2 + 24kn^2 l^2 + 2(18k^2 + 1) l^2 m^2 - 12kl^3 m - 12km^3 l - 16lmn^2 = 0. \quad \text{......(1)}$$

$$\begin{aligned}\psi_t^4 \equiv\ & a_4^2 l^4 + a_0^2 m^4 + 16a_0 a_4 n^4 + 24a_0 a_2 m^2 n^2 + 24a_2 a_4 n^2 l^2 \\ & + 2(18a_2^2 + a_0 a_4 - 16a_1 a_3) l^2 m^2 + 4(a_3^2 - 3a_2 a_4) l^3 m \\ & - 8a_3 a_4 l^3 n - 8a_0 a_1 m^3 n + 4(4a_1^2 - 3a_0 a_2) m^3 l \\ & - 32a_1 a_4 n^3 l - 32a_0 a_3 n^3 m + 24(a_1 a_4 - 2a_2 a_3) l^2 mn \\ & + 24(a_0 a_3 - 2a_1 a_2) lm^2 n + 16(4a_1 a_3 - a_0 a_4) lmn^2 = 0. \quad \text{......(2)}\end{aligned}$$

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Let the conics defining the two tetrads on the Norm-Conic $z^2 - 4xy = 0$ be respectively :

$$S \equiv z^2 - 4xy = 0. \quad \ldots\ldots (3)$$

$$S_1 \equiv x^2 + y^2 + kz^2 + 2kxy = 0. \quad \ldots\ldots (4)$$

$$S_2 \equiv a_0 x^2 + a_4 y^2 + a_2 z^2 + 2a_3 yz + 2a_1 zx + 2a_2 xy = 0. \quad \ldots\ldots (5)$$

Then the Principal Conics belonging to the ϕ- and ψ-tetrads will be respectively :

The ϕ-Principal Conics

$$96 I_2 S_1^2 - 96 H_2 SS_1 + (3A^2 - 8H + 4 I_1 I_2) S^2 = 0. \quad \ldots\ldots (6)$$

The ψ-Principal Conics

$$96 I_1 S_2^2 - 96 H_1 SS_2 + (3A^2 - 8H + 4 I_1 I_2) S^2 = 0. \quad \ldots\ldots (7)$$

Next, after performing the necessary calculations in accordance with the definitions laid down in the paper previously referred to, we find that the Generating-Conics are as follows :

The ϕ-Generating Conic

$$(AH_1 - 6 I_2 J_1) S = (A^2 - 4 I_1 I_2) S_1 . \quad \ldots\ldots (8)$$

The ψ-Generating Conic

$$(AH_2 - 6 I_1 J_2) S = (A^2 - 4 I_1 I_2) S_2 . \quad \ldots\ldots (9)$$

Furthermore, the equation to the Apolar Locus of the two tetrads is easily calculated to be

$$\begin{aligned} S \Big[& \{(\tfrac{5}{2}a_0 k - \tfrac{3}{2}a_4 k - a_2) x + (\tfrac{1}{2}A - a_4 I_1) y + (2ka_1 - 6k^2 a_3) z\} \\ & \qquad\qquad (a_0 x + a_2 y + a_1 z) \\ & + \{(\tfrac{1}{2}A - a_0 I_1) x + (\tfrac{5}{2}a_4 k - \tfrac{3}{2}a_0 k - a_2) y + (2ka_3 - 6k^2 a_1) z\} \\ & \qquad\qquad (a_2 x + a_4 y + a_3 z) \\ & + \{(4ka_3 + 4a_1) x + (4ka_1 + 4a_3) y + (2 I_1 a_2 - kA) z\} \\ & \qquad\qquad (a_1 x + a_3 y + a_2 z) \Big] \\ & - S_2 (2 I_1 S_2 - AS_1) \\ & + 2 (I_1 S_2^2 - H_1 SS_2 + NS^2) = 0, \quad \ldots\ldots (10) \end{aligned}$$

where we have used above the following convenient invariant notation :

$$I_1 \equiv 1 + 3k^2. \quad \ldots\ldots\ldots\ldots \quad (11)$$

$$I_2 \equiv a_0 a_4 - 4a_1 a_3 + 3a_2^2. \quad \ldots\ldots\ldots\ldots \quad (12)$$

$$A \equiv a_0 + 6ka_2 + a_4. \quad \ldots\ldots\ldots\ldots \quad (13)$$

$$H \equiv 6k(a_0 a_2 + a_2 a_4 - a_1^2 - a_3^2) + (1 - 3k^2)(a_0 a_4 + 2a_1 a_3 - 3a_2^2). \quad \ldots \quad (14)$$

$$H_1 \equiv k(a_0 + a_4) + (1 - 3k^2) a_2. \quad \ldots\ldots\ldots\ldots \quad (15)$$

$$H_2 \equiv (a_0 a_2 + a_2 a_4 - a_1^2 - a_3^2) + k(a_0 a_4 + 2a_1 a_3 - 3a_2^2). \quad \ldots\ldots\ldots\ldots \quad (16)$$

$$J_1 \equiv k - k^3. \quad \ldots\ldots\ldots\ldots \quad (17)$$

$$J_2 \equiv a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 - a_2^3. \quad \ldots\ldots\ldots\ldots \quad (18)$$

It will further be convenient to write

$$96N \equiv 3A^2 - 8H + 4I_1 I_2. \quad \ldots\ldots\ldots\ldots \quad (19)$$

3. Now the general equation to all quartics passing through the four common points of the conics S and S' and the vertices of their harmonic triangle is easily seen to be of the form

$$L(SS_x' - S'S_x) + M(SS_y' - S'S_y) + N(SS_z' - S'S_z) = 0,$$

where L, M, N are general linear expressions in x, y, z, inasmuch as a quartic can be made to satisfy 14 conditions, and we have above at our disposal 7 effective constants for L, M, N, together with the fact that the quartic passes through 7 given points, namely, the intersections of the two given conics and the vertices of their harmonic triangle, since the three cubics $SS_x' - S'S_x = 0$, etc., each pass through the above 7 points.

Let therefore the Auxiliary Quartic of the ψ-tetrad be of the form

$$(z^2 - 4xy)\{(l_1 x + m_1 y + n_1 z)(a_0 x + a_2 y + a_1 z) + (l_2 x + m_2 y + n_2 z)$$
$$(a_2 x + a_4 y + a_3 z) + (l_3 x + m_3 y + n_3 z)(a_1 x + a_3 y + a_2 z)\}$$
$$+ (a_0 x^2 + a_4 y^2 + a_2 z^2 + 2a_3 yz + 2a_1 zx + 2a_2 xy)\{2(l_1 x + m_1 y + n_1 z) y$$
$$+ 2(l_2 x + m_2 y + n_2 z) x - (l_3 x + m_3 y + n_3 z) z\} = 0, \quad (20)$$

where l_1, m_1, etc., have to be determined from the assigned conditions.

If we add to (20) the conics (7), viz., $2\rho(I_1 S_2^2 - H_1 SS_2 + NS^2)$, we get, if ρ be suitably determined, the Apolar Locus of the

ϕ- and ψ-tetrads. If, further, we add the conics (6) to the Apolar Locus, viz., $2\sigma(I_2 S_1^2 - H_2 SS_1 + NS^2)$, we shall obtain the Auxiliary Quartic of ϕ_t^4, if σ be suitably determined. Let us consider therefore the equation

$$(z^2 - 4xy)\{(l_1 x + m_1 y + n_1 z)(a_0 x + a_2 y + a_1 z) + (l_2 x + m_2 y + n_2 z)$$
$$(a_2 x + a_4 y + a_3 z) + (l_3 x + m_3 y + n_3 z)(a_1 x + a_3 y + a_2 z)\}$$
$$+ (a_0 x^2 + a_4 y^2 + a_2 z^2 + 2a_3 yz + 2a_1 zx + 2a_2 zy)\{2(l_1 x + m_1 y + n_1 z) y$$
$$+ 2(l_2 x + m_2 y + n_2 z) x - (l_3 x + m_3 y + n_3 z) z\}$$
$$+ 2\rho(I_1 S_2^2 - H_1 SS_2 + NS^2) + 2\sigma(I_2 S_1^2 - H_2 SS_1 + NS^2) = 0. \quad \ldots\ldots (21)$$

We shall now evaluate the various constants in (21). $z^2 = 4xy$ (regarded as an envelope) is apolar to

$$2(l_1 x + m_1 y + n_1 z) y + 2(l_2 x + m_2 y + n_2 z) x - (l_3 x + m_3 y + n_3 z) z = 0$$

in (20) if $-n_3 = l_1 + m_2$. .... (22)

Also the Apolar Locus will cut the norm-conic S in the ϕ-tetrad if

$$2\{2(l_1 t^2 + m_1 + 2n_1 t) + 2(l_2 t^2 + m_2 + 2n_2 t) t^2 - 2(l_3 t^2 + m_3 + 2n_3 t) t\}$$
$$+ 4\rho I_1 (a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4) \equiv 4\tau(t^4 + 6kt^2 + 1).$$

Hence, equating the coefficients of the various powers of t, we obtain

$l_2 + \rho I_1 a_0 = \tau.$ ....(23)

$2n_2 - l_3 + 4\rho I_1 a_1 = 0.$ ....(24)

$l_1 + m_2 - 2n_3 + 6\rho I_1 a_2 = 6k\tau.$ ....(25)

$2n_1 - m_3 + 4\rho I_1 a_3 = 0.$ ....(26)

$m_1 + \rho I_1 a_4 = \tau.$ ....(27)

Hence $\quad l_2 = \tau - \rho I_1 a_0.$ ....(28)

$l_3 = 2n_2 + 4\rho I_1 a_1.$ ....(29)

$n_3 = 2\rho I_1 a_2 - 2k\tau.$ ....(30)

$m_3 = 2n_1 + 4\rho I_1 a_3.$ ....(31)

$m_1 = \tau - \rho I_1 a_4.$ ....(32)

$l_1 + m_2 = 2k\tau - 2\rho I_1 a_2.$ ....(33)

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Substituting in (21), we obtain after some obvious reductions:

$$S[\{l_1 x + (\tau - \rho I_1 a_4) y + n_1 z\}(a_0 x + a_2 y + a_1 z) + \{(\tau - \rho I_1 a_0) x + m_2 y + n_2 z\}(a_2 x + a_4 y + a_3 z) + \{(2n_2 + 4\rho I_1 a_1) x + (2n_1 + 4\rho I_1 a_3) y + (2\rho I_1 a_2 - 2k\tau) z\}(a_1 x + a_3 y + a_2 z)] + 2\tau S_1 S_2 + 2\rho S(NS - H_1 S_2) + 2\sigma(I_2 S_1^2 - H_2 SS_1 + NS^2) = 0. \quad (34)$$

We have still got to express the condition that (20) has the assigned Generating-Conic. We obtain the Generating-Conic of (20) by expressing the condition that $S_2 + \lambda S = 0$, regarded as an envelope, is apolar to

$$2(l_1 x + m_1 y + n_1 z) y + 2(l_2 x + m_2 y + n_2 z) x - (l_3 x + m_3 y + n_3 z) z = 0,$$

i.e., to

$$\tau S_1 - \rho I_1 S_2 = 0 \quad \text{.........................} (35)$$

in virtue of (28), (29), ... , (33).

We obtain as the required ψ-Generating-Conic:

$$(\tau A - 2\rho I_1 I_2) S_2 + (3\rho I_1 J_2 - \tau H_2) S = 0. \quad \text{..............} (36)$$

Hence, comparing (9) and (36), we obtain

$$\rho : \tau = 2 : A \text{ or } \tau = \tfrac{1}{2} A\rho. \quad \text{....................} (37)$$

Substituting from (37) in (34), we get as the Auxiliary Quartic of the ϕ-tetrad:

$$S[\{l_1 x + \rho(\tfrac{1}{2}A - I_1 a_4) y + n_1 z\}(a_0 x + a_2 y + a_1 z) + \{\rho(\tfrac{1}{2}A - I_1 a_0) x + m_2 y + n_2 z\}(a_2 x + a_4 y + a_3 z) + \{(2n_2 + 4\rho I_1 a_1) x + (2n_1 + 4\rho I_1 a_3) y + \rho(2I_1 a_2 - kA) z\}(a_1 x + a_3 y + a_2 z)] + A\rho S_1 S_2 + 2\rho S(NS - H_1 S_2) + 2\sigma(I_2 S_1^2 - H_2 SS_1 + NS^2) = 0. \quad (38)$$

If we write (38) in the form $SC + 2S_1(\rho A S_2 + 2\sigma I_2 S_1) = 0$, we see that it is in standard form regarded as the Auxiliary Quartic of $\phi_t^4 = 0$, since S, regarded as an envelope, is apolar to both S_1 and S_2. To get the Generating-Conic of (38), we have therefore only to express the condition that $S_1 + \mu S = 0$, regarded as an envelope, is apolar to $\rho A S_2 + 2\sigma I_2 S_1 = 0$. We obtain as the Generating-Conic of (38),

$$(\rho A H_1 + 6\sigma I_2 J_1) S - (\rho A^2 + 4\sigma I_1 I_2) S_1 = 0. \quad \text{.........} (39)$$

Comparing (8) and (39), we obtain $\sigma = -\rho$.

Hence the Auxiliary Quartic (38) becomes

$$\begin{aligned}S[\{l_1 x+\rho(\tfrac{1}{2}A-I_1 a_4)y+n_1 z\}(a_0 x+a_2 y+a_1 z)\\ +\{\rho(\tfrac{1}{2}A-I_1 a_0)x+m_2 y+n_2 z\}(a_2 x+a_4 y+a_3 z)\\ +\{(2n_2+4\rho I_1 a_1)x+(2n_1+4\rho I_1 a_3)y+\rho(2I_1 a_2-kA)z\}\\ (a_1 x+a_3 y+a_2 z)]\\ +\rho(AS_1 S_2-2H_1 SS_2+2H_2 SS_1-2I_2 S_1^2)=0. \quad\ldots\ldots\ldots\ldots(40)\end{aligned}$$

We must next express the conditions that (40) shall pass through the points (0, 0, 1) ; (1, 1, 0) ; (1, -1, 0), the vertices of the harmonic triangle of ϕ_t^4. We obtain, after some rather troublesome reduction,

$$l_1+m_2=\rho\{k(a_0+a_4)-2a_2\} \quad\ldots\ldots\ldots\ldots(41)$$

$$n_1 a_1+n_2 a_3=\rho\{2k(a_1^2+a_3^2)-12k^2 a_1 a_3\} \quad\ldots\ldots(42)$$

$$\begin{aligned}a_0 l_1+a_4 m_2+2(n_1 a_3+n_2 a_1)=\rho\{\tfrac{5}{2}k(a_0^2+a_4^2)-3a_0 a_4 k-a_2(a_0+a_4)\\ -12k^2(a_1^2+a_3^2)+8ka_1 a_3\}. \quad (43)\end{aligned}$$

We note that (41) is the same as (33), whence, where four independent equations were to be expected, only three transpire.

Putting

$$a_0 l_1+a_4 m_2=\rho\{\tfrac{5}{2}k(a_0^2+a_4^2)-3a_0 a_4 k-a_2(a_0+a_4)\}+2\mu(a_0-a_4), \quad (44)$$

and solving the set of equations just given, we obtain

$$l_1=\rho\{\tfrac{5}{2}a_0 k-\tfrac{3}{2}a_4 k-a_2\}+2\mu$$

$$m_2=\rho\{-\tfrac{3}{2}a_0 k+\tfrac{5}{2}a_4 k-a_2\}-2\mu$$

$$n_1=\rho\{2ka_1-6k^2 a_3\}+\mu\frac{a_3(a_0-a_4)}{a_1^2-a_3^2}$$

$$n_2=\rho\{2ka_3-6k^2 a_1\}+\mu\frac{a_1(a_0-a_4)}{a_3^2-a_1^2}.$$

Substituting in (40) and comparing with (10), we obtain the following pencil of Quartics that satisfy the assigned conditions (μ being a variable parameter) :—

$$F+2\mu SS'=0,$$

where F is the Apolar Locus of the two given tetrads and

$$\begin{aligned}S'\equiv x(a_0 x+a_2 y+a_1 z)-y(a_2 x+a_4 y+a_3 z)\\ +\frac{a_0-a_4}{a_1^2-a_3^2}(a_3 y-a_1 x)(a_1 x+a_3 y+a_2 z)=0,\end{aligned}$$

i.e. S' is the conic through the two harmonic triangles of the given tetrads.

Comparing this with the previous paper above referred to, we have the following result:—

If we seek to find a quartic curve, which passes through two tetrads of points, viz., $\phi_l^4 = 0$ *and* $\psi_l^4 = 0$, *and is such that*

(i) *the* ϕ*-Principal Conics are the Self-Conjugate Conics of* ϕ *with respect to* ψ_l^4;

(ii) *the* ψ*-Principal Conics are the Self-Conjugate Conics of* ϕ *with respect to* ϕ_l^4;

(iii) *the* ϕ*-Generating Conic is the Polo-Reciprocal Conic of* ϕ *with respect to* ψ_l^4;

(iv) *the* ψ*-Generating-Conic is the Polo-Reciprocal Conic of* ψ *with respect to* ϕ_l^4;

then, subject to these conditions, one and only one quartic curve can in general be found to satisfy these conditions; but if the two tetrads ϕ_l^4 *and* ψ_l^4 *lie on one and the same conic, a pencil of quartic curves can be found to satisfy these conditions, two members of which pencil are the Apolar Locus of* ϕ_l^4 *and* ψ_l^4 *and the Conic-Pair consisting of the conic on which lie the* ϕ*- and* ψ*-tetrads, together with the conic through their harmonic triangles.*