ON ENTIRE FUNCTIONS WITH GAP POWER SERIES

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(Received 23 June, 1969; revised 15 June, 1970)

1. Introduction. In this note we consider transcendental entire functions

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

whose power series contain gaps, i.e.

$$a_n=0 \qquad (n \notin \Lambda), \tag{2}$$

where $\Lambda=\{\lambda_k\}$ is a suitable set of positive integers. We denote the set of all such functions $f(z)$ by $E(\Lambda)$. As usual $M(r)=M(r,f)$ denotes the maximum modulus of $f(z)$ on the circle $|z|=r$. The order ρ and the lower order λ of $f(z)$ are defined by

$$\rho=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r},$$

$$\lambda=\liminf_{r\to\infty}\frac{\log\log M(r)}{\log r},$$

respectively.

The following theorem is due to Macintyre [4].

THEOREM A. *Suppose that* $f(z)\in E(\Lambda)$, *where*

$$\sum_{k=1}^{\infty}\lambda_k^{-1}<\infty. \tag{3}$$

Then $f(z)$ *is unbounded on* $z>0$.

Edrei [2] has shown that, if the order of $f(z)$ is taken into account, then the gap condition (3) may be relaxed. He proves

THEOREM B. *Suppose that* $f(z)\in E(\Lambda)$ *and is of finite order* ρ, *and that*

$$\liminf_{s\to\infty}\frac{1}{\log s}\sum_{\lambda_n\leq s}\lambda_n^{-1}<\frac{1}{2\rho}. \tag{4}$$

Then $f(z)$ *is unbounded on* $z>0$.

From both theorems we may draw the further conclusion that $f(z)$ has no finite radial asymptotic values. Both Macintyre and Edrei use an idea of Pólya [6] to show that, even for this weaker conclusion, their gap conditions are best possible. More precisely, Macintyre shows that, if Λ is any set of positive integers for which (3) does not hold, then there exists an

https://doi.org/10.1017/S0017089500001191 Published online by Cambridge University Press

$f(z) \in E(\Lambda)$ such that $f(x) \to 0$ as $x \to +\infty$. Similarly, Edrei shows that, if Λ is any set of positive integers for which (4) does not hold, then there exists an $f(z) \in E(\Lambda)$ of finite order ρ such that $f(x) \to 0$ as $x \to +\infty$.

In this note we prove a result which contains those of Macintyre and Edrei quoted above. We define

$$\phi(s) = \phi(s, f) = \log M(e^s). \tag{5}$$

It is well-known that $\phi(s)$ is a convex function of s. We shall suppose that $\psi(s)$ is also a convex function of s which satisfies

$$\frac{\psi(s)}{s} \to +\infty \qquad (s \to +\infty) \tag{6}$$

and define

$$\tau_\psi(s) = \max_t \{st - \psi(t)\}. \tag{7}$$

The functions $\psi(s)$ and $\tau_\psi(s)$ display a certain duality in so far as $\tau_\psi(s)$ is a convex function of s which satisfies a relation corresponding to (6) and, moreover,

$$\psi(s) = \max_t \{st - \tau_\psi(t)\}; \tag{8}$$

see [5, p. 7]. We then have

THEOREM 1. *Let Λ be a set of positive integers. Then a necessary and sufficient condition that every $f(z) \in E(\Lambda)$ which satisfies*

$$\phi(s, f) = O(\psi(s)) \qquad (s \to +\infty) \tag{9}$$

be unbounded on $z > 0$ is that

$$\liminf_{s \to \infty} \left\{ 2 \sum_{\lambda_n \leqq s} \lambda_n^{-1} - \frac{\tau_\psi(s)}{s} \right\} = -\infty. \tag{10}$$

We define the ψ-order ρ $(0 \leqq \rho \leqq \infty)$ of an entire function $f(z)$ by

$$\rho = \limsup_{s \to \infty} \frac{\psi^{-1}(\phi(s))}{s}. \tag{11}$$

The ψ-order reduces to the usual notion of order in the case when ψ is the exponential function. We can also define the lower ψ-order of $f(z)$ in a similar way. We then have

COROLLARY 1. *Let Λ be a set of positive integers. Then a sufficient condition that every $f(z) \in E(\Lambda)$ which is of finite ψ-order at most ρ be unbounded on $z > 0$ is that*

$$\liminf_{s \to \infty} \frac{1}{s} \sum_{\lambda_n \leqq \psi(s)} \lambda_n^{-1} < \frac{1}{2\rho}. \tag{12}$$

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If $\log \psi(s)$ *is a convex function of s, the gap condition is also necessary.*

COROLLARY 2. *Let* Λ *be a set of positive integers. A sufficient condition that every* $f(z) \in E(\Lambda)$ *which is of finite lower* ψ*-order at most* λ *be unbounded on* $z > 0$ *is that*

$$\limsup_{s\to\infty} \frac{1}{s} \sum_{\lambda_n \leq \psi(s)} \lambda_n^{-1} < \frac{1}{2\lambda}.$$

If $\log \psi(s)$ *is a convex function of s, the gap condition is also necessary.*

In the case when $\psi(s) = \exp s$, Corollary 1 reduces to Edrei's Theorem B and Corollary 2 to a companion theorem. In [2], Edrei noted that his results, although best possible, contain an imprecision with regard to the type of entire function considered. Such an imprecision is inevitable in the case of Theorem B, since $f(z)$ is bounded on $z > 0$ if and only if $f(Rz)$ $(R > 0)$ is bounded on $z > 0$. However, Edrei's result can still be sharpened somewhat by taking into account the type of $f(z)$. As usual, if $f(z)$ is of finite order ρ, its type τ is defined by

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}.$$

COROLLARY 3. *Let* Λ *be a set of positive integers. A necessary and sufficient condition that every* $f(z) \in E(\Lambda)$ *which is of finite order at most* ρ *and finite type be unbounded on* $z > 0$ *is that*

$$\liminf_{s\to\infty} \left\{ \sum_{\lambda_n \leq s} \lambda_n^{-1} - \frac{1}{2\rho} \log s \right\} = -\infty.$$

We remark that in all the above results the conclusion that $f(z)$ is unbounded on $z > 0$ may be replaced by the assertion that there is no polynomial which majorises $f(z)$ on $z > 0$.

2. Proofs of sufficiency. In this section we show that the gap conditions given are sufficient that $f(z)$ be unbounded on $z > 0$.

For an entire function $F(z)$ which satisfies $F(0) = 0$ we introduce the notation

$$\| F \|_X = \left\{ \int_0^X |F(x)|^2 \frac{dx}{x^2} \right\}^{\frac{1}{2}}.$$

We require three lemmas, the first of which is a variant of Lemma 1 of [**1**].

LEMMA 1. *Let* $0 < \mu_1 < \mu_2 < \ldots < \mu_n$. *Then, for each* $\nu = 1, 2, \ldots n$, *there exists a real-valued function*

$$B_\nu(x) = \sum_{j=1}^{n} \beta_{j,\nu} x^{\mu_j}$$

such that

$$\text{(i)} \int_0^1 x^\alpha B(x) \frac{dx}{x} = \frac{1}{\alpha+\mu_\nu} \prod_{\substack{j=1 \\ j\neq\nu}}^{n} \frac{\alpha-\mu_j}{\alpha+\mu_j},$$

$$\text{(ii)} \int_0^1 B_\nu^2(x) \frac{dx}{x} = (2\mu_\nu)^{-1}.$$

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LEMMA 2. *Let* $\Lambda = \{\lambda_j\}$ *be a set of positive integers and define*

$$\sigma(x) = \exp\left\{2 \sum_{\lambda_j \leqq x} \mu_j^{-1}\right\},$$

where $\mu_j = \lambda_j - \frac{1}{2} (j = 1, \dots n)$. *Then there exist constants* $C(\lambda)$ $(\lambda \in \Lambda)$ *which are independent of x and such that the coefficients of each polynomial* $P(t)$ *of the form*

$$P(t) = \sum_{\lambda_j \leqq x} A_j t^{\lambda_j}$$

satisfy the inequality

$$|A_\nu| \leqq C(\lambda_\nu) \| P \|_{\sigma(x)} \qquad (\lambda_\nu \leqq x).$$

Proof of Lemma 2. We write σ for $\sigma(x)$ and let $P(t) = t^{\frac{1}{2}} Q(t)$. Then, by Lemma 1(i),

$$\begin{aligned}\int_0^\sigma Q(t) B_\nu\left(\frac{t}{\sigma}\right)\frac{dt}{t} &= \sum_{j=1}^n A_j \int_0^\sigma t^{\mu_j} B_\nu\left(\frac{t}{\sigma}\right)\frac{dt}{t} \\ &= \sum_{j=1}^n A_j \sigma^{\mu_j} \int_0^1 t^{\mu_j} B_\nu(t) \frac{dt}{t} \\ &= A_\nu \frac{1}{2\mu_\nu} \prod_{\substack{\lambda_i \leqq x \\ i \neq \nu}} \frac{\mu_\nu - \mu_i}{\mu_\nu + \mu_i} \exp \frac{2\mu_\nu}{\mu_i}.\end{aligned}$$

On employing the inequality $e^y(1-y) < (1+y)$ $(0 < y < 1)$, we deduce that

$$|A_\nu| \leqq 2\mu_\nu \left\{ \prod_{\substack{i=1 \\ i \neq \nu}}^\infty \left| \frac{\mu_\nu + \mu_i}{\mu_\nu - \mu_i} \right| \exp\left(-\frac{2\mu_\nu}{\mu_i}\right)\right\} \int_0^\sigma \left| Q(t) B_\nu\left(\frac{t}{\sigma}\right)\right| \frac{dt}{t}.$$

The convergence of the infinite product follows from that of the infinite series $\sum_{j=1}^\infty \mu_j^{-2}$. The conclusion of Lemma 2 may now be deduced by means of Schwarz's inequality and Lemma 1(ii).

LEMMA 3. *Suppose that* $f(z) \in E(\Lambda)$ *and has a power series expansion of the form* (1). *Suppose also that* (9) *and* (10) *hold. Then, for each fixed* $R > 0$,

$$\liminf_{x \to \infty} \max_{0 \leqq t \leqq R\sigma(x)} \left| \sum_{k > x} a_k t^k \right| = 0,$$

where $\sigma(x)$ *is defined as in Lemma* 2.

Proof of Lemma 3. For each $r > 0$,

$$\begin{aligned}|a_k| &\leqq M(r) r^{-k} \\ &= \exp\{\phi(s) - sk\} \\ &\leqq \exp\{A\psi(s) - sk\},\end{aligned}$$

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where A is an appropriate constant and $r = e^s$. On minimising the right hand side with respect to s, we obtain

$$|a_k| \leqq \exp\left\{-A\tau_\psi\left(\frac{k}{A}\right)\right\},$$

where $\tau_\psi(s)$ is defined by (7). Since $\tau_\psi(s)$ is a convex function of s, $s^{-1}\tau_\psi(s)$ increases. Hence, for $0 \leqq t \leqq R\sigma(x)$,

$$\begin{aligned}\left|\sum_{k>x} a_k t^k\right| &\leqq \sum_{k>x}\left\{R\sigma(x)\exp\left[-\frac{A}{k}\tau_\psi\left(\frac{k}{A}\right)\right]\right\}^k \\ &\leqq \sum_{k>x}\left\{R\sigma(x)\exp\left[-\frac{A}{x}\tau_\psi\left(\frac{x}{A}\right)\right]\right\}^k.\end{aligned}$$

It follows from (10) that, for each fixed $R > 0$, the expression inside the braces is smaller than any pre-assigned positive constant for a set of values of x which is unbounded above. This completes the proof of Lemma 3.

Proof of sufficiency in Theorem 1. We fix $\lambda \in \Lambda$ and apply Lemma 2 to the function

$$P(t) = \sum_{k \leqq x}(a_k R^k)t^k,$$

where $\lambda \leqq x$. We obtain that

$$|a_\lambda R^\lambda| \leqq C(\lambda)\| P \|_{\sigma(x)}.$$

Thus

$$\begin{aligned}|a_\lambda R^{\lambda-\frac{1}{2}}| &\leqq C(\lambda)\left\| \sum_{k \leqq x} a_k t^k \right\|_{R\sigma(x)} \\ &\leqq C(\lambda)\left\{\| f \|_{R\sigma(x)} + \left\| \sum_{k>x} a_k t^k \right\|_{R\sigma(x)}\right\}.\end{aligned}$$

Allowing $x \to +\infty$ through a suitable sequence of values and employing Lemma 3, we obtain that

$$\begin{aligned}|a_\lambda R^{\lambda-\frac{1}{2}}| &\leqq C(\lambda)\limsup_{x\to\infty} \| f \|_{R\sigma(x)} \\ &\leqq C(\lambda)\left\{\int_0^\infty |f(x)|^2\frac{dx}{x^2}\right\}^{\frac{1}{2}}.\end{aligned}$$

But, if $f(x)$ is bounded on $x > 0$, the right hand side is finite. A contradiction then follows on letting $R \to \infty$, unless $a_\lambda = 0$. But this cannot be true for every value of $\lambda \in \Lambda$ because $f(z)$ is transcendental. Thus $f(x)$ is unbounded on $x > 0$ and we have proved that the gap condition of Theorem 1 is sufficient.

Proof of sufficiency in Corollary 1. The inequality

$$\tau_\psi(s) \geqq s\psi^{-1}(s) - s \tag{13}$$

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is obtained from (7) on taking $t = \psi^{-1}(s)$. The gap condition (10) is therefore implied by

$$\liminf_{s\to\infty}\left\{2\sum_{\lambda_n\leqq s}\lambda_n^{-1}-\psi^{-1}(s)\right\}=-\infty, \tag{14}$$

which, in turn, is implied by

$$\liminf_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\psi(s)}\lambda_n^{-1}<\tfrac{1}{2}. \tag{15}$$

Suppose now that

$$\liminf_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\psi(s)}\lambda_n^{-1}<\frac{1}{2\rho}$$

as in Corollary 1. Then, for a suitable $\tau > \rho$,

$$\liminf_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\psi(\tau s)}\lambda_n^{-1}<\tfrac{1}{2},$$

which is inequality (15) with $\psi(s)$ replaced by $\psi(\tau s)$. Moreover, since $f(z)$ has ψ-order ρ,

$$\phi(s,f)=O(\psi(\tau s)) \qquad (s\to+\infty),$$

which is equation (9) with $\psi(s)$ replaced by $\psi(\tau s)$. It then follows from Theorem 1 with $\psi(s)$ replaced by $\psi(\tau s)$ that $f(z)$ is unbounded on $z > 0$.

Proof of sufficiency in Corollary 2. Suppose that

$$\limsup_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\psi(s)}\lambda_n^{-1}<\frac{1}{2\rho}$$

as in Corollary 2. Then, for an appropriate $\tau > \rho$,

$$\limsup_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\psi(\tau s)}\lambda_n^{-1}<\tfrac{1}{2}.$$

But, since $f(z)$ has lower ψ-order ρ, the inequality $\phi(s) < \psi(\tau s)$ holds for a set of values of s which is unbounded above. It follows that

$$\liminf_{s\to\infty}\frac{1}{s}\sum_{\lambda_n\leqq\phi(s)}\lambda_n^{-1}<\tfrac{1}{2}.$$

Since $f(z)$ has ϕ-order 1, it follows from Corollary 1 that $f(z)$ is unbounded on $z > 0$.

Proof of sufficiency in Corollary 3. We take $\psi(s) = e^{\rho s}$ in Theorem 1. The equality (10) holds in this case, as can be seen by writing $\psi(s) = e^{\rho s}$ in (14). Moreover, since $f(z)$ has finite order ρ and finite type,

$$\phi(s,f)=O(e^{\rho s})$$

and so condition (9) holds. Theorem 1 then yields that $f(z)$ is unbounded on $z > 0$.

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3. Proofs of necessity. In this section we use a construction discussed at length in [2] and [4]. If Λ is a given set of positive integers we define the function $F(z)$ by

$$F(z) = \sum_{n=1}^{\infty} \left\{ \frac{G(-\lambda_n)}{(1+\lambda_n)^2 G(\lambda_n)} \right\} z^{\lambda_n}, \tag{16}$$

where $G(z)$ is defined by

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) \exp \frac{z}{\lambda_n}.$$

This is the notation used by Edrei in [2]. The function $F(z) \in E(\Lambda)$ and

$$F(x) \to 0 \qquad (x \to +\infty).$$

The proof is precisely as in Section 7 of [2].

Fuchs [3] has given an estimate for the coefficients in (16). He shows that they are majorised by

$$\exp \left\{ A\lambda_n - 2\lambda_n \sum_{k=1}^{n} \lambda_k^{-1} \right\},$$

where A is a constant depending only on Λ. We let $F_0(z) = F(\gamma z)$ where $\gamma = \exp\{-(a+A+1)\}$, a being a real constant to be chosen later. Then $F_0(z) \in E(\Lambda)$ and is bounded on $z > 0$. Moreover

$$\phi(s, F_0) = O(\max_x \{(s-a)x - x\lambda(x)\}), \tag{17}$$

where

$$\lambda(x) = 2 \sum_{\lambda_k \leqq x} \lambda_k^{-1}.$$

Proof of necessity in Theorem 1. Suppose that (10) does not hold. Then, for an appropriate constant b,

$$\lambda(x) - \frac{\tau_\psi(x)}{x} > -b,$$

for all $x > 0$. Thus

$$\max_x \{(s-a)x - x\lambda(x)\} \leqq \max_x \{(s-a+b)x - \tau_\psi(x)\} = \psi(s-a+b)$$

by (8). Thus, if $a = b$, the function $F_0(z) \in E(\Lambda)$, is bounded on $z > 0$ and satisfies (9). Thus (10) is a necessary condition.

Proof of necessity in Corollaries 1 *and* 2. We take $a = 0$ in (17). Now

$$\max_x \{sx - x\lambda(x)\} = \max_x \{x(s - \lambda(x))\} \leqq \lambda^{-1}(s)s, \tag{18}$$

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because the maximum is evidently attained for a value of x which satisfies $\lambda(x) < s$. Since $\lambda(s)$ is a step function, the notation $\lambda^{-1}(s)$ needs some explanation. We define

$$\lambda^{-1}(s) = \sup\{x \mid \lambda(x) < s\}.$$

Suppose that

$$\liminf_{s\to\infty} \frac{1}{s} \sum_{\lambda_n \leqq \psi(s)} \lambda_n^{-1} \geqq \frac{1}{2\rho},$$

as in Corollary 1. Then, given any $\tau > \rho$,

$$\lambda(\psi(\tau s)) \geqq s$$

for all large values of s. Hence,

$$\psi(\tau s) \geqq \lambda^{-1}(s)$$

for all large s. Also, by (6), given any $\varepsilon > 0$,

$$\psi(\varepsilon s) \geqq s$$

for all large s. Since $\log \psi(s)$ is a convex function of s

$$s\lambda^{-1}(s) \leqq \psi(\varepsilon s)\psi(\tau s) \leqq \psi((\varepsilon+\tau)s)$$

for all large s. It now follows, from (18), that $F_0(z)$ defined above has ψ-order ρ. Since, moreover, $F_0(x) \to 0$ $(x \to \infty)$ this proves that the gap condition in Corollary 1 is necessary.

A similar argument suffices to prove the necessity of the gap condition in Corollary 2.

Proof of necessity in Corollary 3. It is not difficult to show that, if $\log \psi(s)$ is convex, then condition (10) is equivalent to the condition

$$\liminf_{s\to\infty} \left\{ 2 \sum_{\lambda_n \leqq s} \lambda_n^{-1} - \psi^{-1}(s) \right\} = -\infty.$$

If we take $\psi(s) = e^{\rho s}$, the necessity of the gap condition in Corollary 3 follows from that of the gap condition in Theorem 1.

The function $f(z) = z^{-\rho} \sin z^{\rho}$ (with 2ρ a positive integer) is an example of an entire function for which $f(x) \to 0$ $(x \to \infty)$ and such that

$$2 \sum_{\lambda_n \leqq s} \lambda_n^{-1} - \frac{1}{2\rho} \log s = O(1) \qquad (s \to +\infty).$$

A suitable modification yields a simpler proof of the necessity of the gap condition in Corollary 3 in the case $\lambda_n = 2n\rho$ $(n = 1, 2, \ldots)$.

In conclusion, it is a pleasure to thank the referee for his very helpful comments.

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