

AUTOMATIC CONTINUITY OF CERTAIN ISOMORPHISMS BETWEEN REGULAR BANACH FUNCTION ALGEBRAS

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Abstract. Let A and B be regular semisimple commutative Banach algebras; that is to say, regular Banach function algebras. A linear map T defined from A into B is said to be *separating* or *disjointness preserving* if $f \cdot g = 0$ implies $Tf \cdot Tg = 0$, for all $f, g \in A$. In this paper we prove that if A satisfies Ditkin's condition then a separating bijection is *automatically continuous* and its inverse is separating. If also B satisfies Ditkin's condition, then it induces a homeomorphism between the structure spaces of A and B .

Finally, we show that linear isometries between regular uniform algebras are separating. As corollaries, a classical theorem of Nagasawa ([19]) and the Banach-Stone theorem (both for regular uniform algebras) are easily inferred.

1. Introduction. The basic problem in the theory of automatic continuity of linear operators consists of giving algebraic conditions on two Banach algebras \mathcal{A} and \mathcal{B} which ensure that every algebra homomorphism $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ is necessarily continuous. The most important positive result in this context is due to B. E. Johnson ([15]). Every algebra homomorphism of a Banach algebra onto a semisimple Banach algebra is continuous.

In this paper we shall study the automatic continuity of a special type of linear mapping which extends the concept of algebra homomorphism: Let A and B be two Banach algebras. Then a linear map T defined from A into B is said to be *separating* or *disjointness preserving* (also called *d-homomorphisms*, *Lamperti operators*, ...) if $f \cdot g = 0$ implies $Tf \cdot Tg = 0$, for all $f, g \in A$. Linear maps $T: L_p(\mu) \rightarrow L_p(\mu)$ with the property that $f \cdot g = 0$ (μ a.e.) implies $Tf \cdot Tg = 0$ (μ a.e.) were already considered by Banach [7, p. 175]. In [18], J. Lamperti continued Banach's research and later W. Arendt [5] called such maps *Lamperti operators*. In the context of vector lattices, disjointness preserving operators $T: E \rightarrow F$ (with E, F vector lattices) are defined to satisfy the implication $|f| \wedge |g| = 0 \Rightarrow |Tf| \wedge |Tg| = 0$ and have been widely studied: their multiplicative representation in, for example [1], their spectral properties in [6]; recently, Huijsmans-de Pagter [14] and, independently, Koldunov [17] have shown that the inverse of a disjointness preserving bijection is disjointness preserving under certain general conditions.

Disjointness preserving maps were also considered in [8] for spaces of real or complex-valued continuous functions defined on a compact Hausdorff space with the name of separating maps. The main goal in this context is to prove automatic continuity results for separating maps between different kinds of function spaces of the following type.

THEOREM. ([16] and [10].) *A separating bijection $T: C(X) \rightarrow C(Y)$ with X, Y compact (resp. $T: C_0(X) \rightarrow C_0(Y)$ with X, Y locally compact), is continuous and induces a homeomorphism of Y onto X .*

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THEOREM. ([11].) *A separating bijection $T:L^1(G_1)\rightarrow L^1(G_2)$ (with G_1 and G_2 locally compact Abelian groups) is continuous and induces a homeomorphism between their dual groups.*

Our main result here (§ 4) will show that a separating bijection $T:A\rightarrow B$ (where A and B are regular semisimple commutative Banach algebras) is automatically continuous whenever A satisfies Ditkin's condition, its inverse is separating and induces a homeomorphism between their structure spaces if also B satisfies such condition. As a consequence (see [13, § 39]), the results (op. cit.) in [16], [10] and [11] are extended to a wider class of regular Banach function algebras that includes, for instance, Segal algebras [21] or the Banach sequence algebras $l^p(\mathbf{N})$, $p \in (0, \infty)$, in [9]. It is, however, important to remark that a separating map need not be continuous; indeed, K. Jarosz proved [16] that, given two compact spaces X (infinite) and Y , there always exists a *discontinuous* separating map defined from $C(X)$ into $C(Y)$; see also [8].

In § 5, we show, in the spirit of [12, Corollary 4.3] and [10, Theorem 4], that linear isometries between regular uniform algebras are separating. As a corollary of this, we easily derive a classical result of Nagasawa [19] and the Banach-Stone theorem for regular uniform algebras.

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2. Preliminaries. Let A be a commutative Banach algebra which may or may not have an identity element. Let X denote the set of all complex-valued multiplicative linear functionals on A . We can identify X with a subset of the unit sphere of the conjugate space A' of A . Endowed with the *Gelfand topology*, that is, the weak-star topology that it inherits from A' , X is said to be the *structure space* or the *maximal ideal space* of A . If A has a unit, then its structure space is a compact Hausdorff space. Otherwise, it is a locally compact Hausdorff space.

The *Gelfand transform* of $f \in A$ is the complex-valued function \hat{f} on X defined by $\hat{f}(x) = x(f)$. The *Gelfand transform* is a homomorphism of A onto a point-separating subalgebra \hat{A} of $C_0(X)$ (the Banach algebra of all complex-valued continuous functions on X which are zero at infinity provided with the supremum norm $\|\cdot\|_\infty$). In addition, $\|\hat{f}\|_\infty \leq \|f\|_A$ for every $f \in A$, which yields the continuity of the Gelfand transform.

It is said that A is *semisimple* if the set $\bigcap \{\ker(x) : x \in X\}$ is $\{0\}$. In this case, the Gelfand transform is injective. If A is a semisimple commutative Banach algebra, then $(A, \|\cdot\|_A)$ and the Banach function algebra $(\hat{A}, \|\cdot\|_A)$ are indistinguishable as Banach algebras. Indeed, the class of semisimple commutative Banach algebras coincides with the class of Banach function algebras.

Let us now consider the *hull-kernel topology* on X . In general, this topology differs from the Gelfand topology. They coincide if the hull-kernel topology is Hausdorff, which occurs if and only if for every closed subset E of X and every $x \in X \setminus E$, there is $f \in A$ such that $\hat{f}(x) \neq 0$, while \hat{f} vanishes on E . If a Banach algebra A has one of these equivalent properties, then A is said to be *regular*.

Associated with a separating map $T:A\rightarrow B$, we can define a linear mapping $\hat{T}:\hat{A}\rightarrow\hat{B}$ as $\hat{T}(\hat{f}) := \widehat{T(f)}$, for every $f \in A$. If A and B are semisimple, then it is easy

to check that T is a separating map if and only if \hat{T} is separating. In like manner, T is injective (resp. onto) if and only if \hat{T} is injective (resp. onto).

\mathbf{N} (resp. \mathbf{R} , \mathbf{C}) will stand for the set of all natural numbers (resp. real, complex numbers). If X is locally compact, then X^* will denote its Alexandroff compactification; that is, $X^* = X \cup \{\infty\}$, ∞ being an ideal point with a neighbourhood consisting of all sets in X^* whose complement is compact in X . If $f \in C(X)$ (the linear space of all complex-valued continuous functions on X), the cozero of f is the set $\text{coz}(f) := \{x \in X : f(x) \neq 0\}$ and $\text{supp}(f)$ will denote the support of f ; i.e., the closure of $\text{coz}(f)$. When U is any subset of X , we shall denote by $\text{int}(U)$ the interior of U and by $\text{cl}(U)$ the closure of U in X . $f|_U$ stands for the restriction of f to U , for any $f \in C(X)$.

Finally, if $y \in Y$, let $T'y' : \hat{A} \rightarrow \mathbf{C}$ be defined as $T'y'(\hat{f}) = \hat{T}(\hat{f})(y)$, for all $\hat{f} \in \hat{A}$.

3. Previous results. In the sequel we shall use the following two results concerning regular commutative Banach algebras (see e.g.; [13], § 39]).

PROPOSITION 1. *Let \mathcal{A} be a regular commutative Banach algebra. Let E be a compact subset of its structure space and disjoint from a closed subset F . Then there is $f \in \mathcal{A}$ such that $\hat{f} = 1$ on E and $\hat{f} = 0$ on F .*

PROPOSITION 2. *If \mathcal{A} is a regular commutative Banach algebra with identity and $\{U_j\}_{j=1}^n$ is an open cover of its structure space, then there are $f_j \in \mathcal{A}$ such that the support of \hat{f}_j is contained in U_j , and $\sum_{j=1}^n \hat{f}_j = 1$.*

DEFINITION 1. Let A and B be semisimple commutative Banach algebras, X and Y their respective structure spaces and $T : A \rightarrow B$ a separating map. An open subset V of X^* is said to be a *vanishing set* for $T'y'$, $y \in Y$, if for all $\hat{f} \in \hat{A}$ such that $\text{coz}(\hat{f}) \subseteq V$, then $T'y'(\hat{f}) = 0$.

Y_0 will stand for the elements $y \in Y$ such that there exists $f_y \in A$ with $\hat{T}(\hat{f}_y)(y) \neq 0$. Fix $y \in Y_0$. Then we define the set

$$\text{supp } T'y' := X^* \setminus \bigcup \{V \subset X^* : V \text{ is a vanishing set for } T'y'\}.$$

LEMMA 1. *With the same notation as in Definition 1, let $y \in Y_0$. Then the set $\text{supp } T'y'$ contains, at most, one element. If, in addition, A is regular, then $\text{supp } T'y'$ is a singleton.*

Proof. Let us suppose that there exist two distinct elements x_1 and x_2 in $\text{supp } T'y'$. Let U and V be two disjoint open neighbourhoods of x_1 and x_2 , respectively. Then there exist $f, g \in A$ such that

$$\hat{T}(\hat{f})(y) \cdot \hat{T}(\hat{g})(y) \neq 0$$

with $\text{coz}(\hat{f}) \subset U$ and $\text{coz}(\hat{g}) \subset V$, which contradicts the separating property of \hat{T} .

Let us now assume that A is regular. It is well known that we can embed A as an ideal in a regular commutative Banach algebra A_1 with identity. Furthermore, X is a locally compact subspace of the (compact) structure space X_1 of A_1 . Indeed, $X_1 = X^*$.

Let us suppose that $\text{supp } T'y'$ is empty. In this case, we have that

$$X^* = \bigcup \{V \subset X^* : V \text{ is a vanishing set for } T'y'\}.$$

Since X^* is a compact set, there exist finitely many vanishing sets U_1, U_2, \dots, U_n of $T'y'$ such that $X^* = U_1 \cup \dots \cup U_n$. By Proposition 2, there are $f_j \in A_1$ such that the support of \hat{f}_j is contained in U_j , and $\sum_{j=1}^n \hat{f}_j = 1$. Hence, since A is an ideal in A_1 , we have that $\hat{f} = \sum_{j=1}^n \hat{f}_j \cdot \hat{f}$, for all $f \in A$. Furthermore, since $\text{coz}(\hat{f}_j \cdot \hat{f}) \subset U_j$, we deduce from the definition of a vanishing set that $\hat{T}(\hat{f} \cdot \hat{f}_j)(y) = 0$, for $j = 1, \dots, n$. Thus $\hat{T}(\hat{f})(y) = 0$, for all $f \in A$, which contradicts the fact that $y \in Y_0$.

REMARK 1. (1) A simple example shows that, without the assumption of regularity, $\text{supp } T'y'$ may be empty. Let $A(\mathcal{D})$ be the disc algebra, $\mathcal{D} = \{y \in \mathbb{C} : |y| \leq 1\}$. It is well known that $A(\mathcal{D})$ is a non-regular Banach function algebra on \mathcal{D} . Let us define $T(f) = f$, for all $f \in A(\mathcal{D})$. If we fix any $y \in \text{int}(\mathcal{D})$, then it is clear that the only possibility would be $\text{supp } T'y' = y$, but, by the analytic identity theorem, we can find an open neighbourhood V of y such that $\text{coz}(f) \cap (\mathcal{D} \setminus V) \neq \emptyset$, for all $f \in A(\mathcal{D})$. This and the paragraph above show that $\text{supp } T'y' = \emptyset$ for all $y \in \text{int}(\mathcal{D})$.

(2) It is straightforward to check that $\text{supp } T'y' = \{x \in X^* : \text{for any } X^* \text{-neighbourhood } U \text{ of } x, \text{ there exists } f \in A \text{ such that } \hat{T}\hat{f}(y) \neq 0 \text{ and } \text{coz}(\hat{f}) \subset U \cap X\}$.

DEFINITION 2. Let A and B be regular semisimple commutative Banach algebras, X and Y their respective structure spaces and $T : A \rightarrow B$ a separating map. The above lemma lets us define a mapping $h : Y_0 \rightarrow X^*$ with $h(y) := \text{supp } T'y', y \in Y_0$. We call h the *support map* of \hat{T} .

PROPOSITION 3. *With the same notation as in Definition 2, let U be an open subset of X^* and suppose that $f \in A$. Then the following statements hold.*

- (1) *The support map h of \hat{T} is continuous.*
- (2) *$\hat{f}|_{U \cap X} \equiv 0$ implies that $\hat{T}\hat{f}|_{h^{-1}(U)} \equiv 0$.*
- (3) *$h(\text{coz}(\hat{T}\hat{f})) \subset \text{cl}_{X^*}(\text{coz}(\hat{f}))$.*
- (4) *If T is injective, then $h(Y_0)$ is a dense subset of X^* .*

Proof. It can be found basically in [1, Proposition 3.1]. (See also [8] or [16].) □

DEFINITION 3. ([13, 39.23]) A commutative Banach algebra \mathcal{A} is said to satisfy *Ditkin's condition* if, for every $f \in \mathcal{A}$ and $x \in X$ such that $\hat{f}(x) = 0$, there exists a sequence $\{f_n\} \subset \mathcal{A}$ and open neighbourhoods V_n of x such that $\hat{f}_n|_{V_n} \equiv 0$, for all $n \in \mathbb{N}$, and $\lim \|f \cdot f_n - f\|_{\mathcal{A}} = 0$.

ⁿ Furthermore, if \mathcal{A} does not have a unit, then, for every $f \in \mathcal{A}$, there must exist a sequence $\{f_n\} \subset \mathcal{A}$ such that $\{f_n\} \subset C_{00}(X)$ (the subspace of $C_0(X)$ consisting of functions with compact support) and $\lim_n \|f \cdot f_n - f\|_{\mathcal{A}} = 0$.

PROPOSITION 4. *Let A and B be regular semisimple commutative Banach algebras, X and Y their respective structure spaces and $T : A \rightarrow B$ a separating map. Let $y \in Y_0$.*

- (1) *If A satisfies Ditkin's condition and $T'y' : (\hat{A}, \|\cdot\|_A) \rightarrow \mathbb{C}$ is a continuous map, then $h(y) \in X$.*
- (2) *If $T'y' : (\hat{A}, \|\cdot\|_{\infty}) \rightarrow \mathbb{C}$ is a continuous map, then $h(y) \in X$.*

Proof. (1) Let us suppose that $h(y) \notin X$; that is, $h(y) = \infty$. Let $f \in A$. Since A satisfies Ditkin's condition, there is a sequence $\{f_n\} \subset A$ such that $\{\hat{f}_n\} \subset C_{00}(X)$ and $\lim_n \|f \cdot f_n - f\|_A = 0$, which implies that $\{\hat{f} \cdot \hat{f}_n\}$ converges to \hat{f} in $(\hat{A}, \|\cdot\|_A)$.

Furthermore, $(\hat{f}_n \cdot \hat{f}) \equiv 0$ on $(X^* \setminus \text{cl}(\text{coz}(\hat{f}_n))) \cap X$ and $y \in h^{-1}(X^* \setminus \text{cl}(\text{coz}(\hat{f}_n)))$, since $h(y) = \infty$ and $\hat{f}_n \in C_{00}(X)$, for every $n \in \mathbb{N}$. Consequently, from Proposition 3, we have that $\hat{T}(\hat{f}_n \cdot \hat{f})(y) = 0$, for every $n \in \mathbb{N}$.

As $T'y': (\hat{A}, \|\cdot\|_A) \rightarrow \mathbb{C}$ is a continuous mapping, the sequence $\{T'y'(\hat{f}_n \cdot \hat{f})\}$ converges to $T'y'(\hat{f})$. Consequently, since $T'y'(\hat{f}_n \cdot \hat{f}) = 0$, for all $n \in \mathbb{N}$, we infer that $T'y'(\hat{f}) = 0$.

Summing up, $T'y'(\hat{f}) = 0$, for every $f \in A$, which contradicts the fact that $y \in Y_0$. (See Definition 1.)

(2) Let us suppose that $h(y) \notin X$; that is, $h(y) = \infty$. Since $\hat{A} \subseteq C_0(X)$, we can consider the following neighbourhood of $h(y)$ for any $f \in A$:

$$U_n := \{x \in X : |\hat{f}(x)| \leq 1/n\}.$$

Also let

$$K_n := \{x \in X : |\hat{f}(x)| \geq 2/n\},$$

for $n \in \mathbb{N}$. From the definition of $C_0(X)$, each K_n is a compact subset of X .

By Proposition 1, there exists $g_n \in A$ such that $\hat{g}_n|_{K_n} \equiv 1$ and $\hat{g}_n|_{U_n} \equiv 0$, for all $n \in \mathbb{N}$. Hence, the sequence $\{\hat{g}_n \cdot \hat{f}\}$ converges to \hat{f} in $(\hat{A}, \|\cdot\|_\infty)$ since $\|\hat{f} - (\hat{g}_n \cdot \hat{f})\|_\infty < \frac{2}{n}$, for all $n \in \mathbb{N}$.

Furthermore, from Proposition 3, we have that $\hat{T}(\hat{g}_n \cdot \hat{f})(y) = 0$, for all $n \in \mathbb{N}$, since $(\hat{g}_n \cdot \hat{f})|_{U_n \cap X} \equiv 0$ and y belongs to $h^{-1}(U_n)$. The rest of the proof follows from the same arguments as in (1) above. □

REMARK 2. Let A and B be regular semisimple commutative Banach algebras, X and Y their respective structure spaces and $T: A \rightarrow B$ a separating map. Let us define a continuous map $\omega: h^{-1}(X) \subseteq Y_0 \rightarrow \mathbb{C}$ as follows. Given $y \in h^{-1}(X)$, let U be a relatively compact open neighbourhood of $h(y)$ and let $e_{(y,U)}$ be any function in A such that $\hat{e}_{(y,U)} \equiv 1$ on U . These functions are provided by Proposition 1. Let us now define

$$\omega(y) := \hat{T}(\hat{e}_{(y,U)})(y).$$

By Proposition 3, it is easy to check that the definition of ω does not depend on the choice of the function $e_{(y,U)}$; i.e., if $\hat{e}'_{(y,U)} \equiv 1$ on U , then $\hat{T}(\hat{e}'_{(y,U)})(y) = \hat{T}(\hat{e}_{(y,U)})(y)$. On the other hand, if V is another relatively compact neighbourhood of $h(y)$ and $e_{(y,V)}$ is as above, then $\hat{T}(\hat{e}_{(y,U)}) = \hat{T}(\hat{e}_{(y,V)})$ on $h^{-1}(U \cap V)$, by Proposition 3.

Hence, the mapping ω is well defined and, given that for all $y \in h^{-1}(X)$ this mapping is equal to $\hat{T}(\hat{e}_{(y,U)})$ on $h^{-1}(U)$ (which is a neighbourhood of y), we infer that ω is a continuous mapping.

DEFINITION 4. Let A and B be regular semisimple commutative Banach algebras, X and Y their respective structure spaces and $T: A \rightarrow B$ a separating map. We denote by Y_c the subset of Y_0 consisting of all y such that $T'y': (\hat{A}, \|\cdot\|_\infty) \rightarrow \mathbb{C}$ is continuous and by Y_a the complement of Y_c in Y_0 .

PROPOSITION 5. *With the same notation as in Definition 4, let $y \in h^{-1}(X)$. The following statements are equivalent.*

- (1) *The map $T'y':(\hat{A}, \|\cdot\|_\infty) \rightarrow \mathbf{C}$ is continuous; i.e., $y \in Y_c$.*
- (2) *$\hat{T}(\hat{f})(y) = \omega(y) \cdot \hat{f}(h(y))$, for every $f \in A$.*

Proof. (1) \Rightarrow (2). First we claim that if $\hat{f}(h(y)) = 0, f \in A$, then $\hat{T}(\hat{f})(y) = 0$. To this end, and since $\hat{f}(h(y)) = 0$ and \hat{f} is continuous, there exist, for every $n \in \mathbf{N}$, open neighbourhoods U_n of $h(y)$ such that $\sup\{|\hat{f}(x)| : x \in \text{cl}(U_n)\} < \frac{1}{n}$. Let us consider a relatively compact open neighbourhood V_n of $h(y)$ such that $\text{cl}(V_n) \subset U_n$, for all $n \in \mathbf{N}$. By Proposition 1, we can choose $g_n \in A$ such that $\hat{g}_n|_{\text{cl}(V_n)} \equiv 1$ and $\hat{g}_n \equiv 0$ outside U_n . Clearly the sequence $\{\hat{f} \cdot \hat{g}_n\}$ converges to the function 0 and $\hat{f} \cdot \hat{g}_n \equiv \hat{f}$ on V_n , for all $n \in \mathbf{N}$. Hence, by Proposition 3, we have that $\hat{T}(\hat{f})(y) = \hat{T}(\hat{f} \cdot \hat{g}_n)(y)$ and consequently, since $T'y'$ is a continuous mapping, $\hat{T}(\hat{f})(y) = 0$.

On the other hand, let us suppose that $\hat{f}(h(y)) \neq 0$. Define the function $g := f - (\hat{f}(h(y))) \cdot e_{(y,V)}$ ($e_{(y,V)}$ being as in the above remark). Since \hat{T} is linear and $\hat{g}(h(y)) = 0$, we infer that $\hat{T}(\hat{f})(y) = \omega(y) \cdot \hat{f}(h(y))$, for all $f \in A$.

- (2) \Rightarrow (1). Clear □

PROPOSITION 6. *With the same notation as in Definition 4, the following statements hold.*

- (1) Y_c is closed in $h^{-1}(X)$.
- (2) $h(Y_d)$ is a subset of limit points of X^* .
- (3) If A satisfies Ditkin's condition, then the set $h(Y_d) \cap \text{int}(K)$ is finite, for every compact subset K of X .

Proof. (1) It is apparent, by Proposition 4, that $Y_c \subseteq h^{-1}(X)$. Let us consider a net $\{y_\alpha\}$ in Y_c that converges to $y \in h^{-1}(X)$. By Proposition 5, $\hat{T}(\hat{f})(y_\alpha) = \omega(y_\alpha) \cdot \hat{f}(h(y_\alpha))$, for every α and every $f \in A$. Since $\omega, \hat{f} \circ h$, and $\hat{T}(\hat{f})$ are continuous mappings, it is clear that $\hat{T}(\hat{f})(y) = \omega(y) \cdot \hat{f}(h(y))$, for every $f \in A$; that is, $y \in Y_c$ by Proposition 5.

(2) If $h(y) = \infty$, then $h(y)$ is a limit point of X^* . Let us now see that if $h(y) \in X$ is isolated in $X, y \in Y$, then $T'y':(\hat{A}, \|\cdot\|_\infty) \rightarrow \mathbf{C}$ is a continuous map; that is, $y \in Y_c$. Given $f \in A$, let us define the map $\hat{g} := \hat{f}(h(y)) \cdot \hat{e}_{(y,U)}$, for any relatively compact open neighbourhood U of y , as in Remark 2. As $\hat{f}|_{\{h(y)\}} \equiv \hat{g}|_{\{h(y)\}}$, we have that $\hat{T}(\hat{f})(y) = \hat{T}(\hat{g})(y)$, by Proposition 3. Hence, $T'y'(\hat{f}) = \omega(y) \cdot \hat{f}(h(y))$, for every $f \in A$, which implies that $T'y':(\hat{A}, \|\cdot\|_\infty) \rightarrow \mathbf{C}$ is continuous, by Proposition 5.

(3) Let us suppose, contrary to what we claim, that there exists a sequence $\{h(y_n)\}$ of distinct elements of $\text{int}(K)$ such that $y_n \in Y_d$, for every $n \in \mathbf{N}$. As K is a normal space, we can assume, by taking a subsequence if necessary, that $\{U_n\}$ is a sequence of pairwise disjoint open subsets of K such that $h(y_n) \in U_n \subset K$, for every $n \in \mathbf{N}$. Let V_n be a closed neighbourhood of $h(y_n)$ with $V_n \subset U_n$, for every $n \in \mathbf{N}$.

Fix $m \in \mathbf{N}$. By Proposition 5, since the map $T'y'_m: (\hat{A}, \|\cdot\|_\infty) \rightarrow \mathbf{C}$ is discontinuous, there exists a function $f_m \in A$ with $\hat{T}(\hat{f}_m)(y_m) \neq \omega(y_m) \cdot \hat{f}_m(h(y_m))$. Let us now define the function

$$\hat{g}_m := \hat{f}_m - (\hat{f}_m(h(y_m))) \cdot \hat{e}_{(y_m,V)},$$

where $\hat{e}_{(y_m, V)}$ is as in Remark 2 and $K \subseteq V$. Hence, $\hat{T}(\hat{g}_m)(y_m) \neq 0$ and $\hat{g}_m(h(y_m)) = 0$. Since \hat{T} is a linear map, we shall assume without loss of generality that $|\hat{T}(\hat{g}_m)(y_m)| > m$. Since A satisfies Ditkin's condition, there exists $k_m \in A$ such that $\hat{k}_m|_{V_m} \equiv 0$ and $\|\varphi_m\|_A < \frac{1}{m^2}$, where $\varphi_m := g_m - g_m \cdot k_m$. Furthermore, as $\hat{\varphi}_m \equiv \hat{g}_m$ on V_m , we have, by Proposition 3,

$$|\hat{T}(\hat{\varphi}_m)(y_m)| = |\hat{T}(\hat{g}_m)(y_m)| > m.$$

Let us now define $\varphi := \sum_{n \in \mathbb{N}} \varphi_n$. Since $\|\varphi_n\|_A < \frac{1}{n^2}$, for all $n \in \mathbb{N}$, and A is a Banach space, we infer that φ belongs to A . From the fact that the Gelfand transform is a linear continuous mapping, we deduce that $\hat{\varphi} = \sum_{n \in \mathbb{N}} \hat{\varphi}_n$.

On the other hand, since the family $(U_n)_{n \in \mathbb{N}}$ is pairwise disjoint and $\text{coz}(\hat{\varphi}_n) \subset U_n$, for all $n \in \mathbb{N}$, the separating property of \hat{T} yields $T(\hat{\varphi}_n)|_{h^{-1}(U_n)} \equiv 0$, for $n \neq n'$ ($n, n' \in \mathbb{N}$). Thus, $|\hat{T}(\hat{\varphi}_n)(y_n)| = |\hat{T}(\hat{\varphi})(y_n)| > n$, for every $n \in \mathbb{N}$, which is a contradiction since $\hat{T}(\hat{\varphi})$ is bounded. □

4. Main results.

THEOREM 1. *Let A and B be regular semisimple commutative Banach algebras, X and Y their respective structure spaces and $T : A \rightarrow B$ a separating bijection. Assume further that A satisfies Ditkin's condition.*

- (1) *T is continuous.*
- (2) *T^{-1} is separating.*
- (3) *If also B satisfies Ditkin's condition, then X and Y are homeomorphic.*

Proof. (1) Let us first note that $Y = Y_0$ since T is onto. Since $Y_c \cup Y_d = Y$, then $h(Y_c) \cup h(Y_d) = h(Y)$. As T is injective, $h(Y)$ is dense in X^* , by Proposition 3. Hence, given $x \in X$ and a compact neighbourhood U of x , we have, by Proposition 6, $h(Y_c) \cap U \neq \emptyset$. That is, $h(Y_c)$ is dense in X^* .

Now, if $y \in Y_c$, then $\hat{T}(\hat{f})(y) = \omega(y) \cdot \hat{f}(h(y))$, for all $f \in A$, which implies, since \hat{T} is onto and \hat{A} separates the points of X , that $\omega(y) \neq 0$. Hence, if $\hat{T}(\hat{f})|_{Y_c} \equiv 0$, for some $f \in A$, then $\hat{f}|_{h(Y_c)} \equiv 0$. Furthermore, we know, by the above paragraph and Proposition 4, that $h(Y_c)$ is dense in X . As a consequence, if $\hat{T}(\hat{f})|_{Y_c} \equiv 0$, then $\hat{f} \equiv 0$.

Next, we claim both that $Y_c = h^{-1}(X)$ and that Y_c is dense in Y . To this end, suppose that there exists $y \in h^{-1}(X) \setminus Y_c$. As Y_c is, from Proposition 6, a closed subset of $h^{-1}(X)$, there exists, by Proposition 1 and the surjectivity of \hat{T} , a function $g \in A$ such that $\hat{T}(\hat{g})|_{Y_c} \equiv 0$ and $\hat{T}(\hat{g})(y) = 1$. This contradicts the above paragraph. In like manner we prove that Y_c is dense in Y .

Finally, in order to prove the continuity of T , we shall prove that it has a closed graph. Let us suppose that the sequences $\{f_n\} \subset A$ and $\{T(f_n)\}$ converge to $f \in (A, \|\cdot\|_A)$ and $g \in (B, \|\cdot\|_B)$, respectively. Then, given any $y \in Y$, the sequence $\{\hat{T}(\hat{f}_n)(y)\}$ converges to $\hat{g}(y)$. On the other hand, given any $x \in X$, the sequence $\{\hat{f}_n(x)\}$, converges to $\hat{f}(x)$. Consequently, for any $y \in Y_c$, the sequence $\{\hat{T}(\hat{f}_n)(y)\}$, that is to say $\{\omega(y) \cdot \hat{f}_n(h(y))\}$, converges to $\hat{T}(\hat{f})(y) = \omega(y) \cdot \hat{f}(h(y))$. Thus, $\hat{T}(\hat{f}) = \hat{g}$ on Y_c , which is dense in Y . This implies that $\hat{T}(\hat{f}) = \hat{g}$ and, consequently, $T(f) = g$. Finally, since A and B are Banach spaces, the Closed Graph Theorem yields the continuity of T .

(2) Let us first prove that the inverse of \hat{T} , $\hat{T}^{-1}:\hat{B} \rightarrow \hat{A}$, is also a separating bijection. Let \hat{g}_1 and \hat{g}_2 be two elements of \hat{B} such that $\hat{g}_1 \cdot \hat{g}_2 = 0$. Let $\hat{f}_1, \hat{f}_2 \in \hat{A}$ such that $\hat{T}\hat{f}_1 = \hat{g}_1$ and $\hat{T}\hat{f}_2 = \hat{g}_2$. Since ω is non-vanishing on Y_c (see (1) above), we deduce that $\hat{f}_1 \cdot \hat{f}_2 = 0$ on $h(Y_c)$, which is dense in X . As a consequence, $\hat{f}_1 \cdot \hat{f}_2 = 0$ on the whole of X . Hence, both \hat{T}^{-1} and, consequently, $T^{-1}:B \rightarrow A$ are separating maps.

(3) Since T is continuous, it is a routine matter to verify that the map $T'y':(\hat{A}, \|\cdot\|_A) \rightarrow \mathbb{C}$ is continuous, for every $y \in Y$. Thus, by Proposition 4, we know that $h(y) \in X$, for every $y \in Y$. Therefore, since $h^{-1}(X) = Y_c$ (see (1) above), we infer that $Y = h^{-1}(X) = Y_c$ and, consequently, $T'y'$ is a weighted composition map, for all $y \in Y$. (See Proposition 5.)

Next, by (2), we can consider the support map $k:X \rightarrow Y^*$ of \hat{T}^{-1} . Of course, k is continuous and its range is dense in Y^* , by Proposition 3. Furthermore, by (1) and Proposition 4, we know that $(T^{-1})'x'$ is a weighted composition map for all $x \in X$ and $k(X) \subset Y$.

Let us now fix $y \in Y$. We claim that $k(h(y)) = y$. Assume, contrary to what we claim, that $k(h(y)) \neq y$. Then there exist disjoint compact neighbourhoods U and V of $k(h(y))$ and y , respectively. By Remark 1(2), there exists a function $g_0 \in B$ such that $\text{coz}(\hat{g}_0) \subseteq U$ and $\hat{T}^{-1}(\hat{g}_0)(h(y)) \neq 0$. This implies, by Proposition 3(3), that $y \notin \text{cl}(\text{coz}(\hat{g}_0))$ and $k(h(y)) \in \text{cl}(\text{coz}(\hat{g}_0))$. Let $g_1 \in B$ (see Proposition 1) such that $\hat{g}_1(y) \neq 0$ and $\hat{g}_1|_{\text{coz}(\hat{g}_0)} \equiv 0$. Consequently, we have that $\text{coz}(\hat{g}_1) \cap \text{coz}(\hat{g}_0) = \emptyset$ and, since \hat{T}^{-1} is separating, $\text{coz}(\hat{T}^{-1}(\hat{g}_1)) \cap \text{coz}(\hat{T}^{-1}(\hat{g}_0)) = \emptyset$. On the other hand, $\text{coz}(\hat{T}^{-1}(\hat{g}_0))$ is a neighbourhood of $h(y)$, but if we take any $\hat{f} \in \hat{A}$ such that $\text{coz}(\hat{f}) \subseteq \text{coz}(\hat{T}^{-1}(\hat{g}_0))$, then $\text{coz}(\hat{g}_1) \cap \text{coz}(\hat{T}(\hat{f})) = \emptyset$, because \hat{T} is separating. Consequently, $\hat{T}(\hat{f})(y) = 0$, since $\hat{g}_1(y) \neq 0$. This contradicts Remark 1(2). In like manner, we infer that $h(k(x)) = x$, for every $x \in X$; that is, h is a homeomorphism of Y onto X . \square

As a straightforward application of our main results, we have the following result. Let G_1 and G_2 be locally compact Abelian groups. Let \widehat{G}_i ($i = 1, 2$) be the dual group of G_i . Let us recall that a Segal algebra $S(G_i)$ is a dense ideal in the group algebra $L^1(G_i)$ with continuous imbedding, on which G_i acts continuously by isometric translations. These algebras were introduced in [20]. Some time later, it was proved in [21] that a Segal algebra $S(G_i)$ is a regular semisimple commutative Banach algebra (with respect to a certain norm), which satisfies Ditkin's condition and whose structure space is \widehat{G}_i .

COROLLARY 1. *A separating bijection $T:S(G_1) \rightarrow S(G_2)$ is automatically continuous and induces a homeomorphism of \widehat{G}_2 onto \widehat{G}_1 .*

5. Isometries between regular uniform algebras. Let A and B be regular semisimple commutative Banach algebras, and X and Y their respective structure spaces. In this section we shall assume, in addition, that $\|f^2\|_A = \|f\|_A^2$, for every $f \in A$ (resp. $\|g^2\|_B = \|g\|_B^2$, for every $g \in B$), which implies that the Gelfand transform is an isometry. Consequently, \hat{A} (resp. \hat{B}) is a (uniformly) closed subalgebra of $C_0(X)$ (resp. $C_0(Y)$). In this case, A and B are said to be regular *uniform algebras* on X and Y respectively. If \mathcal{X} is a compact subspace of \mathbb{C} and we consider the algebra of functions on \mathcal{X} uniformly approximable by rational functions with poles off \mathcal{X} , then we have an example of a non-trivial regular uniform algebra.

On the other hand, let us recall that $x_0 \in X$ is said to be a *strong boundary point* for A if, for each open neighbourhood U of x_0 , there is $f \in A$ such that $|\hat{f}(x_0)| = \|\hat{f}\|_\infty = 1$ and $|\hat{f}(x)| < 1$ for all $x \in X \setminus U$. The set of strong boundary points for A is dense in the Shilov boundary for A , which here coincides with its structure space X since A is regular. (See [4].)

LEMMA 2. *Let A be a regular uniform algebra on X . If $x_0 \in X$ is a strong boundary point for A then, for each open neighbourhood U of x_0 and $\varepsilon > 0$, there is $g \in A$ such that $|\hat{g}(x_0)| = \|\hat{g}\|_\infty = 1$ and $|\hat{g}(x)| < \varepsilon$, for all $x \in X \setminus U$.*

Proof. Since $X^* \setminus \{U\}$ is compact the function \hat{f} , which is provided by the definition of strong boundary point, attains a maximum value $s < 1$ inside $X^* \setminus \{U\}$. Hence there exists $n \in \mathbf{N}$ such that $|(\hat{f})^n(x)| < \varepsilon$, for all $x \in X \setminus U$. Thus, $g := (f)^n$ has the required properties. □

In [10] and [12], the authors prove that linear isometries between $C_{00}(X)$ -spaces and spaces of vector-valued continuous functions, respectively, are separating. The following result shows that the same is true in the context of regular uniform algebras.

THEOREM 2. *Let A and B be regular uniform algebras. If T is a linear isometry of A onto B , then T and T^{-1} are separating.*

Proof. It is obvious that \hat{T} is a linear isometry of \hat{A} onto \hat{B} , since the Gelfand transform is an isometry. Let us suppose that there exist $\hat{f}, \hat{g} \in \hat{A}$ and $y_1 \in Y$ such that $\hat{f} \cdot \hat{g} \equiv 0$ with $\hat{T}(\hat{f})(y_1) \cdot \hat{T}(\hat{g})(y_1) \neq 0$. Since the set of strong boundary points for B is dense in Y , there exists a strong boundary point y_0 such that $\hat{T}(\hat{f})(y_0) = a \neq 0$ and $\hat{T}(\hat{g})(y_0) = b \neq 0$.

Multiplying by constants if necessary, we shall assume that $a, b \in \mathbf{R}$, $|a| \geq |b|$, and $\|\hat{f}\|_\infty = \|\hat{g}\|_\infty = 1$.

As a, b are reals, we know that $|a| < \max(|a + b|, |a - b|)$. Let us suppose that $|a + b| > |a| \geq |b|$ and let $\lambda \in \mathbf{R}$ such that $0 < \lambda < \frac{|a + b| - |a|}{2}$.

We define

$$V = \{y \in Y : |\hat{T}(\hat{f})(y)| < |a| + \lambda\},$$

$$W = \{y \in Y : |\hat{T}(\hat{g})(y)| < |b| + \lambda\}$$

and $U = V \cap W$. It is clear that U is an open neighbourhood of y_0 and, since y_0 is a strong boundary point for B , Lemma 2 ensures that there exists $h \in B$ such that $0 \leq |\hat{h}| \leq 1$ with $\hat{h}(y_0) = \frac{a + b}{|a + b|}$ and $\hat{h}(Y \setminus U) < |a + b|$. Then

$$|\hat{T}(\hat{f})(y_0) + \hat{T}(\hat{g})(y_0) + \hat{h}(y_0)| = \left| a + b + \frac{a + b}{|a + b|} \right| = 1 + |a + b|.$$

Hence, $\|\hat{T}(\hat{f}) + \hat{T}(\hat{g}) + \hat{h}\|_\infty \geq 1 + |a + b|$.

Since \hat{T} is onto, there exists $k \in B$ such that $\hat{T}(k) = \hat{h}$. Hence, since \hat{T} is an isometry, we infer that $\|\hat{f} + \hat{g} + k\|_\infty \geq 1 + |a + b|$. As $\hat{f} \cdot \hat{g} \equiv 0$, we have that $\max(\|\hat{f} + k\|_\infty, \|\hat{g} + k\|_\infty) \geq 1 + |a + b|$. Therefore, if we assume that $\|\hat{f} + k\|_\infty \geq 1 + |a + b|$, then $\|\hat{T}(\hat{f}) + \hat{h}\|_\infty \geq 1 + |a + b|$.

Given any element y of Y such that $y \in U$, we have that $|\hat{T}(\hat{f})(y) + \hat{h}(y)| \leq |\hat{T}(\hat{f})(y)| + |\hat{h}(y)| < |a| + \lambda + 1 < 1 + |a + b|$. On the other hand, if $y \notin U$, then $|\hat{T}(\hat{f})(y) + \hat{h}(y)| \leq \|\hat{T}(\hat{f})(y)\| + |\hat{h}(y)| \leq 1 + |a + b|$. Hence, for every $y \in Y$, we infer that $|\hat{T}(\hat{f})(y) + \hat{h}(y)| < 1 + |a + b|$, which contradicts the above paragraph. This proves that T and, by Theorem 1(2), T^{-1} are separating. \square

In [19], Nagasawa proved that two unital uniform algebras are isometric if and only if they are (algebra) isomorphic. As a corollary of Theorem 2, we easily derive this classical theorem of Nagasawa and the Banach-Stone theorem ([3]) in the setting of regular uniform algebras.

COROLLARY 2. *Let A and B be regular uniform algebras. Let us suppose that there exists a linear isometry T of A onto B .*

- (1) (Banach-Stone) *Their structure spaces are homeomorphic.*
- (2) ([19]) *If A and B have units, they are (algebra) isomorphic.*

Proof. (1) Combine Theorems 1(3) and 2. (Note that the hypothesis on Theorem 1(3) is redundant whenever T^{-1} is continuous.)

(2) Assume now that 1_A and 1_B are the units of A and B respectively. Then one easily checks that the function ω , which we have defined in Remark 2, turns out to be $\hat{T}(1_A)$, which implies that $\omega \in \hat{B}$. By Proposition 5(2), we have that

$$\omega \cdot \hat{T}(\hat{f} \cdot \hat{g}) = \hat{T}(\hat{f}) \cdot \hat{T}(\hat{g}),$$

for every $f, g \in A$. Hence, if we now choose $k \in A$ such that $\hat{T}(k) = \hat{1}_B$, then $\omega \cdot \hat{T}(k^2) = \hat{1}_B$, which is to say that $\omega^{-1} = \hat{T}(k^2)$ belongs to \hat{B} . Finally, it is clear that $\omega^{-1} \cdot \hat{T}$ is the desired algebra isomorphism of \hat{A} onto \hat{B} . \square

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