Canad. Math. Bull. Vol. **30** (4), 1987

BOREL SETS IN METRIC SPACES WITH SMALL SEPARABLE SUBSETS

BY
P. DANIELS AND G. GRUENHAGE

ABSTRACT. Let X be a metric space such that every separable subspace of X has size less than the continuum. We answer a question of D. H. Fremlin by showing that $MA + \neg CH$ does not necessarily imply that every subset of X is analytic.

1. **Introduction.** D. H. Fremlin [4] asked the following questions: Assume Martin's Axiom and the negation of the continuum hypothesis ($MA + \neg CH$). Let X be a metric space such that every separable subset of X has size smaller than the continuum $\mathfrak{c}$. Does every subset of X have to be Borel? Can X have subsets of all Borel classes $<\omega_1$?

Recall that, if $|X| < \mathfrak{c}$ and X is metric, then $MA + \neg CH$ implies that every subset of X is a relative F_σ. Thus a counterexample to Fremlin's question must have size at least $\mathfrak{c}$. Another relevant result is A. Miller's theorem [6] that if every subset of a metric space is Borel, then in fact the classes are bounded.

In this note, we show that the answer to Fremlin's first question is generally "no" by showing that, assuming $MA_{\omega_1} + \diamondsuit_{\omega_2}(E)$, there exists a subset X of the Baire 0-dimensional space $B(\omega_2) = \omega_2^\omega$ such that every separable subset of X has size $\leqq \omega_1$, but not every subset of X is analytic.

In $B(\kappa)$, a useful notion of "small" is "σ-local weight $<\kappa$ (σ-$LW(<\kappa)$)" (see Stone [7]), meaning the union of countably many discrete collections $\mathscr{D}_n$, where each $D \in \mathscr{D}_n$ has weight $<\kappa$. Let us say $Y \subset B(\kappa)$ is *essentially of class* α if Y is of class α, and the class of Y cannot be lowered by adding and subtracting two σ-$LW(<\kappa)$ sets. In a letter to the second author, V. V. Uspenskii has observed that if $Z \subset \omega^\omega$ is essentially of class α in ω^ω (i.e., its class cannot be lowered by adding and subtracting two countable sets), then $B(\kappa) \times Z$ is essentially of class α in $B(\kappa) \times \omega^\omega \cong B(\kappa)$; hence sets which are essentially of class α exist in $B(\kappa)$.

The subspace X of $B(\omega_2)$ that we construct has the same Borel structure, modulo σ-$LW(<\kappa)$ sets, as $B(\omega_2)$. We use the existence of essentially class α sets

Received by the editors June 10, 1986, and, in final revised form, June 4, 1987.
AMS Classification Number: 54H05.

to show that this X has Borel sets of all classes, hence the answer to Fremlin's second question is "yes".

Our construction does not completely settle Fremlin's questions. There may be another model of $MA + \neg CH$ in which the answer to the first question is "yes", and then by Miller's result the answer to the second question will be "no".

2. **The example.** First, we discuss the axioms we need. If S is a stationary subset of ω_2, then $\diamondsuit_{\omega_2}(S)$ is the following statement:

There is a sequence of $\langle \mathscr{A}_\alpha : \alpha \in S \rangle$ such that

(i) $A \in \mathscr{A}_\alpha \Rightarrow A \subset \alpha$;
(ii) $|\mathscr{A}_\alpha| \leqq \omega_1$;
(iii) If $X \subset \omega_2$, then $\{\alpha \in S : X \cap \alpha \in \mathscr{A}_\alpha\}$ is stationary.

$\diamondsuit^*_{\omega_2}(S)$ is the same statement with (iii) strengthened to:

(iii)′ If $X \subset \omega_2$, then $\{\alpha \in S : X \cap \alpha \in \mathscr{A}_\alpha\} \supset C \cap S$ for some closed and unbounded (club) subset C of ω_2. (We say $C \cap S$ is "club in S".)

An easy exercise shows that $\diamondsuit_{\omega_2}(S) \Rightarrow \mathfrak{c} \leqq \omega_2$. To construct the example, we need $\diamondsuit_{\omega_2}(E)$, where $E = \{\alpha < \omega_2 : \text{cf } \alpha = \omega\}$. Since it will make the description of the example simpler, we will use $\diamondsuit^*_{\omega_2}(E)$ instead, and indicate later how, using a trick due to K. Kunen, the construction can be modified to work with $\diamondsuit_{\omega_2}(E)$. The axiom $MA_{\omega_1} + \diamondsuit^*_{\omega_2}(E)$ holds in the usual model obtained via *ccc* forcing to prove the consistency of $MA + \mathfrak{c} = \omega_2$, as long as $\diamondsuit^*_{\omega_2}(E)$ holds in the ground model. One way to show this is to modify Exercise H8 in Kunen [5].

We use the following facts concerning the structure of subsets of $B(\omega_2)$ – see Stone [7], [8] for proofs of the first three:

1) An analytic subset of $B(\omega_2)$ which is not σ-$LW(<\omega_2)$ contains a homeomorphic copy of $B(\omega_2)$;

2) If $A \subset B(\omega_2)$ is such that $\{\text{sup ran } f : f \in A \text{ and sup ran } f \notin \text{ran } f\}$ is stationary, then A is not σ-$LW(<\omega_2)$;

3) If $H \subset B(\omega_2)$ is homeomorphic to $B(\omega_2)$, then there exists a club $C_H \subset \omega_2$ such that

$$C_H \cap E \subset \{\text{sup ran } f : f \in H\}.$$

For each $\alpha \leqq \omega_2$, let $\Sigma_\alpha = \cup_{n\in\omega} \alpha^n$, and for each $\sigma \in \Sigma_{\omega_2}$, let $[\sigma] = \{f \in B(\omega_2) : \sigma \subset f\}$.

4) If $H \subset B(\omega_2)$ is homeomorphic to $B(\omega_2)$, then there is a function $\theta_H : \Sigma_{\omega_2} \to \Sigma_{\omega_2}$ such that

(a) $\sigma \subset \tau \Rightarrow \theta_H(\sigma) \subset \theta_H(\tau)$ and $\text{sup ran } \theta_H(\sigma) < \text{sup ran } \theta_H(\tau)$;
(b) $\text{sup ran } \sigma \leqq \text{sup ran } \theta_H(\sigma)$;

(c) For each $f \in B(\omega_2)$,

$$\bigcap_{n \in \omega} [\theta_H(f|n)] \subset H.$$

(θ_H can be used to construct the club C_H in 3); see Fleissner [2].)

The idea of the construction is as follows. First, we use a coding of $\diamondsuit^*_{\omega_2}(E)$ to obtain a "$\diamondsuit$*-sequence" $\langle \Theta_\alpha : \alpha \in E \rangle$, where each Θ_α consists of $\leqq \omega_1$ functions $\theta : \Sigma_\alpha \to \Sigma_\alpha$, and such that, given $\theta : \Sigma_{\omega_2} \to \Sigma_{\omega_2}$,

$$\{\alpha \in \omega_2 : \theta \restriction \Sigma_\alpha \in \Theta_\alpha\}$$

contains a club in E. We use Θ_α to pick out a set $X_\alpha \subset \{f \in B(\omega_2)$: sup ran $f = \alpha\}$ of size $\leqq \omega_1$, and we let $X = \cup_{\alpha \in E} X_\alpha$. Because $|X_\alpha| \leqq \omega_1$, each separable subset of X has size $\leqq \omega_1$. Because the Θ_α's "trap" all functions $\theta : \Sigma_{\omega_2} \to \Sigma_{\omega_2}$, by the facts 1)-4) above, the Borel structure of $B(\omega_2)$ is essentially reflected in the set X. To get a non-analytic subset A of X, simply let S be any stationary, co-stationary subset of E, and let $A = \cup_{\alpha \in S} X_\alpha$. The corresponding subset of $B(\omega_2)$ is non-analytic in $B(\omega_2)$ by facts 1) and 2); our construction makes this true in X also.

The X_α, $\alpha \in E$, are defined as follows. Let $g_\alpha \in B(\omega_2)$ be an increasing function with sup ran $g_\alpha = \alpha$. For each $\theta \in \Theta_\alpha$, choose $f_\theta \in \cap_{n \in \omega} [\theta(g_\alpha|n)]$ if possible, such that sup ran $f_\theta = \alpha$. Let $X_\alpha = \{f_\theta : \theta \in \Theta_\alpha\}$.

We now establish the following key fact:

5) If $H \subset B(\omega_2)$ is homeomorphic to $B(\omega_2)$, then {sup ran $f : f \in H \cap X$} contains a club in E.

Let $\theta_H : \Sigma_{\omega_2} \to \Sigma_{\omega_2}$ satisfy the conditions of fact 4). Let C_1 be a club in E such that, for each $\alpha \in C_1$, $\theta_H \restriction \Sigma_\alpha \in \Theta_\alpha$, and let C_2 be a club such that, if $\alpha \in C_2$ and $\sigma \in \Sigma_\alpha$, then $\theta_H(\sigma) \in \Sigma_\alpha$. We complete the proof of 5) by showing

$$C_1 \cap C_2 \cap E \subset \{\text{sup ran } f : f \in H \cap X\}.$$

Let $\alpha \in C_1 \cap C_2 \cap E$. Then $\cap_{n \in \omega} [\theta_H(g_\alpha|n)]$ is a single function f with sup ran $f = \alpha$. Since $\theta_H \restriction \Sigma_\alpha \in \Theta_\alpha$, $f \in X_\alpha$, and by fact 4c), $f \in H$.

Now we show that for any stationary, co-stationary subset S of E, if we let $A = \cup_{\alpha \in S} X_\alpha$, then A is not analytic in X. If A were analytic, then $A = B \cap X$ for some analytic $B \subset B(\omega_2)$. By fact 2), B is not σ-$LW(<\omega_2)$, so by fact 1), B contains a homeomorph H of $B(\omega_2)$. Then $A \supset H \cap X$, so by 5), {sup ran $f : f \in A$} contains a club in E, which is a contradiction.

Now, assume that G is a set of essentially class α in $B(\omega_2)$. We show that $G \cap X$ is exactly of class α in X. If not, there is a set $J \subset B(\omega_2)$ of class β, where $\beta < \alpha$, such that $G \cap X = J \cap X$. Then $G \backslash J$ and $J \backslash G$ do not meet X; hence by facts 1) and 5), they are σ-$LW(<\omega_2)$. This contradicts the fact that G is essentially of class α.

To do the construction of X with just $\diamondsuit_{\omega_2}(E)$, recall that Kunen (see [1; Section 5]) has shown that $\diamondsuit_{\omega_1}$ implies that there is a countably complete normal filter $\mathscr{F}$ on ω_1 containing the club filter, and a sequence $\langle \mathscr{A}_\alpha : \alpha < \omega_1 \rangle$ such that

(i) $A \in \mathscr{A}_\alpha \Rightarrow A \subset \alpha$;
(ii) $|\mathscr{A}_\alpha| \leqq \omega$;
(iii) For each $X \subset \omega_1$, $\{\alpha : X \cap \alpha \in \mathscr{A}_\alpha\} \in \mathscr{F}$.

One may similarly prove that $\diamondsuit_{\omega_2}(E)$ implies that there is a countably complete normal filter $\mathscr{F}$ containing all sets club in E, and a sequence $\langle \Theta_\alpha : \alpha \in E \rangle$ such that each Θ_α consists of $\leqq \omega_1$ functions $\theta : \Sigma_\alpha \to \Sigma_\alpha$, and each $\theta : \Sigma_{\omega_2} \to \Sigma_{\omega_2}$ is "trapped" on a member of $\mathscr{F}$, i.e.,

$$\{\alpha : \theta \restriction \Sigma_\alpha \in \Theta_\alpha\} \in \mathscr{F}.$$

Using these Θ_α's, the X_α's are constructed as before. Just as one uses the normality of the club filter on ω_2 to show that disjoint stationary subsets of ω_2 exist, one can use the normality of $\mathscr{F}$ to obtain a subset S of E such that S and $E \setminus S$ meet every element of $\mathscr{F}$.

Now the rest of the proof proceeds as before, with uses of the club filter replaced by $\mathscr{F}$.

REMARK. Fleissner [3] proves that, assuming $\diamondsuit_{\omega_1}(S)$ for every stationary $S \subset \omega_1$, the following is true: if X is a metric space of weight $\leqq \omega_1$, and $\mathscr{A}$ is an analytic additive (i.e., $\cup \mathscr{A}'$ is analytic for each $\mathscr{A}' \subset \mathscr{A}$) disjoint family of subsets of X, then $\mathscr{A}$ is σ-discretely decomposable; in particular, if every subset of X is analytic, then X is σ-discrete. From his argument it follows that if for each $\alpha < \omega_1$, we choose an increasing function $f_\alpha \in B(\omega_1)$ with sup ran $f_\alpha = \alpha$, then $\diamondsuit$ implies that there is a non-analytic subset of $\{f_\alpha : \alpha < \omega_1\}$. This argument can be modified to show that if for each $\alpha \in E$, we choose an increasing function $f_\alpha \in B(\omega_2)$ with sup ran $f_\alpha = \alpha$, then $\diamondsuit_{\omega_2}(E)$ implies that there is a non-analytic subset $Y = \{f_\alpha : \alpha \in E\}$; further, every separable subset of Y is countable. However, it is not clear that this Y should contain Borel sets of all orders.

REFERENCES

1. Amer Beslagic, *A Dowker Product*, Trans. Amer. Math. Soc. **292** (1985), pp. 519-530.

2. W. G. Fleissner, *Applications of stationary sets in topology*, in *Surveys in General Topology*, edited by G. M. Reed, Academic Press, 1980.

3. ———, *An axiom for nonseparable Borel theory*, Trans. Amer. Math. Soc. **251** (1979), pp. 309-328.

4. D. H. Fremlin, Note of August, '83, question BQ.

5. Ken Kunen, *Set Theory*, North Holland, Amsterdam, 1980.

6. A. Miller, *On the length of Borel hierarchies*, Annals Math. Logic **16** (1979), pp. 233-267.

7. A. H. Stone, *Non-separable Borel sets*, Dissertationes Math. **18**, 1962.
8. ———, *Non-separable Borel sets II*, Gen. Topology Appl. **2** (1972), pp. 249-270.

DEPARTMENT OF MATHEMATICS
AUBURN UNIVERSITY
AUBURN, AL 36849