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ON SUMS OF TWO PRIME SQUARES, FOUR PRIME CUBES AND POWERS OF TWO

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Abstract

We prove that every sufficiently large even integer can be represented as the sum of two squares of primes, four cubes of primes and 28 powers of two. This improves the result obtained by Liu and Lü ['Two results on powers of 2 in Waring–Goldbach problem', *J. Number Theory* **131**(4) (2011), 716–736].

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1. Introduction

In the 1950s, Linnik [3, 4] proved that every large even integer N is a sum of two primes and a bounded number of powers of two,

$$N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_1}}.$$

Throughout the paper, p and v, with or without subscripts, denote a prime number and a positive integer, respectively. The famous Goldbach conjecture implies that $k_1 = 0$. The explicit value for the number k_1 was improved by many authors.

In 1999, Liu *et al.* [8] proved that every sufficiently large even integer N can be represented in the form

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_2}} \tag{1.1}$$

and they also showed that there is a representation of the form (1.1) for some finite value of v_{k_2}. The best result so far is by Zhao [12], who obtained $k_2 = 39$.

In 2001, Liu and Liu [7] proved that every large even integer N can be written as a sum of eight cubes of primes and powers of two,

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_3}}. \tag{1.2}$$

The value $k_3 = 330$ was determined by Platt and Trudgian [10].

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In 2011, Liu and Lü [9] considered the hybrid problem combining (1.1) and (1.2), that is,

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k},$$

and proved that every sufficiently large even integer can be written as a sum of two squares of primes, four cubes of primes and 211 powers of two. In 2015, Platt and Trudgian [10] improved the value of ν_k to 205. In this paper, we obtain a further improvement of the value of ν_k.

THEOREM 1.1. *Every sufficiently large even integer is a sum of two squares of primes, four cubes of primes and* 28 *powers of two.*

2. Notation and preliminary lemmas

In this section, we introduce the necessary notation and lemmas for the proof of Theorem 1.1.

Throughout, N denotes a sufficiently large even integer. We fix a positive constant $\eta < 10^{-10}$ and let $\varepsilon < 10^{-10}$ be an arbitrarily small positive constant not necessarily the same in different formulae. The letter p, with or without subscripts, is reserved for a prime number. We write $e(\alpha) = e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. By $A \sim B$, we mean that $B < A \leqslant 2B$. We denote by (m, n) the greatest common divisor of m and n. As usual, $\varphi(n)$ stands for Euler's function. Let

$$P_2 = \frac{1}{2}\sqrt{(1-\eta)N}, \quad U_3 = \frac{1}{2}\Big(\frac{\eta N}{2}\Big)^{1/3}, \quad V_3 = \frac{1}{2}\Big(\frac{\eta N}{2}\Big)^{5/18}, \quad L = \frac{\log(N/\log N)}{\log 2},$$

$$F(\alpha) = \sum_{p\sim P_2}(\log p)e(\alpha p^2), \quad S(\alpha) = \sum_{p\sim U_3}(\log p)e(\alpha p^3),$$

$$T(\alpha) = \sum_{p\sim V_3}(\log p)e(\alpha p^3), \quad H(\alpha) = \sum_{\nu\le L} e(\alpha 2^\nu), \quad \mathcal{E}(\lambda) = \{\alpha \in (0,1] : |H(\alpha)| \ge \lambda L\}.$$

For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this purpose, we set

$$Q_1 = N^{(1/9)-2\varepsilon}, \quad Q_2 = N^{(8/9)+\varepsilon}$$

and, for $(a, q) = 1$, $1 \le a \le q$, we define the major and minor arcs by

$$\mathfrak{M}(q,a) = \Big(\frac{a}{q} - \frac{1}{qQ_2}, \frac{a}{q} + \frac{1}{qQ_2}\Big], \quad \mathfrak{M} = \bigcup_{1\le q\le Q_1} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(q,a),$$

$$\mathfrak{I}_0 = \Big(\frac{1}{Q_2}, 1 + \frac{1}{Q_2}\Big], \quad \mathfrak{m} = \mathfrak{I}_0 \setminus \mathfrak{M}.$$

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Then it follows from orthogonality that

$$R(N) := \sum_{\substack{N=p_1^2+p_2^2+p_3^3+p_4^3+p_5^3+p_6^3+2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_k} \\ p_1\sim P_2,\, p_2\sim P_2,\, p_3\sim U_3,\, p_4\sim U_3,\, p_5\sim V_3,\, p_6\sim V_3,\, \nu_k\le L}} (\log p_1)(\log p_2)\cdots(\log p_6)$$
$$= \int_0^1 F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha$$
$$= \Big(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\Big)F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha. \tag{2.1}$$

Now we state the lemmas required in the proof of the main theorem. Lemma 2.1 will be used in the estimation of the integral over $\mathfrak{M}$ and Lemmas 2.2–2.5 will be used in the estimation of the integral over $\mathfrak{m}$.

Lemma 2.1 [5, Theorem 1.1]. *For $\frac{1}{2}N < n \leqslant N$,*

$$\int_{\mathfrak{M}} F^2(\alpha)S^2(\alpha)T^2(\alpha)e(-\alpha n)\,d\alpha = \frac{1}{2^2\cdot 3^4}\mathfrak{S}(n)\mathfrak{J}(n) + O\Big(\frac{N^{11/9}}{L}\Big).$$

Here $\mathfrak{S}(n)$ is defined by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{S_2^{*2}(q,a)S_3^{*4}(q,a)e_q(-an)}{\varphi^6(q)}, \quad S_k^*(q,a) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e_q(ar^k)$$

and satisfies $\mathfrak{S}(n) \gg 1$ for $n \equiv 0 \pmod 2$; $\mathfrak{J}(n)$ is defined by

$$\mathfrak{J}(n) = \sum_{\substack{n=m_1+m_2+m_3+m_4+m_5+m_6\\ P_2{}^2<m_1,m_2\le 4P_2{}^2,\ U_3{}^3<m_1,m_2\le 8U_3{}^3,\ V_3{}^3<m_1,m_2\le 8V_3{}^3}} (m_1m_2)^{-1/2}(m_3m_4m_5m_6)^{-2/3},$$

and satisfies

$$N^{11/9} \ll \mathfrak{J}(n) \ll N^{11/9}.$$

Lemma 2.2 [9, Lemma 4.3]. *For $\alpha\in\mathfrak{m}$, $F(\alpha) \ll N^{4/9+\varepsilon}$.*

Lemma 2.3. *We have*

$$\text{(i)}\quad \int_0^1 |S(\alpha)T(\alpha)|^4\,d\alpha \ll U_3V_3{}^4;$$
$$\text{(ii)}\quad \int_0^1 |F(\alpha)T^2(\alpha)|^2\,d\alpha \ll N^{10/9}L^c;$$
$$\text{(iii)}\quad \int_0^1 |F(\alpha)S(\alpha)T(\alpha)|^2\,d\alpha \le 6.4894513U_3{}^2V_3{}^2.$$

Proof. For (i), see [11, Equation (2.7)].

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For (ii),

$$\int_0^1 |F(\alpha)T^2(\alpha)|^2 d\alpha \ll L^6 \sum_{p_1,p_2 \le P_2} 1 \sum_{\substack{q_1,q_2,q_3,q_4 \sim V_3 \\ p_1^2 - p_2^2 = q_1^3 + q_2^3 - q_3^3 - q_4^3}} 1$$
$$\ll L^6 \Big(\sum_{p_1 = p_2} \sum_{\substack{q_1,q_2,q_3,q_4 \sim V_3 \\ q_1^3 + q_2^3 - q_3^3 - q_4^3 = 0}} 1 + \sum_{\substack{q_1,q_2,q_3,q_4 \sim V_3 \\ q_1^3 + q_2^3 - q_3^3 - q_4^3 \neq 0}} \tau(|q_1^3 + q_2^3 - q_3^3 - q_4^3|) \Big)$$
$$\ll L^6 V_3^4 + L^c N^{10/9}$$
$$\ll N^{10/9} L^c,$$

where we have used [1, Equation (2.1)] in the second sum.

For (iii), we recall the definition of $\mathfrak{I}(n)$:

$$\mathfrak{I}(n) = \frac{1}{2^2 \cdot 3^4} \sum_{\substack{m_1+m_2+m_3 = n_1+n_2+n_3 \\ P_2{}^2 < m_1, n_1 \le 4P_2{}^2 \\ U_3{}^3 < m_2, n_2 \le 8U_3{}^3 \\ V_3{}^3 < m_3, n_3 \le 8V_3{}^3}} (m_1 n_1)^{-1/2} (m_2 n_2 m_3 n_3)^{-2/3}.$$

Noting that

$$m_1 = n_1 + n_2 + n_3 - m_2 - m_3 \ge n_1 + U_3^3 + V_3^3 - 8U_3^3 - 8V_3^3 \ge (1 - 3\eta) n_1$$

and

$$\sum_{U_3{}^3 < m \le 8U_3{}^3} m^{-2/3} \sim 3U_3, \qquad \sum_{V_3{}^3 < m \le 8V_3{}^3} m^{-2/3} \sim 3V_3, \qquad \sum_{P_2{}^2 < m \le 4P_2{}^2} m^{-1} \sim 2\log 2, \tag{2.2}$$

we obtain

$$\mathfrak{I}(n) \le \frac{1}{2^2 \cdot 3^4} \sum_{\substack{m_1+m_2+m_3 = n_1+n_2+n_3 \\ P_2{}^2 < m_1, n_1 \le 4P_2{}^2 \\ U_3{}^3 < m_2, n_2 \le 8U_3{}^3 \\ V_3{}^3 < m_3, n_3 \le 8V_3{}^3}} (1 - 3\eta)^{-1/2} n_1^{-1} (m_2 n_2 m_3 n_3)^{-2/3}$$
$$\le \frac{1 + o(1)}{2^2 \cdot 3^4} (1 + 3\eta) \cdot 2\log 2 \cdot (3U_3)^2 (3V_3)^2$$
$$\le \Big(\frac{\log 2}{2} + o(1) \Big) U_3{}^2 V_3{}^2.$$

We deduce from [14, Lemmas 3.1 and 4.1] that

$$\int_0^1 |F(\alpha)S(\alpha)T(\alpha)|^2 \, d\alpha \le 2.3405748 \times (8 + o(1)) \Big(\frac{\log 2}{2} + o(1) \Big) U_3{}^2 V_3{}^2$$
$$\le 6.4894513 U_3{}^2 V_3{}^2.$$

This completes the proof of (iii). □

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In the following lemma, we chose the exponent $\frac{13}{18}$ instead of the earlier exponent $\frac{53}{63}$ in Liu and Lü [9] because it leads to a smaller λ and fewer powers of two.

LEMMA 2.4. *We have* $\mathrm{meas}(\mathcal{E}(\lambda)) \ll N^{E(\lambda)}$, *with*

$$E(0.8709277) > \tfrac{13}{18} + 10^{-10}.$$

PROOF. Let

$$H_h(\alpha) = \sum_{0\le n\le h-1} e(\alpha 2^n),$$

$$G(\xi, h) = \frac{1}{2^h}\sum_{r=0}^{2^h-1} \exp\Big\{\xi \mathrm{Re}\Big(H_h\Big(\frac{r}{2^h}\Big)\Big)\Big\},$$

$$E(\lambda) = \frac{\xi\lambda}{\log 2} - \frac{\log G(\xi, h)}{h\log 2} - \frac{\varepsilon}{\log 2}.$$

Then, for any $h \in \mathbb{N}$, $\varepsilon > 0$ and $\xi > 0$, we have $\mathrm{meas}(\mathcal{E}(\lambda)) \ll N^{E(\lambda)}$. For the proof, we refer the reader to Heath-Brown and Puchta [2]. We confirmed the estimate $E(0.8709277) > \frac{13}{18} + 10^{-10}$ by taking $\xi = 1.174$ and $h = 23$ and running the computation on a PC for about 15 minutes. □

LEMMA 2.5 [13, Lemma 3.1]. *For* $k \ge 3$ *and* $\mathcal{A} \subseteq (P, 2P) \cap \mathbb{N}$, *we define*

$$g(\alpha) = \sum_{x\in\mathcal{A}} e(\alpha x^k)$$

and

$$\omega_k(p^{uk+v}) = \begin{cases} kp^{-u-\frac{1}{2}}, & u \ge 0,\ v = 1,\\ p^{-u-1}, & u \ge 0,\ 2 \le v \le k.\end{cases}$$

Let $\mathcal{M}$ *be the union of the intervals* $\mathcal{M}(q, a)$ *for* $1 \le a \le q \le P^{k2^{1-k}}$ *and* $(a, q) = 1$, *where* $\mathcal{M}(q, a) = \{\alpha : |q\alpha - a| \le P^{k(2^{1-k}-1)}\}$. *Let*

$$\mathcal{J}_0 = \sup_{\gamma\in[0,1]} \int_{\mathcal{M}} \frac{\omega_k^2(q)|h(\alpha+\gamma)|^2}{(1 + P^k|\alpha - a/q|)^2}\, d\alpha.$$

Suppose that $G(\alpha)$ *and* $h(\alpha)$ *are integrable functions of period one. Let* $\mathfrak{n} \subseteq [0, 1)$ *be a measurable set. Then*

$$\int_{\mathfrak{n}} g(\alpha)G(\alpha)h(\alpha)\, d\alpha \ll P\mathcal{J}_0^{1/4}\Big(\int_{\mathfrak{n}} |G(\alpha)|^2\, d\alpha\Big)^{1/4}\mathcal{J}(\mathfrak{n})^{1/2} + P^{1-2^{-k}+\varepsilon}\mathcal{J}(\mathfrak{n}),$$

where

$$\mathcal{J}(\mathfrak{n}) = \int_{\mathfrak{n}} |G(\alpha)h(\alpha)|\, d\alpha.$$

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3. Auxiliary estimates

We are now equipped to establish the auxiliary estimates needed in this paper. We initiate our proof by recalling the Farey dissection (2.1):

$$\begin{aligned} R(N) &= \int_0^1 F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha \\ &= \Big(\int_{\mathfrak{M}} + \int_{\mathfrak{m}\bigcap\mathcal{E}(\lambda)} + \int_{\mathfrak{m}\backslash\mathcal{E}(\lambda)}\Big)F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha. \end{aligned}$$

3.1. The evaluation of the integral over $\mathfrak{M}$.

Lemma 3.1 [6, Lemma 4.2]. *Let*

$$\Xi(N,k) = \{(1-\eta)N \le n \le N : n = N - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_k}\}.$$

For $k \ge 2$ and $N \equiv 0 \pmod 2$,

$$\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 0 \pmod 2}} 1 \ge (1-\varepsilon)L^k.$$

Lemma 3.2 [9, Section 5]. *We have $\mathfrak{S}(n) > 0.592836481$.*

Lemma 3.3. *For $(1-\eta)N \leqslant n \leqslant N$,*

$$\mathfrak{J}(n) \ge 84.8230017{U_3}^2{V_3}^2.$$

Proof. The domain of $\mathfrak{J}(n)$ can be written as

$$\mathfrak{D} = \left\{(m_1,\ldots,m_6) : \begin{array}{ll} P_2^2 < m_1, m_2 \le (2P_2)^2, & U_3^3 < m_3, m_4 \le (2U_3)^3, \\ V_3^3 < m_3, m_4 \le (2V_3)^3, & m_1 = n - m_2 - \cdots - m_6 \end{array}\right\}.$$

Define

$$\mathfrak{D}^* = \left\{(m_1,\ldots,m_6) : \begin{array}{ll} P_2^2 < m_1, m_2 \le 3{P_2}^2 - 3\eta N, & U_3^3 < m_3, m_4 \le (2U_3)^3, \\ V_3^3 < m_3, m_4 \le (2V_3)^3, & m_1 = n - m_2 - \cdots - m_6 \end{array}\right\}.$$

For $(m_1,\ldots,m_6) \in \mathfrak{D}^*$, we can deduce from $(1-\eta)N \leqslant n \leqslant N$ that

$$P_2^2 < n - m_2 - 2\eta N \le n - m_2 - \cdots - m_6 = m_1 < (2P_2)^2.$$

Thus, $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$. Since $(1-\eta)N \leqslant n \leqslant N$, a simple calculation gives

$$\begin{aligned} &\mathfrak{J}(n) \\ &\ge \sum_{(m_1,\ldots,m_6)\in\mathfrak{D}^*} (m_1m_2)^{-1/2}(m_3m_4m_5m_6)^{-2/3} \\ &\ge \sum_{P_2^2<m_2\le 3{P_2}^2-3\eta N} ((n-m_2-2\eta N)m_2)^{-1/2} \sum_{\substack{U_3^3<m_3\le(2U_3)^3\\ U_3^3<m_4\le(2U_3)^3}} (m_3m_4)^{-2/3} \sum_{\substack{V_3^3<m_5\le(2V_3)^3\\ V_3^3<m_6\le(2V_3)^3}} (m_5m_6)^{-2/3} \\ &\ge \sum_{P_2^2<m_2\le 3{P_2}^2-3\eta N} m_2^{-1/2}((1-\eta)N-m_2)^{-1/2} \sum_{\substack{U_3^3<m_3\le(2U_3)^3\\ U_3^3<m_4\le(2U_3)^3}} (m_3m_4)^{-2/3} \sum_{\substack{V_3^3<m_5\le(2V_3)^3\\ V_3^3<m_6\le(2V_3)^3}} (m_5m_6)^{-2/3}. \end{aligned}$$

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Replacing $m_2/(1-\eta)N$ by t and applying Euler–Maclaurin summation and (2.2) gives

$$\begin{aligned}\mathfrak{J}(n) &\geq \Big(\int_{1/4}^{3/4} t^{-1/2}(1-t)^{-1/2}\,dt - \int_{3/4-3\eta/(1-\eta)}^{3/4} t^{-1/2}(1-t)^{-1/2}\,dt\Big)(3U_3)^2(3V_3)^2\\ &\geq \Big(\int_{1/4}^{3/4} t^{-1/2}(1-t)^{-1/2}\,dt - 10\eta\Big)(3U_3)^2(3V_3)^2\\ &\geq 84.8230017{U_3}^2{V_3}^2. \qquad \square\end{aligned}$$

Proposition 3.4. *We have*

$$\int_{\mathfrak{M}} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha \geq 0.1552042U_3^2V_3^2L^k.$$

Proof. Lemmas 2.1 and 3.1–3.3 reveal that

$$\begin{aligned}\int_{\mathfrak{M}} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha &= \sum_{n\in\Xi(N,k)}\int_{\mathfrak{M}} F^2(\alpha)S^2(\alpha)T^2(\alpha)e(-\alpha n)\,d\alpha\\ &= \frac{1}{2^2\cdot 3^4}\sum_{n\in\Xi(N,k)}\mathfrak{S}(n)\mathfrak{J}(n) + O(N^{11/9}L^{k-1})\\ &\geq 0.1552042U_3^2V_3^2\sum_{n\in\Xi(N,k)} 1 + O(N^{11/9}L^{k-1})\\ &\geq 0.1552042U_3^2V_3^2L^k. \qquad \square\end{aligned}$$

3.2. The estimation of the integrals over $\mathfrak{m}$.

Lemma 3.5. *We have*

$$\int_{\mathfrak{m}} |F(\alpha)|^2|S(\alpha)T(\alpha)|^{5/2}\,d\alpha \ll N^{(107/72)+\varepsilon}.$$

Proof. Consider

$$\int_{\mathfrak{m}} |F(\alpha)|^2|S(\alpha)T(\alpha)|^{5/2}\,d\alpha = \int_{\mathfrak{m}} S(\alpha)T(\alpha)\overline{S(\alpha)T(\alpha)}|S(\alpha)T(\alpha)|^{1/2}|F(\alpha)|^2\,d\alpha.$$

By taking $g(\alpha) = S(\alpha)$, $h(\alpha) = T(\alpha)$ and $G(\alpha) = \overline{S(\alpha)T(\alpha)}|S(\alpha)T(\alpha)|^{1/2}|F(\alpha)|^2$ in Lemma 2.5, we see that the integral is

$$\begin{aligned}&\ll U_3{\mathcal{J}_0}^{1/4}\Big(\int_{\mathfrak{m}} |F^4(\alpha)S^3(\alpha)T^3(\alpha)|\,d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{m}} |F^2(\alpha)S^{3/2}(\alpha)T^{5/2}(\alpha)|\,d\alpha\Big)^{1/2}\\ &\quad + {U_3}^{(7/8)+\varepsilon}\int_{\mathfrak{m}} |F^2(\alpha)S^{3/2}(\alpha)T^{5/2}(\alpha)|\,d\alpha\\ &=: I_1 + I_2, \qquad (3.1)\end{aligned}$$

where

$$\mathcal{J}_0 = \sup_{\gamma\in[0,1]}\int_{\mathcal{M}} \frac{\omega_3^2(q)|T(\alpha+\gamma)|^2}{(1+U_3^3|\alpha-\frac{a}{q}|)^2}\,d\alpha, \quad \mathcal{M} = \bigcup_{1\leq q\leq U_3^{3/4}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\Big[\frac{a}{q}-\frac{1}{U_3^{3/4}}, \frac{a}{q}+\frac{1}{U_3^{3/4}}\Big].$$

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By arguments similar to those of Zhao [14],

$$\begin{aligned}\mathcal{J}_0 &\le V_3^2 \sum_{q\le U_3^{3/4}} \omega_3^2(q)\tau^c(q) \int_{|\beta|\le U_3^{-9/4}} \frac{1}{(1+U_3^3|\beta|)^2}\, d\alpha \\ &\le V_3^2 \sum_{q\le U_3^{3/4}} \omega_3^2(q)\tau^c(q)\Big(\int_{|\beta|\le U_3^{-3}} 1 + U_3^{-6}\int_{U_3^{-3}<|\beta|\le U_3^{-9/4}} \frac{1}{|\beta|^2}\, d\alpha\Big) \\ &\ll V_3^2 U_3^{-3}(\log N)^c \\ &\ll N^{-4/9+\varepsilon}, \end{aligned} \tag{3.2}$$

where Lemma 2.1 in [14] is used in the estimation of $\sum_{q\le U_3^{3/4}} \omega_3^2(q)\tau^c(q)$. From the Cauchy–Schwarz inequality and Lemmas 2.2 and 2.3(i) and (iii),

$$\begin{aligned}&\int_{\mathfrak{m}} |F^4(\alpha)S^3(\alpha)T^3(\alpha)|\, d\alpha \\ &\quad\ll \max_{\alpha\in\mathfrak{m}} |F(\alpha)|^3\Big(\int_0^1 |F(\alpha)S(\alpha)T(\alpha)|^2\, d\alpha\Big)^{1/2}\Big(\int_0^1 |S(\alpha)T(\alpha)|^4\, d\alpha\Big)^{1/2} \\ &\quad\ll N^{(8/3)+\varepsilon}. \end{aligned} \tag{3.3}$$

Similarly, it follows from the Hölder inequality and Lemma 2.3(ii) and (iii) that

$$\begin{aligned}\int_{\mathfrak{m}} |F^2(\alpha)S^{3/2}(\alpha)T^{5/2}(\alpha)|\, d\alpha &\ll \Big(\int_0^1 |F(\alpha)S(\alpha)T(\alpha)|^2\, d\alpha\Big)^{3/4}\Big(\int_0^1 |F(\alpha)T^2(\alpha)|^2\, d\alpha\Big)^{1/4} \\ &\ll N^{(43/36)+\varepsilon}. \end{aligned} \tag{3.4}$$

Combining (3.1)–(3.2) together with (3.3)–(3.4) yields

$$I_1 \ll N^{(107/72)+\varepsilon}, \quad I_2 \ll N^{(107/72)+\varepsilon}.$$

Hence,

$$\int_{\mathfrak{m}} |F(\alpha)|^2 |S(\alpha)T(\alpha)|^{5/2}\, d\alpha \ll N^{(107/72)+\varepsilon}$$

and Lemma 3.5 is proved. □

Proposition 3.6 (The estimation over $\mathfrak{m}\bigcap\mathcal{E}(\lambda)$). *We have*

$$\int_{\mathfrak{m}\bigcap\mathcal{E}(\lambda)} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\, d\alpha \ll U_3^2 V_3^2 L^{k-1}.$$

Proof. An application of Hölder's inequality together with Lemmas 2.2, 2.4 and 3.5 yields

$$\begin{aligned}&\int_{\mathfrak{m}\bigcap\mathcal{E}(\lambda)} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\, d\alpha \\ &\quad\ll L^k \max_{\alpha\in\mathfrak{m}} |F(\alpha)|^{2/5}\Big(\int_{\mathfrak{m}} |F(\alpha)|^2|S(\alpha)T(\alpha)|^{5/2}\, d\alpha\Big)^{4/5}\Big(\int_{\mathcal{E}_\lambda} 1\, d\alpha\Big)^{1/5} \ll U_3^2 V_3^2 L^{k-1}.\end{aligned}$$

This completes the proof of Proposition 3.6. □

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PROPOSITION 3.7 (The estimation over $\mathfrak{m}\backslash\mathcal{E}(\lambda)$). *We have*

$$\int_{\mathfrak{m}\backslash\mathcal{E}(\lambda)} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha \le 6.4894513U_3^2V_3^2\lambda^kL^k.$$

PROOF. By Lemma 2.3(iii),

$$\int_{\mathfrak{m}\backslash\mathcal{E}(\lambda)} F^2(\alpha)S^2(\alpha)T^2(\alpha)H^k(\alpha)e(-\alpha N)\,d\alpha \le (\lambda L)^k\Big(\int_0^1 |F(\alpha)S(\alpha)T(\alpha)|^2\,d\alpha\Big)$$
$$\le 6.4894513U_3^2V_3^2\lambda^kL^k. \qquad \square$$

4. Proof of Theorem 1.1

On combining Propositions 3.4, 3.6 and 3.7,

$$R(N) \ge (0.1552042 - 6.4894513\lambda^k)U_3^2V_3^2L^k.$$

When $k \ge 28$ and $\lambda = 0.8709277$,

$$R(N) > 0$$

for all sufficiently large even integers N. This completes the proof of the theorem.

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