DISTRIBUTIVE SUBLATTICES OF A FREE LATTICE

FRED GALVIN AND BJARNI JÓNSSON

The purpose of this note is to characterize those distributive lattices that can be isomorphically embedded in free lattices. If it is known (cf. **(2)**) that in a free lattice every element is either additively or multiplicatively irreducible, and consequently every sublattice of a free lattice must also have this property. We therefore begin by studying the class of all those distributive lattices in which this condition is satisfied.

The notion of a linearly indecomposable lattice will play a fundamental role in these investigations. Given two non-empty subsets B and C of a partially ordered set A, we write $B \leqslant C$ if and only if either $B = C$ or else $b < c$ whenever $b \in B$ and $c \in C$. It is obvious that under this relation the non-empty subsets of A form a partially ordered set. A lattice A is said to be linearly indecomposable if there do not exist sublattices B and C of A such that $A = B \cup C$ and $B < C$. Clearly every lattice A is the union of a unique linearly ordered family $\mathscr{C}$ of linearly indecomposable lattices. Furthermore, A is distributive if and only if each member of $\mathscr{C}$ is distributive, and in order for A to have the property that each of its elements is either additively or multiplicatively irreducible it is necessary and sufficient that each member of $\mathscr{C}$ have this property. We therefore need only consider the case of a linearly indecomposable lattice.

LEMMA 1. *Suppose D is a distributive lattice with the property that every element of D is either additively or multiplicatively irreducible. If the elements $x_1, x_2, x_3 \in D$ are such that no two of them are comparable, then they generate an eight-element Boolean algebra.*

Proof. Since the element

$$(x_2 + x_3)(x_3 + x_1)(x_1 + x_2) = x_2x_3 + x_3x_1 + x_1x_2$$

cannot be both additively and multiplicatively reducible, either one of the factors on the left must be contained in the other two factors, or else one of the summands on the right must contain the other two summands. By symmetry and duality we may assume that x_2x_3 and x_3x_1 are contained in x_1x_2, so that

$$x_2x_3 = x_3x_1 \leqslant x_1x_2. \tag{1}$$

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Considering the element

$$(x_1 + x_3)(x_2 + x_3) = x_1x_2 + x_3,$$

we see that one of the following four inclusions must hold:

$$(2) \qquad x_1 + x_3 \leqslant x_2 + x_3, \qquad x_2 + x_3 \leqslant x_1 + x_3, \qquad x_1x_2 \geqslant x_3, \quad x_3 \geqslant x_1x_2.$$

If the first inclusion holds, then $x_1 = x_1x_2 + x_1x_3$, and it follows by (1) that $x_1 = x_1x_2 \leqslant x_2$, contrary to our hypothesis that x_1, x_2, x_3 be incomparable. Similarly the second inclusion in (2) leads to a contradiction, and obviously so does the third. Finally, if $x_3 \geqslant x_1x_2$, then the inclusion in (1) can be replaced by an equality, and it follows that x_1, x_2, and x_3 are the atoms of a Boolean algebra with eight elements.

LEMMA 2. *Suppose D is a linearly indecomposable distributive lattice with the property that every element of D is either additively or multiplicatively irreducible. Then the width of D is at most* 3. *Furthermore, if the width of D is* 3, *then D is a Boolean algebra with eight elements.*

Proof. By Lemma 1, if the width of D is 3 or more, then D contains as a sublattice a Boolean algebra B with eight elements. Let z and u be the zero and the unit of B. We shall show that if d is an element of D which does not belong to B, then either $d < z$ or $d > u$.

First observe that if p is an atom in B, then there exists no element $d \in D$ such that $z < d < p$. In fact, if such an element d exists, and if q and r are the other two atoms of B, then the element

$$q + d = (q + p)(q + d + r)$$

is both additively and multiplicatively reducible.

Now consider any element d of D and let

$$p' = z + pd, \qquad q' = z + qd, \qquad \text{and} \qquad r' = z + rd$$

where p, q, and r are the atoms of B. Then $z \leqslant p' \leqslant p$, hence $p' = z$ or $p' = p$. Since p is multiplicatively reducible (in B and therefore also in D), it must be additively irreducible. It follows that if $p' = p$, then $p = pd \leqslant d$. Similarly, either $q' = z$ or else $q \leqslant d$, and either $r' = z$ or $r \leqslant d$. By symmetry we need only consider four out of the eight cases that may arise.

If $p' = q' = r' = z$, then $u(z + d) = z$. Hence $d \leqslant z$, for otherwise the element

$$p + d = (p + d + q)(p + d + r)$$

would be both additively and multiplicatively reducible.

If $p \leqslant d$ and $q' = r' = z$, then $(q + d)u = q + du = q + p$. Since $q + p$ is multiplicatively irreducible it follows that

$$q + p = q + d, \qquad d = d(q + d) = d(q + p) = dq + dp = p.$$

If $p \leqslant d$, $q \leqslant d$, and $r' = z$, then $ud = p + q$, hence $d = p + q$.
If $p \leqslant d$, $q \leqslant d$, and $r \leqslant d$, then $u \leqslant d$.

Thus we see that if d is not an element of B, then either $d < z$ or else $d > u$.

The element z is multiplicatively reducible and must therefore be additively irreducible, whence it follows that if there exist elements $d \in D$ with $d < z$, then the set A consisting of all these elements must be a sublattice of D. The set $C = D - A$ is precisely the set of all elements $d \in D$ with $z \leqslant d$, and we therefore have $A < C$. We therefore see that if A were non-empty, then D would not be linearly indecomposable as required by the hypothesis. Similarly, the assumption that there exists $d \in D$ with $u < d$ leads to a contradiction, and we conclude that $D = B$.

LEMMA 3. *Suppose D is a linearly indecomposable distributive lattice with the property that every element of D is either additively or multiplicatively irreducible. If the width of D is 2, then D is isomorphic to a direct product of two chains, one of which has exactly two elements.*

Proof. We consider two cases depending on whether D does or does not have a zero element. In each case the proof will be divided into several parts.

Case I. D has a zero element z.

Statement Ia. There exists an atom p of D which is multiplicatively irreducible

Proof. The zero element z must be multiplicatively reducible, for otherwise the set $D - \{z\}$ would be a sublattice of D, and D would not be linearly indecomposable. Thus there exist $p, q \in D$ such that $z = pq$, $z < p$, and $z < q$. If neither p nor q were an atom, then there would exist $x, y \in D$ such that $z < x < p$ and $z < y < q$, and the elements p, q, and $x + y$ would be incomparable, which is impossible because the width of D is only 2. We may therefore assume that p is an atom.

If p is multiplicatively reducible, $p = ab$ with $p < a$ and $p < b$, then two of the three elements a, b, and q must be comparable. Since ab is properly contained in a and in b, a and b cannot be comparable, and since p is not contained in q, neither a nor b can be contained in q. Therefore either a or b must contain q, and we can assume that $q \leqslant a$.

For any $x \in D$ with $z < x \leqslant q$ we have $x + p = a(x + b)$. Now $x < x + p$ and $p < x + p$. Also, the equality $x + p = x + b$ is excluded because it would imply that $b \leqslant x + b = x + p \leqslant a$. We must therefore have $x + p = a$, $q \leqslant x + p$, $q = x + pq = x + z = x$. Thus q is an atom of D.

If q is also multiplicatively reducible, $q = cd$ with $q < c$ and $q < d$, then p is contained in either c or d, say $p \leqslant c$. Observe that b does not contain q, and therefore contains neither d nor $p + q$. Similarly, d contains neither b nor $p + q$. Furthermore, $b(p + q) = p < b$ and $d(p + q) = q < d$, so that

$p + q$ contains neither b nor d. Consequently b, d, and $p + q$ are incomparable. This contradicts our hypothesis, and we conclude that either p or q must be a multiplicatively irreducible atom.

Statement Ib. If p is a multiplicatively irreducible atom of D, then the set

$$C = \{x \mid x \in D \quad \text{and} \quad px = z\}$$

is a chain and an ideal of D, and D is the inner direct product of C and of the two-element chain $C' = \{z, p\}$.

Proof. Clearly C is an ideal of D, and if $x, y \in C$, then either $x \leqslant y$ or $y \leqslant x$, because otherwise the three elements x, y, and p would be incomparable. Since C and C' are ideals of D and have only the zero element in common, their inner direct product $A = C' \times C$ exists and is an ideal of D. The proof will be completed by showing that if the set $B = D - A$ were non-empty, then B would be a sublattice of D and $A < B$.

Given $x \in B$ we have $x \notin C$, whence $px \neq z$, and thus $p < x$. For all $y \in C$ we have $x(y + p) = xy + p$, whence it follows that $p \leqslant xy$ or $xy \leqslant p$ or $x \leqslant y + p$ or $y + p \leqslant x$. The first case is excluded because $py = z < p$, and the third case is ruled out because it would yield $x = xy + p \in A$. The case $xy \leqslant p$ yields $xy = z$, $x(y + p) = p$, and since p is multiplicatively irreducible it follows that $y + p = p$, $y \leqslant p$, $y = z$, $y + p = p < x$. Finally, in the last case the equality $y + p = x$ is ruled out since $y + p \in A$. Thus $p + y < x$ whenever $x \in B$ and $y \in C$, whence it follows that $A < B$.

Clearly, if $x_1 \in B$ and $x_1 \leqslant x_2$, then $x_2 \in B$. To show that B is a sublattice of D it is therefore sufficient to show that if $x_1, x_2 \in B$, then $x_1x_2 \in B$. If this fails, then $x_1x_2 \in A$. Since every member of B contains p, we have $p \leqslant x_1x_2$, and therefore $x_1x_2 = p + y$ for some $y \in C$. But since x_1x_2 is multiplicatively reducible, and is therefore additively irreducible, it follows that $y = z$, $p = x_1x_2$. However, this is excluded because p is multiplicatively irreducible.

The next statement will be needed in the treatment of Case II below.

Statement Ic. The set C in Ib consists of all the additively irreducible elements of D, except the element p.

Proof. Since C is a chain, every element of C is additively irreducible in C, and since C is an ideal of D, it follows that every element of C is additively irreducible in D. On the other hand, if $a \in D$, $a \notin C$, and $a \neq p$, then $a = p + y$ for some $y \in C$, and therefore a is additively reducible.

Case II. D does not have a zero element.

Statement IIa. If $z \in D$ is multiplicatively reducible, then the dual ideal generated by z is linearly indecomposable.

Proof. There exist $a, b \in D$ such that $z = ab$, $z < a$, and $z < b$. Let D_z be the dual ideal generated by z, and suppose there exist sublattices A and B of D_z such that $D_z = A \cup B$ and $A < B$. Clearly $a, b \in A$. If $x \in D$ and $x \notin D_z$, then $x \leqslant a$ or $x \leqslant b$, for otherwise the elements a, b, and x would be incomparable. It readily follows that if A' is the set of all those elements $x \in D$ which are contained in some member of A, then $D = A' \cup B$ and $A' < B$, contrary to our hypothesis.

Statement IIb. Every element of D contains a multiplicatively reducible element.

Proof. If $x \in D$ is not itself multiplicatively reducible, then either x is the largest element of D, or else there exists $y \in D$ such that x and y are incomparable, for otherwise D would be the union of the two sublattices

$$A = \{y \mid x \geqslant y \in D\} \qquad \text{and} \qquad B = \{y \mid x < y \in D\}$$

with $A < B$. Thus $xy \leqslant x$, and xy is multiplicatively reducible.

Statement IIc. The set A consisting of all the additively irreducible elements of D is a chain, and every member of A is covered by a unique member of $D - A$.

Proof. Suppose $a, b \in A$. Since D does not have a zero element, there exist $x, y \in D$ such that $x < y < ab$, and by IIb there exists a multiplicatively reducible element z with $z \leqslant x$. Let D_z be the dual ideal generated by z. In view of IIa we can apply Ia, b, c with D replaced by D_z. Let p and C be as in Ib. Then $a \neq p \neq b$ because a and b do not cover z, and it follows by Ic that $a, b \in C$. Since C is a chain, we conclude that $a \leqslant b$ or $b \leqslant a$. Thus A is a chain. Finally, by Ib, a is covered by $p + a$ and by no other additively reducible element.

Statement IId. Let A be the set consisting of all the additively irreducible elements of D, and for each $a \in A$ let a' be the unique member of $D - A$ that covers a. Then the mapping $\langle 0, a\rangle \to a$, $\langle 1, a\rangle \to a'$ is an isomorphism of the outer direct product $\{0, 1\} \times A$ onto D.

Proof. For each multiplicatively reducible element z of D let D_z be the dual ideal generated by z and let $A_z = A \cap D_z$. In view of IIa we may apply Ia, b, c with D replaced by D_z. Observe that $p = z'$ satisfies the hypothesis of Ib, and denote by C_z the corresponding set C defined in Ib. Clearly $A_z \subseteq C_z$.

If z_0 and z_1 are multiplicatively irreducible elements of D with $z_0 \leqslant z_1$, then we see by IIb that $z_1' = z_0' + z_1$ and $z_0 = z_0' z_1$, and hence that $C_{z_1} \subseteq C_{z_0}$. Now suppose z is multiplicatively reducible, $a \in D_z$ and $a \notin A_z$. Then there exist $b, c \in D$ such that $a = b + c$, $b < a$, and $c < a$. We can then find a multiplicatively reducible element z_0 with $z_0 \leqslant bc$. Then a is additively reducible in D_{z_0} so that $a \notin C_{z_0}$. Consequently $a \notin C_z$. Thus we see that $C_z \subseteq A_z$, hence $A_z = C_z$.

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By Ib, the mapping $\langle 0, a\rangle \to a$, $\langle 1, a\rangle \to a' = z' + a$ is an isomorphism of $\{0, 1\} \times C_z$ onto D_z. The lattices $\{0, 1\} \times C_z$ form a chain whose union is $\{0, 1\} \times A$, and the lattices D_z form a chain whose union is D. Consequently the indicated mapping is an isomorphism of $\{0, 1\} \times A$ onto D.

THEOREM 4. *For any distributive lattice D the following conditions are equivalent:*

(i) *Every element of D is either additively or multiplicatively irreducible.*

(ii) *D is the union of a linearly ordered family $\mathscr{C}$ of sublattices such that each member of $\mathscr{C}$ is either a one-elememt lattice or an eight-element Boolean algebra, or else is isomorphic to a direct product of two chains, one of which consists of exactly two elements.*

Proof. As we observed in the introduction, D is the union of a simply ordered family of linearly indecomposable sublattices. That (i) implies (ii) therefore follows from Lemmas 2 and 3, together with the obvious observation that a lattice of width 1 (a chain) is linearly indecomposable if and only if it consists of just one element.

Conversely, it is easy to show that under the hypothesis of (ii) each member C of $\mathscr{C}$ has the property that every element of C is either additively or multiplicatively irreducible, whence it follows that D also has this property.

LEMMA 5. *Every simply ordered subset of a free lattice is denumerable.**

Proof. Let F be a free lattice generated by a set X. The alternative case being trivial, we assume that X is non-denumerable. Let X_0 be a denumerably infinite subset of X, and let F_0 be the sublattice of F generated by X_0.

For $a, b \in F$ write $a \equiv b$ if and only if there exists an automorphism f of F such that $f(a) = b$. Clearly $\equiv$ is an equivalence relation over F. For each $a \in F$ there exists a finite subset Y of X such that a belongs to the sublattice of F generated by Y. We can find a permutation p of X which maps Y into X_0, and p can be extended to an automorphism f of F. Consequently $a \equiv f(a) \in F_0$. Thus every equivalence class modulo $\equiv$ contains a member of F_0. The number of equivalence classes must therefore be denumerable, and the proof will be completed if we show that no simply ordered subset of F contains more than one element from any one equivalence class. That is, it suffices to show that if $a \equiv b$ and $a \leqslant b$, then $a = b$.

Suppose f is an automorphism of F such that $f(a) = b$. There exists a finite subset Y of X such that a belongs to the sublattice of F generated by Y. If Z is the image of Y under f, then there exists a permutation p of X such that $p(x) = f(x)$ whenever $x \in Y$, and $p(x) = x$ whenever $x \in X - (Y \cup Z)$. If g is the autmorophism of F such that $g(x) = p(x)$ whenever

*A somewhat more involved argument can be used to show that if F is a free lattice and if Y is a subset of F with $\aleph_\alpha$ elements, where $\aleph_\alpha$ is a non-denumerable, regular cardinal, then Y contains a subset Z with $\aleph_\alpha$ elements such that Z generates a free sublattice of F.

$x \in X$, then $g(a) = f(a) = b$, and g is of some finite order n. If now $a \leqslant b$, then

$$a \leqslant g(a) \leqslant g^2(a) \leqslant \ldots \leqslant g^n(a) = a,$$

hence $a \leqslant b \leqslant a$, hence $a = b$. This completes the proof.

THEOREM 6. *For any distributive lattice D the following conditions are equivalent:*

(i) *D is isomorphic to a sublattice of a free lattice.*

(ii) *D is isomorphic to a sublattice of a free lattice with three generators.*

(iii) *D is denumerable, and every element of D is either additively or multiplicatively irreducible.*

(iv) *D is the union of a denumerable, linearly ordered family $\mathscr{C}$ of sublattices where each member of $\mathscr{C}$ is either a one-element lattice or an eight-element Boolean algebra, or else is isomorphic to a direct product of a two-element chain and a denumerable chain.*

Proof. Clearly (ii) implies (i) and, as we observed in the introduction, (i) implies that every element of D is either additively or multiplicatively irreducible. Using Theorem 4 and Lemma 5, we therefore see that (i) implies that D is denumerable. Thus (i) implies (iii). Since (iii) and (iv) are equivalent by Theorem 4, it remains only to prove that (iv) implies (ii).

If F is a free lattice generated by x, y, and z, then it is easy to check that the elements yz, zx, and xy generate an eight-element Boolean algebra. Also, F contains as a sublattice a free lattice F' with five generators x_0, x_1, x_2, x_3, x_4. If C is a denumerable chain, then there exists an isomorphism f of C into the sublattice generated by x_2, x_3, and x_4. Defining the mapping g of $A = \{0, 1\} \times C$ into F' by the conditions

$$g(\langle 1, c\rangle) = x_0 + x_1 f(c), \qquad g(\langle 0, c\rangle) = (x_0 + x_1 f(c))x_1,$$

for all $c \in C$, we shall see that g is an isomorphism of A into F'.

Let h be the endomorphism of F' such that

$$h(x_0) = 0, \quad h(x_1) = 1, \quad \text{and} \quad h(x_i) = x_i \quad \text{for} \quad i = 2, 3, 4.$$

Then

$$hg(\langle 1, c\rangle) = f(c) = hg(\langle 0, c\rangle)$$

for all $c \in C$. Consequently g is one-to-one on the set of elements of the form $\langle 1, c\rangle$, and also on the set of elements of the form $\langle 0, c\rangle$. Furthermore, if $c, c' \in C$, then $g(\langle 0, c\rangle) \leqslant x_1$ and $g(\langle 1, c'\rangle) \not\leqslant x_1$, so that $g(\langle 0, c\rangle) \neq g(\langle 1, c'\rangle)$. Thus g is one-to-one.

If $c, c' \in C$ and $c \leqslant c'$, then it is easy to check that

$$g(\langle 1, c\rangle) + g(\langle 0, c'\rangle) = g(\langle 1, c'\rangle),$$
$$g(\langle 1, c\rangle)g(\langle 0, c'\rangle) = g(\langle 0, c'\rangle),$$

and since g is obviously order-preserving, it follows that g is an isomorphism.

Thus we see that, under the hypothesis of (iv), every member of $\mathscr{C}$ is isomorphic to a sublattice of a free lattice with three generators, and we conclude by (**1**, Theorem 2.4) that (ii) holds. This completes the proof.

References

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University of Minnesota