On the application of the operational calculus to the expansion of a function in a series of Legendre's functions of the second kind

By D. P. Banerjee.

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In the present note we shall obtain the expansion in a series of Legendre functions of the second kind of an integral function $\phi(\omega)$ represented by Laplace's integral

$$\phi(\omega) = \int_0^\infty e^{-\omega x} f(x)\, dx, \tag{1}$$

where $f(x)$ is an analytic function of x, regular in the circle $|x| < a + \delta$ $(\delta > 0)$. The expansion is $\phi(\omega) = \sum_0^\infty a_n\, q_n(\omega)$ (2)

where a_n are constants and $q_n(\omega) = i^{n+1} Q_n(i\omega)$.

From our assumptions it follows that[1]

$$\left(\frac{2x}{\pi}\right)^{\frac{1}{2}} f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!}\, x^{n+\frac{1}{2}} = \sum_0^\infty a_n\, J_{n+\frac{1}{2}}(x)$$

is convergent for $|x| \leqq a$. (3)

Substituting from (3) in (1), and assuming term-by-term integration to be permissible, we have[2]

$$\phi(\omega) = \sum_0^\infty a_n \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-\omega x}\, J_{n+\frac{1}{2}}(x)\, x^{-\frac{1}{2}}\, dx = \sum_0^\infty a_n\, q_n(\omega).$$

The reader may find it interesting to determine the various series in a number of special cases. It is easy to verify, for example, that

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[1] Watson, *Theory of Bessel functions* (Cambridge, 1922), § 16, 11 (3).

[2] Nicholson, "Electrification of parallel circular discs." *Phil. Trans. Royal Soc.* (A), 224 (1924), 315.

$$\frac{1}{\omega^m} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2^{m-\frac{1}{2}}}{(m-1)!} \sum_{s=0}^{\infty} \frac{(m - \frac{1}{2} + 2s)\, \Gamma\,(m + s - \frac{3}{2})}{s\,!}\, q_{m-1+2s}(\omega) \tag{4}$$

$$= \frac{2^m}{\pi^{\frac{1}{2}}\,(m-1)!} \sum_{s=0}^{\infty} \frac{(m - \frac{1}{2} + 2s)\, \Gamma\,(m + s - \frac{3}{2})}{s\,!}\, (-1)^s\, Q_{m-1+2s}(\omega).$$

$$\frac{1}{(\omega + a)^m} = \frac{2^m\, \Gamma\,(m - \frac{3}{2})}{\pi^{\frac{1}{2}}\,(m-1)!} \sum_{s=0}^{\infty} (m - \tfrac{1}{2} + s)\, C_s^{m-\frac{1}{2}}(a)\, (-1)^s\, Q_{m-1+s}(\omega). \tag{5}$$

$$\frac{\Gamma\,(n + \frac{1}{2})}{2\,(1 + \omega^2)^{n+\frac{1}{2}}} = \sum_{s=0}^{\infty} \frac{(2n + \frac{1}{2} + 2s)}{s\,!}\, \frac{\Gamma\,(2n + \frac{1}{2} + s)\, \Gamma\,(\frac{1}{2} - n)}{\Gamma\,(n + s)\, \Gamma\,(\frac{1}{2} - n - s)}\, q_{2n+2s}(\omega). \tag{6}$$

ANANDAMOHAN COLLEGE,
MYMENSINGH, BENGAL.

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