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ON CONFORMALLY FLAT SPACES WITH COMMUTING CURVATURE AND RICCI TRANSFORMATIONS

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Let (M, g) be a C^∞ Riemannian manifold and A be the field of symmetric endomorphisms corresponding to the Ricci tensor S; that is,

$$S(X, Y) = g(AX, Y).$$

We consider a condition weaker than the requirement that A be parallel ($\nabla A = 0$), namely, that the "second exterior covariant derivative" vanish ($\nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A = 0$), which by the classical interchange formula reduces to the property

$$(P) \qquad R(X, Y) \circ A = A \circ R(X, Y),$$

where $R(X, Y)$ is the curvature transformation determined by the vector fields X and Y.

The property (P) is equivalent to

$$(Q) \qquad R(AX, X) = 0.$$

To see this we observe first that a skew symmetric and a symmetric endomorphism commute if and only if their product is skew symmetric. Thus we have

$$\begin{aligned}(P) &\Leftrightarrow R(Z, W)A \text{ is skew symmetric}\\ &\Leftrightarrow g(R(Z, W)AX, X) = 0\\ &\Leftrightarrow g(R(AX, X)Z, W) = 0\\ &\Leftrightarrow (Q).\end{aligned}$$

Let M be a connected conformally flat manifold of dimension n, $n \geqq 3$. Then the Ricci endomorphisms determine the curvature according to the formula

$$(1) \qquad R(X, Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y,$$

where $r = \operatorname{trace} A$ and $X \wedge Y$ denotes the endomorphism

$$Z \rightarrow g(Y, Z)X - g(X, Z)Y.$$

In this paper the connected conformally spaces satisfying (P) are classified.

LEMMA 1. *Let M be an n-dimensional conformally flat space satisfying (P).*

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Then

$$A^2 - \frac{r}{n-1} A = \rho I, \tag{2}$$

where ρ is a C^∞ function on M and I is the identity field.

Proof. Setting $Y = AX$ in (1) and then applying (Q) gives

$$BX \wedge X = 0, \tag{3}$$

where

$$B = A^2 - \frac{r}{n-1} A.$$

Since (3) may be interpreted as an exterior product, we conclude that every X is an eigenvector of B, so $B = \rho I$ for some scalar field ρ.

LEMMA 2. *Under the conditions in Lemma 1, A has at most the two eigenvalues*

$$\frac{r \pm [r^2 + 4(n-1)\rho]^{\frac{1}{2}}}{2(n-1)}.$$

Let M' be the open subset of M on which $r^2 + 4(n-1)\rho \neq 0$. Then the eigenspaces of A form smooth complementary orthogonal distributions on each connected component of M'.

The eigenvalues are the roots of

$$\mu^2 - \frac{r}{n-1}\mu - \rho = 0;$$

the rest is also routine.

Let us fix notation as follows: The eigenvalues of A are μ_1 and μ_2. They are defined and continuous on all of M and distinct on M'. The eigenspaces on M' are D_1 and D_2, of dimensions k and $n - k$. We shall use adapted orthonormal frames and coframes $\{X_a, X_\alpha\}$ and $\{\omega_a, \omega_\alpha\}$, $a, b = 1, \ldots, k$ and $\alpha, \beta = k + 1, \ldots, n$; moreover, $i, j = 1, \ldots, n$. The corresponding connection and curvature forms are ω_{ab}, etc. and Ω_{ab}, etc.

LEMMA 3. *Let $K = (\mu_1 - \mu_2)/(n - 2)$. On D_1 the sectional curvature is K, on D_2 it is $-K$, and on mixed sections it vanishes; that is,*

$$\Omega_{ab} = K\omega_a \wedge \omega_b,$$
$$\Omega_{\alpha\beta} = -K\omega_\alpha \wedge \omega_\beta,$$
$$\Omega_{a\beta} = 0.$$

Proof. Noting that $r/(n-1) = \mu_1 + \mu_2$, formula (1) becomes

$$R(X, Y) = \frac{1}{n-2}\{AX \wedge Y + X \wedge AY - (\mu_1 + \mu_2)X \wedge Y\}.$$

The rest follows by taking orthonormal eigenvectors for X and Y.

Note that M' is just the set on which $K \neq 0$.

THEOREM. *Let M be an n-dimensional connected conformally flat space satisfying (P), $n \geqq 3$. Then M is one of four types:*

(a) *M is flat (M' empty).*

In the remaining cases $M = M'$; that is, M is either flat everywhere or has no flat points; moreover, k is constant.

(b) *M has constant curvature ($k = 0$ or n).*

(c) *M is locally the Riemannian product of a k-dimensional space of constant curvature K and an $(n-k)$-dimensional space of constant curvature $-K (1 \leqq k \leqq n-1)$.*

(d) *There is an open C^∞ map $t : M \to \mathbf{R}^+$ (positive reals) such that $K = K_0/t^2$ for some constant K_0. The map t is a Riemannian submersion having fibres which are totally umbilical hypersurfaces of constant (intrinsic) curvature $(1 + K_0)/t^2$ ($k = 1$ or $n - 1$).*

Proof. Define the vector valued 1-form $F = F^i \otimes X_i$ by

$$F^i = A^i{}_j \omega^j - \frac{r}{2(n-1)} \omega^i,$$

where $A^i{}_j$ are the components of A. (The summation convention is employed here and in the sequel.) The X_i and ω^i are any local vector field basis and the dual basis of 1-forms, respectively. If $\omega^i{}_j$ are the connection forms for this basis we define the exterior covariant derivative DF of F as the vector-valued 2-form $(DF)^i \otimes X_i$, where

$$(DF)^i = dF^i + \omega^i{}_j \wedge F^j.$$

It is easily checked that DF is independent of the choice of basis. Using the first structural equation viz., $d\omega^i = -\omega^i{}_j \wedge \omega^j$, and the coefficients Γ^i_{kj} of $\omega^i{}_j (\omega^i{}_j = \Gamma^i_{kj}\omega^k)$, we obtain

$$\begin{aligned}(DF)^i &= \left(X_k A^i{}_j + A^i{}_h \Gamma^h_{kj} + A^h{}_j \Gamma^i_{kh} - \frac{1}{2(n-1)} \delta^i{}_j X_k r\right) \omega^k \wedge \omega^j \\ &= \left(\nabla_k A^i{}_j - \frac{1}{2(n-1)} \delta^i{}_j \nabla_k r\right) \omega^k \wedge \omega^j,\end{aligned}$$

where $\delta^i{}_j$ is the Kronecker delta. As a tensor, this has the components

$$(n-2) C^i{}_{jk} = \nabla_k A^i{}_j - \nabla_j A^i{}_k - \frac{1}{2(n-1)} (\delta^i{}_j \nabla_k r - \delta^i{}_k \nabla_j r),$$

where $C^i{}_{jk}$ is Weyl's 3-index tensor. For a conformally flat space it is known that $C^i{}_{jk} = 0$. We use this by calculating DF in terms of an orthonormal basis adapted to the distributions D_i. In particular we can lower all superscripts.

Thus,

$$F_a = A_{ai}\omega_i - \frac{r}{2(n-1)}\omega_a$$
$$= \mu_1\omega_a - \tfrac{1}{2}(\mu_1 + \mu_2)\omega_a$$
$$= L\omega_a,$$

where $L = (n-2)K/2$, and

$$F_\alpha = A_{\alpha i}\omega_i - \frac{r}{2(n-1)}\omega_\alpha$$
$$= \mu_2\omega_\alpha - \tfrac{1}{2}(\mu_1 + \mu_2)\omega_\alpha$$
$$= -L\omega_\alpha,$$

from which

$$\begin{aligned} dF_a &= dL \wedge \omega_a + Ld\omega_a + \omega_{ab} \wedge L\omega_b - \omega_{a\beta} \wedge L\omega_\beta \\ &= dL \wedge \omega_a - L\omega_{ai} \wedge \omega_i + L(\omega_{ab} \wedge \omega_b - \omega_{a\beta} \wedge \omega_\beta) \\ &= dL \wedge \omega_a - 2L\omega_{a\beta} \wedge \omega_\beta \\ &= 0, \end{aligned} \tag{4}$$

$$\begin{aligned} dF_\alpha &= -dL \wedge \omega_\alpha + 2L\omega_{\alpha b} \wedge \omega_b \\ &= 0. \end{aligned} \tag{5}$$

When $k = n$, K is constant and $M' = M$ follows immediately from Schur's theorem (or (4)).

Otherwise, by Cartan's lemma, (4) says that for each a, dL and the $\omega_{a\beta}$ are dependent at most on ω_a and the ω_α and (5) says that the same forms are dependent at most on ω_α and the ω_b. Thus if $2 \leqq k \leqq n-2$ we can make two choices of α for each a and vice-versa, showing that $dL = 0$ and $\omega_{a\beta} = 0$. Consequently, L and $K = 2L/(n-2)$ are constant and D_1 and D_2 are parallel (in particular, completely integrable).

When $k = 1$ we still have by (5) that dL and $\omega_{\alpha 1}$ are dependent at most on ω_α and ω_1. Making two choices of α, we get $dL = H\omega_1$ for some C^∞ function H. Then, (4) reduces to $\omega_{1\beta} \wedge \omega_\beta = 0$, so the $\omega_{1\beta}$ cannot depend on ω_1. Hence $\omega_{1\alpha} = C_\alpha\omega_\alpha$ (α not summed) for some scalar field C_α. But then by (5) again

$$-H\omega_1 \wedge \omega_\alpha + 2L(-C_\alpha\omega_\alpha) \wedge \omega_1 = 0;$$

that is, $C = C_\alpha = H/2L$ is the same for all α. The geometrical interpretation of the relation $\omega_{1\alpha} = C\omega_\alpha$ is that D_2 (the distribution annihilated by ω_1) is completely integrable and has totally umbilical leaves. In fact, $d\omega_1 = -\omega_{1\alpha} \wedge \omega_\alpha = 0$, so locally ω_1 has a primitive u; that is, $du = \omega_1$.

A differential equation for C may be obtained from the fact that the curvature of the section $X_1 \wedge X_\alpha$ vanishes:

$$\begin{aligned}\Omega_{1\alpha} &= d\omega_{1\alpha} + \omega_{1\beta} \wedge \omega_{\beta\alpha} \\ &= dC \wedge \omega_\alpha + Cd\omega_\alpha + \omega_{1\beta} \wedge \omega_{\beta\alpha} \\ &= dC \wedge \omega_\alpha - C\omega_{\alpha i} \wedge \omega_i + C\omega_\beta \wedge \omega_{\beta\alpha} \\ &= \left(\frac{dC}{du} - C^2\right) \omega_1 \wedge \omega_\alpha.\end{aligned}$$

Therefore,

$$\frac{dC}{du} - C^2 = 0.$$

Solving this, we obtain either $C = 0$ or

$$C = -\frac{1}{u - u_0} = -\frac{1}{t},$$

where u_0 is a constant and hence t is another primitive for ω_1. The signs of C and ω_1 can be changed, if necessary, so as to make $t > 0$.

If $C = 0$, then it must be so on connected sets. Hence $H = dL/du = 2LC = 0$ and L, and hence K, are constant. Moreover, $C = 0$ says D_1 and D_2 are parallel so we are back in case (c).

If $C \neq 0$, then we solve $H = 2LC$ for L, obtaining $L = L_0/t^2$, and hence $K = K_0/t^2$ for constants L_0 and K_0. Thus, $t = (K_0/K)^{\frac{1}{2}}$ is a primitive for ω_1 in each component of M'. We don't know yet whether there is only one component, so K_0 might have several values. As a map $t : M' \rightarrow \mathbf{R}^+$, t is clearly a Riemannian submersion whose fibres are the leaves of D_2. As such it is distance-non-increasing. Now suppose that $M' \neq M$. Let γ be a curve entirely in M' except for the last point $\gamma(1) \in M - M'$. The length of $t\gamma$ is at most that of γ and is therefore bounded. Hence $t\gamma(1) = \lim_{s \to 1^-} t\gamma(s)$ exists and is not ∞. It cannot be 0 either, for then there would be a sequence of plane sections converging to a section at $\gamma(1)$ and having curvatures diverging to $\lim_{t \to 0} K_0/t^2$. A similar difficulty is presented at any other finite limit for $t\gamma(1)$, since we would then have curvatures converging to nonzero values contradicting the fact that $M - M'$ is flat. Hence, $M = M'$.

To complete the proof we calculate the intrinsic curvature of the leaves of D_2. The connection forms $\omega_{\alpha\beta}$, restricted to a leaf, become the connection forms of the leaf. Thus, denoting the curvature forms of a leaf by $\Phi_{\alpha\beta}$, the second structural equation for a leaf is

$$\begin{aligned}d\omega_{\alpha\beta} &= -\omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Phi_{\alpha\beta} \\ &= -\omega_{\alpha i} \wedge \omega_{i\beta} + \Omega_{\alpha\beta} \\ &= -\omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + (C^2 + K)\omega_\alpha \wedge \omega_B.\end{aligned}$$

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Evidently the curvature forms of the leaf are

$$\Phi_{\alpha\beta} = (C^2 + K)\omega_\alpha \wedge \omega_\beta,$$

so the curvature of the leaves of D_2 is $(1 + K_0)/t^2$.

Remark. If M is complete, then the case (d) cannot occur, since the base of a complete Riemannian submersion must be complete.

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