SOME CHARACTERIZATIONS OF c_o AND ℓ^1

J.R. Retherford

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1. Introduction. The space c_o consists of the sequences tending to zero with addition and scalar multiplication defined coordinate-wise and with the sup norm. The space ℓ^1 consists of the sequences $b = (b_i)$ under coordinate-wise arithmetic for which $\|b\| = \sum_{i=1}^{\infty} |b_i| < +\infty$.

Several answers to the question

* When is a Banach space X linearly homeomorphic to c_o or ℓ^1 ?

have appeared in the literature since Banach [4] showed that c, the space of convergent sequences with the sup norm, is linearly homeomorphic to c_o.

Our purpose in this paper is to examine the question * above from the point of view of similar bases. While this point of view is certainly not new, our discussion has the advantage of being unified by the use of a theorem due to Osgood, Kuratowski and Banach. It seems, to the author, that this theorem has been undeservedly neglected.

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2. Definitions and Notation. While all of our work could be carried out over the complex field, we assume, for simplicity,

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that all of the spaces in question are over the field of real numbers.

By a Schauder basis for a Banach space X we mean a sequence (x_i) of elements of X such that for each $x \in X$ there is a unique sequence of scalars (a_i) such that

$$x = \lim_n \Sigma_1^n a_i x_i \quad . \tag{2.1}$$

If X has a Schauder basis and if $f_i(x)$ is defined by $f_i(x) = a_i$, where $x = \Sigma_1^\infty a_i x_i$ then $f_i(x_j) = \delta_{ij}$, the Kronecker delta and [4, p.107] each f_i is a continuous linear functional on X. Suppose that (x_i) and (y_i) are Schauder bases for the Banach spaces X and Y respectively. Then (x_i) and (y_i) are similar provided that

$$\{(a_i) \mid \Sigma_1^\infty a_i x_i \text{ converges}\} = \{(a_i) \mid \Sigma_1^\infty a_i y_i \text{ converges}\} . \tag{2.2}$$

It is obvious that if T is a linear homeomorphism from X onto Y, (x_i) a Schauder basis for X and $T(x_i) = y_i$, then (y_i) is a Schauder basis for Y and (x_i) and (y_i) are similar. The converse of this fact is also true (see § 3 below) and is the foundation for all our work.

The concept we need is that of an unconditional basis. A series $\Sigma_1^\infty y_i$ in a Banach space X converges unconditionally (to y) if for every permutation τ of the positive integers, $\Sigma_1^\infty y_{\tau(i)} = y$. A series $\Sigma_1^\infty y_i$ in X is subseries convergent if for each increasing sequence (n_i) of positive integers the series $\Sigma_1^\infty y_{n_i}$ converges to some element of X. It is well-known (see, e.g. [7, p.59]) that subseries convergence and unconditional convergence are equivalent in Banach spaces and that each is equivalent to the following: $\Sigma_1^\infty a_i y_i$ converges where $a_i = \pm 1$ (arbitrarily).

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A Schauder basis (x_i) for X is an unconditional basis if each expansion (2.1) converges unconditionally to x.

Let Σ denote the collection of all finite subsets of the positive integers directed by inclusion.

2.3 THEOREM. *Let (x_i) be a Schauder basis for the Banach space X.*

The following are equivalent:

(a) *The basis (x_i) is unconditional;*

(b) *there is a K such that for σ, $\sigma' \in \Sigma$, $\sigma \subseteq \sigma'$ and arbitrary scalars $(a_i)_{i \in \sigma'}$ we have*
$\| \sum_{i \in \sigma} a_i x_i \| \leq K \| \sum_{i \in \sigma'} a_i x_i \|$; *and*

(c) $\Sigma_1^\infty |f_i(x)| x_i$ *converges for each* $x = \sum_{i=1}^{\infty} f_i(x) x_i \in X$ *and there is an* $M > 0$ *such that*
$\| \Sigma_1^\infty f_i(x)\, x_i \| \leq M \| \Sigma_1^\infty |f_i(x)|\, x_i \|$, M *independent of* x.

The equivalence of (a) and (b) is due to M.M. Grinblyum [8] and the equivalence of (a) and (c) is found in the work of B.E. Veic [13].

It is readily seen that the unit vector bases (e_i) of c_o and ℓ^1 (i.e. e_i consists of all zero's except the i^{th} entry which is 1) are unconditional and thus a space linearly homeomorphic to either c_o or ℓ^1 must have an unconditional basis.

3. The Osgood-Kuratowski-Banach Theorem. With obvious intentions we refer to the following as the OKB Theorem.

3.1 THEOREM. *If X and Y are Banach spaces and (T_n) a sequence of continuous linear operators from X to Y such that $T(x) = \lim_n T_n(x)$ exist for each $x \in X$ then T is a continuous linear operator.*

Proof. The proof is merely an application of

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[12, Thm. 4.4-E, p.204-205].

Let us observe how easily the isomorphism theorem we need follows from the OKB theorem.

3.2 THEOREM. *If* (x_i) *and* (y_i) *are similar bases for Banach spaces* X *and* Y *respectively then there is a linear homeomorphism* T *from* X *onto* Y *such that* $T(x_i) = y_i$ *for each* i.

Proof. By hypothesis we may represent an aribtrary point $x \in X$ as $x = \Sigma_1^\infty f_i(x)x_i$. Define $T_n(x) = \Sigma_1^n f_i(x)y_i$ and $T(x) = \Sigma_1^\infty f_i(x) y_i$, convergence being insured by the similarity property. It is clear that each T_n is continuous, T is one-one and onto and that $\lim_n T_n(x) = T(x)$ for each $x \in X$. By the OKB theorem T is continuous. A symmetric argument shows that T^{-1} is also continuous.

For generalizations of the isomorphism theorem see [3] and [9].

3.3 DEFINITION. *A series* $\Sigma_1^\infty y_i$ *in a Banach space* X *is* w.u.c. *(weakly unconditionally convergent) if for each permutation* τ *of the positive integers and each* $f \in X^*$, $\lim_n f(\Sigma_1^n y_{\tau(i)})$ *exists.* (Observe that we do *not* require that the limit element exist.)

We now prove two well-known lemmas before proceeding to the characterizations of c_o and ℓ^1. The proofs are included to illustrate the scope of the OKB theorem.

3.4 LEMMA. *If* $\Sigma_1^\infty a_i$ *is a series of reals such that* $\Sigma_1^\infty t_i a_i$ *converges whenever* $(t_i) \in c_o$ *then* $\Sigma_1^\infty |a_i| < +\infty$.

Proof. For $(t_i) \in c_o$ let $t_i' = |t_i a_i|/a_i$ if $a_i \neq 0$, 0 if $a_i = 0$. The $(t_i') \in c_o$ and so by hypothesis

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$\Sigma_1^\infty |t_i a_i| = \Sigma_1^\infty t_i' a_i < +\infty$. Thus we may define $T = c_o \to \ell^1$ by $T(t) = (t_i a_i)$ where $t = (t_i) \in c_o$. Also define $T_n(t)$ by $T_n(t) = (t_1 a_1, \ldots, t_n a_n, 0, 0, \ldots) \in \ell^1$. Clearly each T_n is continuous and $\lim_n T_n(t) = T(t)$ for each $t \in c_o$. Thus by the OKB theorem T is continuous. Thus for $t^{(n)} = (\underbrace{1, 1, 1, \ldots 1}_{n \text{ terms}}, 0, 0, 0, \ldots) \in c_o$ we have

$$\Sigma_1^n |a_i| = \|T(t^{(n)})\| \leq \|T\| \; \|t^{(n)}\| = \|T\| ,$$

whence $\Sigma_1^\infty |a_i| < +\infty$.

3.5 LEMMA. (see [5, p.159]). The following conditions on a series $\Sigma_1^\infty x_n$ in a Banach space are equivalent:

(i) $\Sigma_1^\infty x_n$ <u>is</u> w.u.c. ;

(ii) <u>there is a constant</u> C <u>such that for every bounded real sequence</u> (b_n) <u>the inequality</u>

$\sup_n \|\Sigma_1^n b_i x_i\| \leq C \sup_i |b_i|$ <u>holds; and,</u>

(iii) <u>for every</u> $(t_i) \in c_o$ <u>the series</u> $\Sigma_1^\infty t_i x_i$ <u>converges.</u>

<u>Proof.</u> (i) $\to$ (ii) : For each n define $T_n : X^* \to \ell^1$ by $T_n(f) = (f(x_1)), f(x_2), \ldots, f(x_n), 0, 0, \ldots)$ and define $T(f) = (f(x_i))$, for each $f \in X^*$. By (i) $\Sigma_1^\infty |f(x_i)| < +\infty$ and so T is well defined. Clearly each T_n is continuous and $\lim_n T_n(f) = T(f)$ for each $f \in X^*$. By the OKB theorem T is continuous. Let $C = \|T\|$. If (b_i) is a bounded sequence of reals we have

$$\|\Sigma_1^n b_i x_i\| = \sup \{|f(\Sigma_1^n b_i x_i)| : \|f\| \leq 1\}$$
$$\leq \sup_i |b_i| ,$$

i.e. (ii) holds.

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(ii) → (iii) : This implication is obvious.

(iii) → (i) : This implication follows trivially from lemma 3.4.

4. Characterizations of c_o. In view of theorem 3.2, in order for a Banach space X to be linearly homeomorphic to c_o or ℓ^1 it is necessary and sufficient that X have a basis similar to the unit vector basis (e_i) of c_o or ℓ^1. Thus all of the characterizations below place conditions on a Schauder basis (x_i) for X which forces (x_i) to be equivalent to the unit vector basis.

4.1 THEOREM (see [5, Lemma 3, p. 160]) : If (x_n) is a Schauder basis for X and if $\inf_n \|x_n\| > 0$ and $\Sigma_1^\infty x_n$ is w.u.c. then (x_n) is similar to the unit vector basis of c_o.

Proof. By lemma 3.5 (iii), $\Sigma_1^\infty t_i x_i$ converges for every $(t_i) \epsilon c_o$. Also, since $\inf_n \|x_n\| > 0$, if $\Sigma_1^\infty t_n x_n$ converges then $(t_n) \epsilon c_o$. Thus (x_n) and (e_n) are similar.

4.2 DEFINITION. A basis (x_i) for a Banach space X is of type P if and only if $\inf_n \|x_n\| > 0$ and $\sup_n \|\Sigma_1^n x_n\| < +\infty$. This leads to the second characterization of c_o.

4.3 THEOREM (see [11, p. 358]) : If a Banach space X has an unconditional basis (x_n) of type P then (x_n) is similar to the unit vector basis of c_o.

Proof. Let $K_1 = \sup_n \| \Sigma_1^n x_i \|$. By Theorem 2.3 (b) there is a K such that $\| \sum_{i \epsilon \sigma} t_i x_i \| \leq K \| \sum_{i \epsilon \sigma'} t_i x_i \|$ for arbitrary t_i, $i \epsilon \sigma'$ where σ, $\sigma' \epsilon \Sigma$, $\sigma \leq \sigma'$. Thus, if n_σ is chosen so that $\sigma \subset \{1, 2, 3, \ldots, n_\sigma\}$ then

$$\| \sum_{i \epsilon \sigma} x_i \| \leq K \| \Sigma_1^{n_\sigma} x_i \| \leq K K_1 .$$ Thus

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(4.4) $$\sup_{\sigma\in\Sigma} \left\| \sum_{i\in\sigma} x_i \right\| \le K K_1 < +\infty .$$

Let (b_i) be a bounded sequence and for each positive integer n choose $f^{(n)} \in X^*$ such that

$$\|f^{(n)}\| = 1 \text{ and } \left\| \Sigma_1^n b_i x_i \right\| = f^{(n)}(\Sigma_1^n b_i x_i) .$$

Letting $\sigma_+(n) = \{i \le n \mid f^{(n)}(x_i) > 0\}$ and $\sigma_-(n) = \{i \le n \mid f^{(n)}(x_i) \le 0\}$ we obtain

$$\left\| \Sigma_1^n b_i x_i \right\| \le \sum_{i\in\sigma_+(n)} |b_i| f^{(n)}(x_i) - \sum_{i\in\sigma_-(n)} |b_i| f^{(n)}(x_i)$$

$$\le \sup_{1\le i\le n} |b_i| \, \|f^{(n)}\| \left(\left\| \sum_{i\in\sigma_+(n)} x_i \right\| + \left\| \sum_{i\in\sigma_-(n)} x_i \right\| \right)$$

$$\le 2KK_1 \sup_n |b_n| .$$

Thus by Lemma 3.5 (ii), $\Sigma_1^\infty x_i$ is w.u.c. and so by Theorem 4.1 (x_i) is similar to (e_i).

The next characterization of c_o was given, without proof, by José Abdelhay [1].

4.5 THEOREM. Suppose (x_i) is a basis for a Banach space with the following properties:

(i) $\|x_i\| = 1$, $i = 1, 2, \ldots$,

(ii) there is a constant $C > 0$ such that $\|\Sigma_1^n x_i\| \le C$, $n = 1, 2, \ldots$,

(iii) if (f_i) is the associated sequence of coefficient functionals and $|f_i(x)| \ge |f_i(y)|$ for each i then $\|x\| \ge \|y\|$; and,

(iv) for each $y \in X$ there is an $x \in X$ such that $f_i(x) = \max[0, f_i(y)]$, $i=1,2,\ldots$.

Then (x_i) is similar to the unit vector basis of c_o.

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Proof. Let $x = \Sigma_1^\infty f_i(x) x_i \in X$. By applying (iv) to both x and $-x$ we see that $\Sigma_1^\infty |f_i(x)| x_i$ converges. By (iii) we have

$$\| \Sigma_1^\infty f_i(x) x_i \| \leq \| \Sigma_1^\infty |f_i(x)| x_i \|$$

and thus by Theorem 2.3 (c) (x_n) is an unconditional basis for X. By (i) and (ii) we see that (x_n) is of type P. The result follows from Theorem 4.3.

In all the above characterizations we have assumed that (x_i) is a basis for X. We now give a characterization of (c_o) using seemingly weaker hypotheses.

4.6 DEFINITION. Let (x_i, f_i) be a biorthogonal system in a Banach space X (i.e. $(x_i) \subset X$, $(f_i) \subset X^*$, $f_i(x_j) = \delta_{ij}$). The system (x_i, f_i) is of type Y if

(i) $X = [x_i]$, the closed linear span of (x_i);

(ii) there is an $M > 0$ such that $\|f_i\| \leq M$ for each i and

(iii) there is a constant $\nu > 0$ such that $\sup_n |f_n(s)| \geq \nu$ for each $s \in S = \{x \in X \mid \|x\| = 1\}$.

S. Yamazaki [14] showed that if X admits a biorthogonal system of type Y then X must be non-reflexive. The following theorem shows that much more is true.

4.7 THEOREM. *If X admits a biorthogonal system (x_i, f_i) of type Y then (x_i) is a basis for X similar to the unit vector basis of c_o.*

Proof. For each $x \in X$ let $|||x||| = \sup |f_n(x)|$. From (ii) and (iii), $|||x||| \leq M \|x\|$ and $|||x||| \geq \nu \|x\|$. Thus X is linearly homeomorphic with the space of all sequences $\{f_i(x)\}$, $x \in X$, with the c_o norm. The set of all finite linear combinations of members of (x_n) corresponds to the dense subset of c_o consisting of the finitely non-zero sequences. This

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implies X is linearly homeomorphic with c_o and that (x_n) is similar to (e_n).

5. Characterizations of ℓ^1. We give three characterizations of ℓ^1.

5.1 THEOREM (see [6, p. 165]). *If (x_n) is an unconditional basis for a Banach space X and if $\sup_n \|x_n\| < +\infty$ and if there is an $f \in X^*$ such that $\inf_n |f(x_n)| > 0$ then (x_n) is similar to the unit vector basis of ℓ^1.*

Proof. If $(a_i) \in \ell^1$ then, since $\sup_n \|x_n\| < \infty$, $\Sigma_1^\infty a_i x_i$ converges. On the other hand if $\Sigma_1^\infty a_i x_i$ converges it converges unconditionally and thus for any $g \in X^*$, $\Sigma_1^\infty |g(a_i x_i)| < +\infty$. Thus for the f in the hypotheses we have $\Sigma_1^\infty |a_i| \, |f(x_i)| < +\infty$ and since $\inf_n |f(x_n)| > 0$ we infer that $\Sigma_1^\infty |a_i| < +\infty$. Thus (x_n) and (e_n) are similar.

5.2 DEFINITION. A basis (x_i) for a Banach space X is of type P^* if and only if $\sup_n \|x_n\| < +\infty$ and $\sup_n \|\Sigma_1^n f_i\| < +\infty$, where (f_i) is the associated sequence of coefficient functionals.

Let us recall two facts from the theory of linear topological spaces.

(i) Let E be a linear topological space and B a convex circled compact subset of E. Let F be a family of continuous linear functionals on E. Then there is a point $x_o \in B$ with $f(x_o) = 1$ for all $f \in F$ if and only if

(5.3) $$|\Sigma_1^n a_i| \leq \sup_{x \in B} |\Sigma_1^n a_i f_i(x)|$$

for arbitrary $f_1, \ldots, f_n$ in F and arbitrary scalars $a_1, \ldots, a_n$; and

(ii) If E is a locally convex space and C an equicontinuous subset of E^* then the $w(E^*, E)$ - closed convex circled extension

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of C is $w(E^*, E)$ - compact. (For (i) see [10, p. 151] and for (ii) see [10, p. 170].)

5.4 THEOREM. (see [11, p. 358]). *If (x_n) is an unconditional basis of type P^* for a Banach space X then (x_n) is similar to the unit vector basis of ℓ^1.*

Proof. If (f_i) is the sequence of coefficient functionals for (x_i) then, since by hypothesis $(\Sigma_1^n f_i)$ is norm-bounded, $(\Sigma_1^n f_i)$ is equicontinuous. Thus by (ii) above, C, the $w(X^*, X)$ - closed convex circled extension of $(\Sigma_1^n f_i)$ is $w(X^*, X)$ - compact. Thus by (i) above there is an $f_o \in C$ such that $f_o(x_n) = 1$ for each n ((5.3) holds for
$\sup_{f \in C} |\Sigma_1^n a_i x_i(f)| \geq |\Sigma_1^n a_i x_i(\Sigma_1^n f_j)| = |\Sigma_1^n a_i|$). The result now follows from Theorem 5.1.

Our last result is a dual to Theorem 4.7.

5.5 DEFINITION. A biorthogonal system (x_i, f_i) in a Banach space X is of *type* Y^* if and only if

(i) $X = [x_i]$;

(ii) there is an $M > 0$ such that $\|x_i\| \leq M$ for all i, and,

(iii) there is a constant $\delta > 0$ such that $\sup_n |f(x_n)| \geq \delta$ for each $f \in S^* = \{f \in X^* \mid \|f\| = 1\}$.

5.6 THEOREM. *If X admits a biorthogonal system of type Y^* then (x_i) is a basis for X similar to the unit vector basis of ℓ^1.*

Proof. For $x \in X$ let $|||x|||$ denote the formal sum $\Sigma_1^\infty |f_n(x)|$. For an arbitrary *finite* linear combination of (x_n), say $x = \Sigma_1^q f_n(x) x_n$, we have $\|x\| \leq M \; \Sigma_1^q |f_n(x)| \leq M \; |||x|||$.

For this same x, $x \neq 0$, let $f = \Sigma_1^q \dfrac{\mathrm{sgn}[f_n(x)]}{\Sigma_1^q |f_n(x)|} f_n$.

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Let $g = \dfrac{\Sigma_1^q \operatorname{sgn}[f_n(x)]f_n}{\|\Sigma_1^q \operatorname{sgn}[f_n(x)]f_n\|}$. From 5.5 (iii) we have

$\sup_n |g(x_n)| \geq \delta$ whence $\|\Sigma_1^q \operatorname{sgn}[f_n(x)]f_n\| \leq \frac{1}{\delta}$. Thus

$\|f\| \leq \dfrac{1}{\delta \Sigma_1^q |f_n(x)|}$. Also, $1 = f(x) \leq \|f\| \, \|x\|$ and so

$1 \leq \dfrac{\|x\|}{\delta \Sigma_1^q |f_n(x)|}$, i.e. $|||x||| \leq \frac{1}{\delta} \|x\|$. The inequalities clearly

hold if $x = 0$. Now, by 5.5 (i), the set of all finite linear combinations of members of (x_n) is dense in X. With the norm $|||\;|||$ these finite linear combinations correspond to a dense subspace of ℓ^1 whose members have only a finite number of non-zero components. From $\frac{1}{M} \|x\| \leq |||x||| \leq \frac{1}{\delta} \|x\|$ on this dense subspace we infer that X is linearly homeomorphic to ℓ^1 and that (x_n) and (e_n) are similar.

APPENDIX

We give here alternate proofs of Theorems 4.7 and 5.6. The proofs given in the main body of the text are perhaps more elegant but the following proofs seem, to the author, to be more instructive.

Again let (x_i, f_i) be a biorthogonal system and let Σ denote the collection of all finite subsets of the positive integers, ω, directed by inclusion. For $\sigma \in \Sigma$ let $L_\sigma = [x_i = i \in \sigma]$, the linear span of $\{x_i : i \in \sigma\}$, $L^\sigma = [x_i : i \in \omega \setminus \sigma]$, $S_\sigma = \{x \in L_\sigma : \|x\| = 1\}$ and $S^\sigma = \{x \in L^\sigma : \|x\| = 1\}$. Similarly, let $F_\sigma = [f_i : i \in \sigma]$, $F^\sigma = [f_i : i \in \omega \setminus \sigma]$, $T_\sigma = \{f \in F_\sigma : \|f\| = 1\}$ and $T^\sigma = \{f \in L^\sigma : \|f\| = 1\}$.

Then Theorem 2.3 (b) is easily seen to be equivalent to the following : (*) dist $(S_\sigma, S^\sigma) \geq \beta > 0$, β <u>a constant independent of</u> $\sigma \in \Sigma$.

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The proof of the following lemma is straightforward and is omitted.

LEMMA. If (x_i, f_i) is a biorthogonal system then

(i) $\|f_n\|^{-1} \leq \text{dist}\,(x_n, L^{\{n\}})$, and

(ii) $\|x_n\|^{-1} \leq \text{dist}\,(f_n, F^{\{n\}})$.

Proof of 4.7. Suppose (x_n, f_n) is a biorthogonal system of type Y. Then by 4.6 (ii) and the lemma dist $(x_n, L^{\{n\}}) \geq \frac{1}{M}$, i.e., $\inf_n \|x_n\| \geq \frac{1}{M}$. Let $\sigma \in \Sigma$ and $s \in S_\sigma$. By 4.6 (iii) there is an $n \in \omega$ such that $|f_n(s)| \geq \frac{\gamma}{2}$. Since $s \in L_\sigma$, $s = \sum_{i \in \sigma} f_i(s)x_i$ and thus $n \in \sigma$, for otherwise, $f_n(s) = 0$ contradicting the above. Thus for $t \in S^\sigma$ we have

$$\|s + t\| = |f_n(s)| \, \Big\| \sum_{i \in \sigma \setminus \{n\}} f_i(s)x_i/f_n(s) + x_n + \frac{t}{f_n(s)} \Big\|$$

$$\geq \frac{\gamma}{2} \text{ dist}\,(x_n, L^{\{n\}}) \geq \frac{\gamma}{2M} .$$

Thus by (*) and 4.6 (i) (x_i) is an unconditional basis for X. For each $p \in \omega$, $\sup_n \|f_n(\Sigma_1^p x_i / \|\Sigma_1^p x_i\|)\| \geq \gamma$ and it follows that $\|\Sigma_1^p x_i\| \leq \frac{1}{\gamma}$. Thus (x_i) is an unconditional basis of type P and the theorem follows from 4.3.

Proof of 5.6. Suppose (x_i, f_i) is a biorthogonal system of type Y^*. An argument similar to the above shows dist $(T_\sigma, T^\sigma) \geq \frac{\delta}{M}$ and so by (*) (f_i) is an unconditional basis for $[f_n : n \in \omega]$. If ϕ denotes the canonical map from X into $[f_n : n \in \omega]^*$ defined by $\phi(x)(f) = f(x)$ for each $f \in [f_n : n \in \omega]$ then it follows that $(\phi(x_i))$ is an unconditional basis for $[\phi(x_i) : i \in \omega]$ and hence (x_i) is an unconditional basis for X (since $[x_i] = X$). As above it follows that $\inf \|x_n\| \geq \frac{\delta}{2}$ and $\sup \|\Sigma_1^n f_i\| \leq \frac{1}{\delta}$,

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i.e. (x_i) is an unconditional basis of type p^*, and the theorem follows from 5.4.

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