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On Maximal k-Sections and Related Common Transversals of Convex Bodies

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Abstract. Generalizing results from [MM1] referring to the intersection body IK and the cross-section body CK of a convex body $K \subset \mathbb{R}^d$, $d \geq 2$, we prove theorems about maximal k-sections of convex bodies, $k \in \{1, \dots, d-1\}$, and, simultaneously, statements about common maximal $(d-1)$- and 1-transversals of families of convex bodies.

1 Introduction

Continuing [Ha1, Ha2, PC, MM1, MM2, MM3, among others], the present paper collects some theorems on maximal k-sections of d-dimensional convex bodies, where k is an integer between 1 and $d-1$ and d is the dimension of the space. A *convex body* $K \subset \mathbb{R}^d, d \geq 2$, is a compact, convex set with interior points in $\mathbb{R}^d$, and we write int (rel int) and bd (rel bd) for interior (relative interior) and boundary (relative boundary) of K, respectively (relative means with respect to the affine hull of K). A flat is an affine plane in $\mathbb{R}^d$, and subspaces in $\mathbb{R}^d$ are always considered as linear. A *maximal k-section* of K is the intersection of K and a k-dimensional flat L_k such that $V_k(K \cap L_k)$ is maximal among the k-volumes of all intersections of K with translates $L_k + x$, $x \in \mathbb{R}^d$, where V_k denotes *k-dimensional Lebesgue measure*. The investigations of maximal $(d-1)$- and 1-sections of convex bodies as well as basic relations between certain star bodies (defined in the following and associated with a given convex body $K \subset \mathbb{R}^d$) give a natural motivation for the results presented here. For $0 \in \operatorname{int} K$, the *intersection body IK* of K is the star body with (necessarily continuous) radial function $V_{d-1}(K \cap u^\perp)$ for $u \in S^{d-1}$, where $u^\perp$ is the orthocomplement of the unit vector u. This notion is due to Lutwak [Lu], see also [Ga, Definition 8.1.1], and intersection bodies have various applications in the field of convexity (dual mixed volumes, Busemann-Petty problem, *etc.*, *cf.* again [Ga, Chapter 8]). The *cross-section body CK* of K is the star body with (necessarily continuous) radial function $\max_{\lambda \in \mathbb{R}} V_{d-1}(K \cap (u^\perp + \lambda u))$, $u \in S^{d-1}$. This notion was introduced in [Ma2], *cf.* also [Ga, Definition 8.3.1 and Section 8.3] for various properties and applications. On the other hand, for $0 \in \operatorname{int} K$ the *chordal symmetral* $\widetilde{\triangle} K$ of K is the star body whose radial function is given by $V_1(K \cap (u\mathbb{R}))/2$, $u \in S^{d-1}$, with $u\mathbb{R}$ the linear 1-subspace of $\mathbb{R}^d$ spanned by u, see [Ga, Definition 5.1.3]. It is obvious that $2\widetilde{\triangle} K$ is the analogue of the intersection body for 1-dimensional sections. Finally, the *difference body* $DK = K + (-K)$ (see *e.g.*, [Ga, Section 3.2]) is the analogue of

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the cross-section body for 1-dimensional sections. Evidently we have the relations $IK \subset CK$ and $2\widetilde{\triangle}K \subset DK$ for $0 \in \operatorname{int} K$.

It was shown in [MM1] that each $x \in \mathbb{R}^d$ belongs to a hyperplane generating a maximal $(d-1)$-section of a convex body $K \subset \mathbb{R}^d$. Thus for $0 \in \operatorname{int} K$ we have $[\operatorname{bd}(IK)] \cap \operatorname{bd}(CK) \neq \varnothing$ (actually this last relation was a joint observation of R. J. Gardner and the second author). On the base of [MMÓ] this was used in [MM1] to characterize convex bodies centred at the origin or even centred balls. It was P. C. Hammer (*cf.* [Ha1, Theorem 1, Ha2, Theorem 3.1, PC, proof of Theorem 4]) who proved that each $x \in \mathbb{R}^d$ belongs to a line generating a maximal 1-section of a convex body $K \subset \mathbb{R}^d$. Thus for $0 \in \operatorname{int} K$, in our terms also $[\operatorname{bd}(2\widetilde{\triangle}K)] \cap \operatorname{bd}(DK) \neq \varnothing$ holds. Analogously, one can use this to characterize convex bodies centred at the origin and centred balls, see [MM1, Proposition 1]. In the present paper we are going to extend these results to maximal k-sections of convex bodies $K \subset \mathbb{R}^d$, $1 < k < d-1$, $d \geq 4$. Moreover, we obtain statements on common hyperplane transversals and common line transversals of convex bodies that generate maximal $(d-1)$-sections and maximal 1-sections of each body, respectively. Our results are obtained by elementary methods from algebraic topology (not surpassing tools from the nice expository paper [Wh]). However, extensions of our Theorems 4 and 5 (and statements close to our Theorem 3) are contained in the very recent paper [MVŽ], but they are derived by advanced methods from algebraic topology. In addition, the theorems given here were obtained in essence earlier, see also our final remark in [MM1], where a slightly weaker form of our Theorem 3 was already announced as a proved statement.

The following notations and definitions will also be useful. We write $K|L_k$ for the *orthogonal projection* of a convex body $K \subset \mathbb{R}^d$ to a k-flat L_k. For a metric space X and $m \geq 0$, the *m-dimensional Hausdorff measure H^m* is an outer measure defined on all subsets of X as follows: for $A \subset X$

$$H^m(A) = \sup_{\delta>0} \Big(\inf \Big\{ \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^m \cdot \pi^{m/2}/(2^m \Gamma(1+\frac{m}{2})) \ \Big| \ A \subset \bigcup_{i=1}^{\infty} A_i \subset X, \forall i \ \operatorname{diam}(A_i) \leq \delta \Big\} \Big),$$

where diam means diameter, *cf.* [Fe, 2.10.1.–2], or also [MM1, p. 449]. All closed subsets of X are H^m-measurable (see [Fe, pp. 54, 170]). If m is a positive integer, one calls $A \subset X$, with $H^m(A) < \infty$, (H^m, m)-*rectifiable* if

$$\forall \varepsilon > 0 \quad \exists A_\varepsilon \subset X, \ H^m(A \backslash A_\varepsilon) < \varepsilon$$

and A_ε is the image of a bounded subset of $\mathbb{R}^m$ by a Lipschitz map defined on this subset, see [Fe, pp. 251–252]. If X is a Euclidean space and A is a compact C^1 m-submanifold, then A is (H^m, m)-rectifiable, and $H^m(A)$ coincides with the differential geometric m-volume ([Fe, Theorems 3.2.26 and 3.2.39]).

2 Results

As direct generalizations of Theorem 1 from [Ha1] (see also [Ha2, Theorem 3.1, PC, proof of Theorem 4]) and [MM1, Theorem 1], which concern the cases $k = 1$ and $k = d - 1$, we ask the following. Does each $x \in \mathbb{R}^d$ belong to a k-flat generating a maximal k-section of a convex body $K \subset \mathbb{R}^d$? Observe that the proof of Theorem 1 from [MM1] has shown actually that each $(d - 2)$-flat is a subset of a $(d - 1)$-flat generating a maximal $(d - 1)$-section of K. This hints of the possibility that also for $1 < k < d - 1$ each $(k - 1)$-flat is a subset of a k-flat generating a maximal k-section of K. This will be confirmed in Theorems 1, 2 and 3 below (for $d \geq 4$ rather than $d \geq 2$). Theorem 3 also contains the statement that the k-flats generating maximal k-sections form a "large" set. This is a generalization of the corresponding statement of Theorem 1 from [MM1] (except that in Theorem 3 the constant $c_{d,k}$ is not sharp, while the constant was sharp in Theorem 1 of [MM1]). Moreover, Corollary 1 below is a generalization of Theorems 2 and 3 from [MM1], which are based on [MMÓ] and Proposition 1 from [MM1], which concern the cases $k = d - 1$ and $k = 1$.

Theorem 1 *Let $L_{k-1} \subset \mathbb{R}^d$, $d \geq 4$, be a fixed $(k - 1)$-subspace, $1 < k < d - 1$, such that for a given convex body $K \subset \mathbb{R}^d$ the relation $(\operatorname{int} K) \cap L_{k-1} \neq \varnothing$ holds. Then there exists a k-subspace $L_k \supset L_{k-1}$ such that*

$$V_k(K \cap L_k) = \max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}.$$

This statement implies analogues of Theorems 2 and 3 and Proposition 1 from [MM1] with the same proofs, *i.e.,* we have

Corollary 1 *Let $d \geq 4$, $1 < k < d - 1$, and $K \subset \mathbb{R}^d$ be a convex body. If, for each k-subspace L_k, we have $V_k(K \cap L_k) = c \cdot \max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}$, where c is a constant independent of L_k, then K is centred* (i.e., *$K = -K$). If both $V_k(K \cap L_k)$ and $\max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}$ are constant, then K is a centred ball.*

An analogue of Theorem 1 above can be formulated, namely

Theorem 2 *Let $L_{k-1} \subset \mathbb{R}^d$, $d \geq 4$, be a fixed $(k - 1)$-subspace, $1 < k < d - 1$, supporting or disjoint to a given convex body $K \subset \mathbb{R}^d$. Then there exists a k-subspace $L_k \supset L_{k-1}$ satisfying*

$$V_k(K \cap L_k) = \max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}.$$

It should be noticed that the separated formulation of these two theorems is also motivated by the ways of proving them, see below.

Remark The statements of Theorems 1 and 2 are sharp in the sense that in general there are no two such L_ks. For example, let K be a ball with centre not in L_{k-1}. ■

The *Grassmannian* $Gr_{d,k}$ is the set of all k-subspaces L_k of $\mathbb{R}^d$. An $O(d)$-invariant Riemannian metric on $Gr_{d,k}$ is given by

$$ds^2 = Tr(dT^* \cdot dT),$$

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where the linear operator $dT\colon L_k \to L_k^{\perp}$ is identified with its graph, that is a k-subspace of $\mathbb{R}^d$ close to L_k. (Tr, *, and $^{\perp}$ denote trace, transposition and ortho-complement, respectively.) About the existence and uniqueness of this Riemannian metric see *e.g.*, [MVŽ].

Nevertheless, one can summarize Theorems 1 and 2 by

Theorem 3 *Let $L_{k-1} \subset \mathbb{R}^d$, $d \geq 4$, be an arbitrary, fixed $(k-1)$-subspace, $1 < k < d-1$, and let $K \subset \mathbb{R}^d$ be a convex body. Then there exists a k-subspace $L_k \supset L_{k-1}$ such that*

$$V_k(K \cap L_k) = \max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}.$$

Moreover, the set of all k-subspaces L_k satisfying the last equality (but not the inclusion $L_k \supset L_{k-1}$) cannot be included, in the sense of the above Riemannian metric ds^2, in a $H^{(k-1)(d-k)}$-measurable, $\big(H^{(k-1)(d-k)}, (k-1)(d-k)\big)$-rectifiable subset of the Grassmannian $Gr_{d,k}$, of $(k-1)(d-k)$-dimensional Hausdorff measure less than some positive constant $c_{d,k}$. This is sharp in the following sense: there exists some convex body K such that the above set of k-subspaces L_k is a smooth, compact $(k-1)(d-k)$-dimensional submanifold of $Gr_{d,k}$, of finite $(k-1)(d-k)$-volume, in the sense of the above Riemannian metric.

It was proved by P. C. Hammer (*cf.* [Ha1, Theorem 1, Ha2, Theorem 3.1, PC, proof of Theorem 4]) that each $x \in \mathbb{R}^d$ belongs to an *affine diameter* (*i.e.,* to a maximal 1-section) of a given convex body $K \subset \mathbb{R}^d$. The following theorem is a natural generalization of Hammer's theorem since, if K_1 is a ball, in fact it is Hammer's statement. As we have been recently informed, this theorem was obtained about 1980 by V. L. Dol'nikov (unpublished).

Theorem 4 *Let $K_1, K_2 \subset \mathbb{R}^d$, $d \geq 2$ be convex bodies. Then there exists a line l such that $K_1 \cap l$ is an affine diameter of K_1 and $K_2 \cap l$ is an affine diameter of K_2.*

Remark The statement of Theorem 4 is sharp in the sense that in general there are no two such lines (each carrying a pair of affine diameters with respect to the pair K_1, K_2), *e.g.,* one can see this for K_1, K_2 being non-concentric balls. ■

On the other hand, replacing k by $d-1$ in Theorem 3 (*cf.* also [MM1, Theorem 1]) one gets the following: Let $K_1 \subset \mathbb{R}^d$ be a convex body, and $K_2, \dots, K_d$ be balls with centres in general position (*i.e.,* these centres span an arbitrarily given, non-degenerate $(d-2)$-flat L_{d-2}). Then there exists a hyperplane $L_{d-1} \supset L_{d-2}$ cutting $K_1, K_2, \dots, K_d$ in maximal $(d-1)$-sections. This observation gives a motivation for (and is generalized by)

Theorem 5 *Let $K_1, \dots, K_d \subset \mathbb{R}^d$ be convex bodies. Then there exists a hyperplane L_{d-1} such that for each $i \in \{1, \dots, d\}$ the intersection $K_i \cap L_{d-1}$ is a maximal $(d-1)$-section of K_i.*

Remark The statement of Theorem 5 is sharp in the sense that in general there are no two such hyperplanes (*e.g.*, let the convex bodies $K_1, \dots, K_d$ be balls whose centres are in general position). ■

3 Proofs of the Theorems

Proof of Theorem 1 It is enough to prove Theorem 1 for smooth and strictly convex bodies $K \subset \mathbb{R}^d$. (Namely, by the evident continuity property of k-dimensional sections through fixed interior points of bodies, in the Hausdorff metric, one can use a limit process for the general case.) When considering $L_k + x$, we will suppose $x \in L_k^{\perp}$, the orthocomplement of L_k, and we seek L_k in the form $L_k = L_{k-1} + u\mathbb{R}$, where $u \in L_{k-1}^{\perp}$, $||u|| = 1$.

For $x \in$ rel bd $(K|L_k^{\perp})$ we have $V_k(K \cap (L_k + x)) = 0$ by strict convexity, so $\max_x V_k(K \cap (L_k + x))$ is attained at some $x \in$ rel int$(K|L_k^{\perp})$. By the Brunn-Minkowski inequality (see, *e.g.*, [BF]), for $x \in$ rel int$(K|L_k^{\perp})$ the function $f_u(x) = V_k(K \cap (L_k + x))^{1/k}$ is concave and, by smoothness of K, differentiable. So it suffices to find $u \in L_{k-1}^{\perp} \cap S^{d-1}$ such that the derivative at $x = 0$ equals 0, i.e., $f_u'(0) = 0$. However, $f_u'(0)$ depends continuously on the radial function of K and its first derivatives relative to a point in $(\operatorname{int} K) \cap L_{k-1}$ (see, *e.g.*, [MMÓ, Lemma 3.5], or (1) below). Therefore $f_u'(0)$ is a continuous function of u, and $f_u'(0) \in L_k^{\perp}$ implies $\langle u, f_u'(0)\rangle = 0$, and $f_u'(0) = f_{-u}'(0)$. That is, $f_u'(0)$ can be considered as an even, continuous tangent vector-field on the unit sphere of $L_{k-1}^{\perp}$. By Grünbaum's theorem (see [Grü, p. 40, Sz, Theorem 1]) this implies that there exists a u such that $f_u'(0) = 0$. ■

Proof of Theorem 2 For L_k supporting K, say, at p, we can apply an approximation argument. Choose $K_n \to K$, $p \in \operatorname{int} K_n$, with k-subspaces $(L_k)_n \supset L_{k-1}$ having the maximum property. We may assume that $(L_k)_n$ tends to some linear k-subspace $L_k \supset L_{k-1}$. By concavity of $f_u(x)$ it suffices to show the (local) maximum property only among linear arrays of translates $L_k + x$, say $\{L_k + \lambda x_0\}$ with $x_0 \in L_k^{\perp}$ and $\lambda \geq 0$, thus for k-dimensional sections of a $(k + 1)$-dimensional convex body. The derivative of the k-volume of these sections with respect to λ is a continuous function of the radial function of this section and the first derivative of the radial function in the direction of x_0, the radial function taken with respect to a centre c in the relative interior of the respective section (*cf. e.g.*, [MMÓ, Lemma 3.5], or (1) below). It will suffice to consider the case $c \in \operatorname{int} K$ only. In fact, if L_k satisfies $V_k(K \cap L_k) \geq V_k(K \cap (L_k + x))$ for each x such that $(\operatorname{int} K) \cap (L_k + x) \neq \varnothing$, then it satisfies the same inequality for all x. In particular it suffices to consider linear arrays $\{L_k + \lambda x_0\}$ such that for $\lambda > 0$ small $(\operatorname{int} K) \cap (L_k + \lambda x_0) \neq \varnothing$. For almost all λ these derivatives exist a.e., (*cf.* [Sch, 2.2.4, Fe, 2.10.27 and 3.2.35]). It is enough to prove that, for $\lambda > 0$ small, if the derivative of the section volume with respect to λ exists (that happens a.e.), it is nonpositive a.e. However, for $c \in \operatorname{int} K$ convergence of K_n to K implies convergence of the derivatives as well, where these exist for K and each K_n. So the required inequality for the derivative of the section volume follows from a limit procedure. Thus L_k has the required maximum property for K.

For the case $K \cap L_{k-1} = \varnothing$ we may assume by the above approximation argu-

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ment and by [We, p. 335], that bd K is analytic, with everywhere positive principal curvatures. Let $D = \{u \in L_{k-1}^{\perp} : ||u|| = 1, K \cap (L_{k-1} + u\mathbb{R}^+) \neq \varnothing\}$, that is a smooth, strictly convex domain on an open half $(d-k)$-sphere of the $(d-k)$-sphere $L_{k-1}^{\perp} \cap S^{d-1}$ (where $\mathbb{R}^+ = [0, \infty)$ and $u\mathbb{R}^+ = \{\lambda u | \lambda \in \mathbb{R}^+\}$). Then for $u \in D$ we have $(\operatorname{int} K) \cap (L_{k-1} + u\mathbb{R}^+) \neq \varnothing$ if and only if $u \in \operatorname{rel\,int} D$, and hence $L_{k-1} + u\mathbb{R}^+$ supports K if and only if $u \in \operatorname{rel\,bd} D$ (rel int and rel bd meant with respect to $L_{k-1}^{\perp} \cap S^{d-1}$). Thus, for $u \in \operatorname{rel\,int} D$ the derivative $f_u'(0)$, and hence also $(f_u^2)'(0) = \frac{d}{dx}(f_u(x)^2)|_{x=0}$ $(x \in L_k^{\perp})$, exists and is a continuous function of u, by smoothness of K. (Recall that pointwise convergence of differentiable convex functions to a differentiable convex function implies pointwise convergence of the derivatives.) Again it suffices to prove that there exists a $u \in \operatorname{rel\,int} D$ such that $f_u'(0) = 0$, or, equivalently, $(f_u^2)'(0) = 0$.

We assert that $(f_u^2)'(0)$ has an extension to a continuous function $D \longrightarrow \mathbb{R}^d$. That is (by regularity of the involved topology, and using [Bo, Ch. I, § 8.5]), if some $(L_k)_n$ converge to an L_k, where $(\operatorname{int} K) \cap (L_k)_n \neq \varnothing$, and L_k supports K, then we have convergence of the respective expressions $(f_u^2)'(0) = \frac{d}{dx}(V_k(K \cap ((L_k)_n + x))^{2/k})|_{x=0}$. (In this proof we will not use $(L_k)_n \supset L_{k-1}$.) It suffices to prove convergence of $d-k$ directional derivatives for $d-k$ orthogonal directions in $(L_k)_n^{\perp}$, these $d-k$ directions converging to some $d-k$ directions as $n \to \infty$. Below we will choose n sufficiently large.

We have

$$V_k(K \cap (L_k)_n) = \frac{1}{k} \int\limits_{S^{d-1} \cap (L_k)_n} \varrho_n^k d\sigma,$$

where $d\sigma$ is the surface area element on $S^{d-1} \cap (L_k)_n$, and ϱ_n is the radial function of K with respect to some relative interior point of $K \cap (L_k)_n$. Moreover, for $(\operatorname{int} K) \cap (L_k)_n \neq \varnothing$, we have

$$\frac{d}{dx} V_k \left(K \cap ((L_k)_n + x)\right)|_{x=0} = \int\limits_{S^{d-1} \cap (L_k)_n} \varrho_n^{k-2} \frac{\partial \varrho_n}{\partial \psi} \, d\sigma, \tag{1}$$

where ψ is the geographic latitude in $S^{d-1} \cap ((L_k)_n + x\mathbb{R})$, with north pole $\frac{x}{\|x\|}$, *cf.* [MMÓ, Lemma 3.5]. We consider x varying in a one-dimensional subspace orthogonal to $(L_k)_n$, and differentiation is meant in this sense. Then

$$\begin{aligned} (f_u^2)'(0) &= \frac{2}{k} \int\limits_{S^{d-1} \cap (L_k)_n} \varrho_n^{k-2} \frac{\partial \varrho_n}{\partial \psi} d\sigma \Big/ \Big(\frac{1}{k} \int\limits_{S^{d-1} \cap (L_k)_n} \varrho_n^k d\sigma \Big)^{1-2/k} \\ &= \frac{2}{k} \int\limits_{S^{d-1} \cap (L_k)_n} \Big(\frac{\varrho_n}{\sqrt{\varepsilon}} \Big)^{k-2} \frac{\partial \varrho_n}{\partial \psi} d\sigma \Big/ \Big(\frac{1}{k} \int\limits_{S^{d-1} \cap (L_k)_n} \Big(\frac{\varrho_n}{\sqrt{\varepsilon}} \Big)^k d\sigma \Big)^{1-2/k} \end{aligned} \tag{2}$$

($\varepsilon > 0$ will be chosen later). If $(L_k)_n$ is close to L_k, which is contained in a supporting hyperplane of K, then by a small translation $(L_k)_n$ can be moved to a position disjoint to K. Thus a nearest translate $(L_k)_n^0$ of $(L_k)_n$, supporting K, is close to $(L_k)_n$, and so to L_k, too. In a suitable new coordinate system K has a tangent hyperplane at the point $K \cap (L_k)_n^0 = \{(0, \dots, 0)\}$, which is orthogonal to the d-th basic unit vector e_d. Moreover, near $K \cap (L_k)_n^0$ the boundary of K has a local representation $x_d = F(x_1, \dots, x_{d-1}) = \sum_{i=1}^{d-1} \frac{x_i^2}{a_i^2}$ + higher order terms, where $a_i > 0$. By the choice of $(L_k)_n^0$ we have $(L_k)_n = (L_k)_n^0 + y,\ y \in ((L_k)_n^0)^\perp$, where y is orthogonal to the tangent hyperplane of K at $K \cap (L_k)_n^0$. Thus $y = \varepsilon e_d$, where $\varepsilon > 0$ is small, and $(L_k)_n^0$ lies in the $x_1 \dots x_{d-1}$-hyperplane. Then $(L_k)_n^0 \ni (0, \dots, 0, 0)$ implies $(L_k)_n \ni (0, \dots, 0, \varepsilon)$, and so $(L_k)_n$ lies in the hyperplane $x_d = \varepsilon$. Choose as centre of polar coordinates the point $(0, \dots, 0, \varepsilon) \in$ rel int$(K \cap (L_k)_n)$. Then we have that ϱ_n is asymptotically the same as for the surface $x_d = \sum_{i=1}^{d-1} \frac{x_i^2}{a_i^2}$ (their quotient tends uniformly to 1, for each direction and any choice of $L_k, (L_k)_n, (L_k)_n^0$). In particular, $(0, \dots, 0, \varepsilon) \in K$.

We recall that x varies in a one-dimensional subspace orthogonal to $(L_k)_n$, and it suffices to consider only $d - k$ such mutually orthogonal one-dimensional subspaces.

One choice is when x is a multiple of e_d (the direction in which we differentiate will be that of the positive e_d-axis). Then $\frac{\partial \varrho_n}{\partial \psi} = \varrho_n \cdot \frac{\partial \varrho_n}{\varrho_n \partial \psi} = \varrho_n / \frac{\partial F}{\partial r}$, where $\frac{\partial F}{\partial r}$ is the partial derivative in radial direction, in a coordinate system in the $x_1 \cdots x_{d-1}$-hyperplane, with origin $(0, \dots, 0)$. We will consider r as the signed distance to $(0, \dots, 0)$, along a line in the $x_1 \cdots x_{d-1}$-hyperplane, passing through $(0, \dots, 0)$. On such a line, F is nearly a quadratic function, and for $r > 0$ the expression $\frac{\partial F}{\partial r}$ is asymptotically $\frac{\partial^2 F}{\partial r^2} \cdot r = \frac{\partial^2 F}{\partial r^2} \cdot \varrho_n$ (their quotient tends uniformly to 1, for each such line, and any choice of $L_k, (L_k)_n, (L_k)_n^0$). Thus $\frac{\varrho_n}{\sqrt{\varepsilon}}$ is close to the radial function of the set $\sum_{i=1}^{d-1} \frac{x_i^2}{a_i^2} \le 1$, and $\frac{\partial \varrho_n}{\partial \psi}$ is close to $1 / \frac{\partial^2 F}{\partial r^2}$. These depend on the second derivatives of the function representing bd K at the point $K \cap (L_k)_n^0$, which in turn are close to the corresponding second derivatives at the point $K \cap L_k$. Hence we have by (2) convergence of $(f_u^2)'(0)$ to a positive value, for x a multiple of e_d.

We have still to consider $d - 1 - k$ further choices for $x\mathbb{R}$, orthogonal to $(L_k)_n$, to each other and to e_d, and thus being parallel to the $x_1 \cdots x_{d-1}$-hyperplane. Let us consider one of these. For ϱ_n we use the same asymptotics as above. Furthermore, we have $\frac{\partial \varrho_n}{\partial \psi} = \varrho_n / \frac{\partial G}{\partial r}$, where $x_d' = G(x_1', \dots, x_{d-1}')$ is a local representation of the boundary of K in a new coordinate system, with the x_d'-axis being parallel to $x\mathbb{R}$ (and oriented some way). More exactly, in general G has two branches with different values on its domain of definition, and in the formula for $\frac{\partial \varrho_n}{\partial \psi}$ we consider that branch which passes through the respective point of rel bd $(K \cap (L_k)_n)$. The other possibility is that the values of the two branches coincide at the respective point, and both have $\left| \frac{\partial G}{\partial r} \right| = \infty$; then $\frac{\partial \varrho_n}{\partial \psi} = 0$.

By the above results on approximation of ϱ_n by the radial function of an ellipsoid we have $\varrho_n <$ const $\cdot \sqrt{\varepsilon}$, the constant only depending on K. Hence, letting $H^\varepsilon = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d = \varepsilon\}$, $K \cap H^\varepsilon$ is contained in the ball about $(0, \dots, 0, \varepsilon)$, of radius const $\cdot \sqrt{\varepsilon}$. Recall that $(0, \dots, 0, \varepsilon) \in (L_k)_n \subset H^\varepsilon$, as noted after the introduction of $(L_k)_n^0$. So

(A): Also $K \cap (L_k)_n$ contains $(0, \dots, 0, \varepsilon)$ and is contained in the ball about $(0, \dots, 0, \varepsilon)$, of radius const $\cdot \sqrt{\varepsilon}$.

By the same reason as above, $K \cap H^\varepsilon$ contains a $(d - 1)$-ball in H^ε about

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$(0, \dots, 0, \varepsilon)$ of radius const $\cdot\sqrt{\varepsilon}$, the constant being positive and only depending on K. Hence

(B): $K \cap H^\varepsilon \ni (0, \dots, 0, \varepsilon) \pm$ const $\cdot\sqrt{\varepsilon} \cdot x/ \parallel x \parallel$.

From **(A)** and **(B)**, by convexity, for the considered finite value $\frac{\partial G}{\partial r}$ we have $|\frac{\partial G}{\partial r}| \geq$ const > 0 for $\frac{\partial G}{\partial r}$ taken in directions lying in L_k, the constant only depending on K. Hence $|\frac{\partial \varrho_u}{\partial \psi}| \leq$ const $\cdot\sqrt{\varepsilon}$. Then (2) implies $|(f_u^2)'(0)| \leq$ const $\cdot\sqrt{\varepsilon}$.

Recapitulating, we have investigated the $d-k$ orthogonal components of $(f_u^2)'(0) \in (L_k)_n^\perp$. The component in the direction of e_d converges to a non-zero vector having the direction of the interior normal of K at the point $K \cap L_k$. The other $d-k-1$ components converge to 0. Hence $(f_u^2)'(0)$ converges to a non-zero vector having the direction of the interior normal of K at the point $K \cap L_k$. This proves our claim that the function $(f_u^2)'(0)$, defined and continuous for $u \in$ rel int D (*i.e.,* for $u \in L_{k-1}^\perp$, $\parallel u \parallel = 1$, $(\operatorname{int} K) \cap (L_{k-1} + u\mathbb{R}^+) \neq \varnothing$), has a continuous extension to

$$D = \{u \in L_{k-1}^\perp \colon \parallel u \parallel = 1,\ K \cap (L_{k-1} + u\mathbb{R}^+) \neq \varnothing\}.$$

We denote this continuous extension by $g\colon D \to \mathbb{R}^d$, which also satisfies $g(u) \in L_k^\perp = (L_{k-1} + u\mathbb{R})^\perp = L_{k-1}^\perp \cap (u\mathbb{R})^\perp$. Thus g can be considered in a natural way as a tangent vector-field on the strictly convex and smooth domain $D \subset L_{k-1}^\perp \cap S^{d-1}$, which is actually contained in an open half $(d-k)$-sphere of this $(d-k)$-sphere. As shown above, for $u \in$ rel bd D, $g(u)$ is non-zero and has the direction of the interior normal of K at the point $K \cap L_k$. Hence $g(u)$ has also the direction of the interior normal of $K|L_{k-1}^\perp$ at the point $(K \cap L_k)|L_{k-1}^\perp$, and so that of the interior normal of D at u.

If, for $u \in$ rel int D, $g(u) = (f_u^2)'(0)$ vanished nowhere, then we could define a retraction $h\colon D \to$ rel bd D (*i.e.,* a continuous map, identical on rel bd D) in the usual way, see, *e.g.,* [HW, Ch. IV, § 1, C]. Namely, to $u \in D$ we associate the point $h(u) \in$ rel bd D which is the first intersection point of rel bd D with the geodesic, on the above $(d-k)$-sphere, starting from u, in the direction opposite to that of $g(u)$. Since a retraction $D \to$ rel bd D does not exist (*cf., e.g.,* [HW, Ch. IV, § 1, B]), therefore there exists a $u \in$ rel int D such that $(f_u^2)'(0) = 0$. As stated above, this suffices to prove our statement for the case $K \cap L_{k-1} = \varnothing$. ■

Proof of Theorem 3 The first part of Theorem 3 simply summarizes Theorems 1 and 2.

The second part of this theorem follows from results of [MVŽ]. We will use on $Gr_{d,k}$ the $O(d)$-invariant Riemannian metric given before Theorem 3. Let

$$C = \{L_k : L_k \subset \mathbb{R}^d \text{ is a } k\text{-subspace and } V_k(K \cap L_k) = \max\{V_k(K \cap (L_k + x)) : x \in \mathbb{R}^d\}\}.$$

Further, let $C \subset M$, where M is an $H^{(k-1)(d-k)}$-measurable, $(H^{(k-1)(d-k)}, (k-1)(d-k))$-rectifiable subset of the Grassmannian $Gr_{d,k}$. Further, for $L_{k-1} \in Gr_{d,k-1}$, let $S_k(L_{k-1}) = \{L_k \in Gr_{d,k} : L_k \supset L_{k-1}\}$. Then for each $L_{k-1} \in Gr_{d,k-1}$ we have $C \cap S_k(L_{k-1}) \neq \varnothing$ by the first part of Theorem 3. This implies, by the proof of Theorem 7 from [MVŽ], that the $(k-1)(d-k)$-dimensional Hausdorff measure of M is at least some positive constant depending on d and k (in the notations of [MVŽ], this constant is $c_{d-1,k-1,0}$).

It remains to show that there exists a convex body K such that the above set C of k-subspaces L_k is a smooth compact $(k-1)(d-k)$-dimensional submanifold of $Gr_{d,k}$; then it necessarily has a finite $(k-1)(d-k)$-volume. Such a K is e.g. a ball with centre different from 0, since then C is diffeomorphic to $Gr_{d-1,k-1}$. ■

Proof of Theorem 4 As in the proof of Theorem 2, suitable approximation methods allow the restriction to smooth and strictly convex bodies K_1, K_2. We have to show that there exists a $u \in S^{d-1}$ such that the affine diameter of K_1 in direction u belongs to the affine hull of the affine diameter of K_2 in direction u. (For each $u \in S^{d-1}$, the uniqueness of affine diameters of smooth, strictly convex bodies parallel to u is easily verified, see also [Ha1, Ha2].) Denoting by $f_i(u)$ the orthogonal projection on $u^{\perp}$ of the affine diameter of K_i in direction $u \in S^{d-1}$, we therefore have to show that there exists a $u \in S^{d-1}$ such that

$$u\mathbb{R} + f_1(u) = u\mathbb{R} + f_2(u)\,,$$

where for each $u \in S^{d-1}$ we have $f_1(u), f_2(u) \in u^{\perp}$. It is obvious that f_1, f_2 are well-defined even functions which are also continuous. Thus we can consider $f_1(u) - f_2(u)$ as an even, continuous tangent vector-field on S^{d-1}. Then, by Grünbaum's theorem (*cf.* [Grü, p. 40; Sz, Theorem 1]), there exists a $u_0 \in S^{d-1}$ such that $f_1(u_0) - f_2(u_0) = 0$. So we have $f_1(u_0) = f_2(u_0)$, which implies $u_0\mathbb{R} + f_1(u_0) = u_0\mathbb{R} + f_2(u_0)$. ■

Proof of Theorem 5 Again, as in the proof of Theorem 2, by analogous approximation arguments we may assume that the considered convex bodies $K_1, \dots, K_d$ are smooth and strictly convex. A hyperplane section of K_i, having maximal $(d-1)$-volume among all hyperplane sections of normal $u \in S^{d-1}$, is of the form $K_i \cap \{x \in \mathbb{R}^d : \langle x, u\rangle = f_i(u)\}$. Here the function f_i is well-defined. In fact, suppose there were two distinct parallel hyperplanes H_1, H_2 of normal u, with the maximum volume section property. Then, by the equality case in the Brunn-Minkowski inequality, the sections of K_i with H_1 and H_2 would be translates of each other. Hence, by $K_i \cap [(H_1 + H_2)/2] \supset [(K_i \cap H_1) + (K_i \cap H_2)]/2$, also $K_i \cap [(H_1 + H_2)/2]$ would be a translate of $K_i \cap H_j$, $j = 1, 2$. So bd K_i would contain segments. Moreover, f_i is continuous and odd. Namely, $\langle x, u\rangle = f_i(u)$ if and only if $\langle x, -u\rangle = -f_i(u)$, yielding $f_i(-u) = -f_i(u)$. We have to show that there exists a direction $u \in S^{d-1}$ such that $f_1(u) = \cdots = f_d(u)$, *i.e.*, such that the above d hyperplanes $\{x \in \mathbb{R}^d : \langle x, u\rangle = f_i(u)\}$ coincide.

We suppose the contrary and let $f = (f_1, \dots, f_d)\colon S^{d-1} \to \mathbb{R}^d \backslash (1, \dots, 1)\mathbb{R}$, which is odd. Here $\mathbb{R}^d \backslash (1, \dots, 1)\mathbb{R}$ is *isomorphic* to $\mathbb{R}^d \backslash (e_d\mathbb{R})$, where the usual basis of $\mathbb{R}^d$ is denoted by $\{e_1, \dots, e_d\}$. Moreover, we have $\mathbb{R}^d \backslash (e_d\mathbb{R}) = (\mathbb{R}^{d-1} \backslash \{0\}) \times \mathbb{R}$. Denoting the composition of f and the isomorphism above by g, we see that $g\colon S^{d-1} \to (\mathbb{R}^{d-1} \backslash \{0\}) \times \mathbb{R}$ is obviously odd and continuous. We can write $g(u) = (g_1(u), g_2(u))$, with $g_1(u) \in \mathbb{R}^{d-1} \backslash \{0\}$, $g_2(u) \in \mathbb{R}$ (thus in complementary subspaces). So g_1, g_2 are components of an odd, continuous function; they are themselves odd and continuous. First we only consider g_1 as a map from S^{d-1} to $\mathbb{R}^{d-1}$ (by $\mathbb{R}^{d-1} \backslash \{0\} \subset \mathbb{R}^{d-1}$). By continuity and the Borsuk-Ulam theorem (see, *e.g.*, [Wh, Corollary 12]) one knows that there exists a $u \in S^{d-1}$ such that $g_1(u) = g_1(-u)$. Moreover, by oddness we have $g_1(u) = -g_1(-u)$. These together imply that $g_1(u) = 0$, a contradiction. ■

4 Final Remark

Unfortunately, the proof of the very last announced statement from [MM1] (on k-dimensional sections and projections, where correctly $V_k(K|L_k')^{-1}$ stands) could not be reproduced by us; thus it remains a conjecture. Anyway, it is equivalent to the statement that, for $1 < k < d-1$, most convex bodies, in the sense of Baire category, have no generalized plane shadow-boundaries with respect to illumination from any projective $(d-k-1)$-subspace of the hyperplane at infinity, as can be shown by considerations analogous to those in [Ma1]. (The *shadow boundary of a convex body K with respect to illumination from a projective $(d-k-1)$-subspace P_{d-k-1} of the hyperplane at infinity is* $\bigcup\{K \cap L_{d-k} : L_{d-k}$ is a supporting $(d-k)$-flat of K, the projective extension of which contains $P_{d-k-1}\}$. The shadow boundary is a *generalized plane shadow boundary* if the following holds: letting L^0_{d-k} be the $(d-k)$-subspace whose projective extension contains P_{d-k-1}, there exists a k-flat L_k intersecting L^0_{d-k} transversally, such that $L_k \cap \mathrm{bd}(K + L^0_{d-k})$ is a subset of the shadow boundary.) We yet remark that, for the case of 0-symmetric convex bodies, an analogous statement was announced, without proof, in [Gr, Theorem 31].

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