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RESEARCH ARTICLE

The inverse scattering theory of Kadomtsev–Petviashvili II equations

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Abstract

This overview discusses the inverse scattering theory for the Kadomtsev–Petviashvili II equation, focusing on the inverse problem for perturbed multi-line solitons. Despite the introduction of new techniques to handle singularities, the theory remains consistent across various backgrounds, including the vacuum, 1-line and multi-line solitons.

1. Introduction

The Kadomtsev–Petviashvili II (KPII) equation [16],

$$(-4u_{x_3} + u_{x_1x_1x_1} + 6uu_{x_1})_{x_1} + 3u_{x_2x_2} = 0 \tag{1.1}$$

a (2 + 1)-dimensional extension of the Korteweg-de Vries equation, models small amplitude, long-wavelength, weakly two-dimensional waves in weakly dispersive media. It has applications in mathematics and physics and is integrable via the Lax pair [],

$$\begin{cases} (-\partial_{x_2} + \partial_{x_1}^2 + u)\Phi(x,\lambda) = 0, \\ (-\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} - \lambda^3)\Phi(x,\lambda) = 0. \end{cases} \tag{1.2}$$

The initial value problem for the KPII equation can be solved using inverse scattering theory (IST), with early work on the IST involving the $\overline{\partial}$-method for vacuum backgrounds [1, 14, 15, 20, 27]. Around 2000, Boiti *et al.*, Villarroel and Ablowitz extended the IST to backgrounds with 1-line solitons [7, 26]. Boiti *et al*. then integrated the Sato theory and set the foundation of the IST of the KPII equation for multi-line soliton backgrounds. Their achievements at least include: deriving an explicit formula of the Green function, L^∞ estimates for the discrete part of the Green function and the $\mathcal{D}$-symmetry, the relation between values of the eigenfunction at multi value points [4–6, 8–10, 21]. Building on their work, we have completed a rigorous IST for smooth perturbations of multi-line solitons, obtaining the first rigorous IST for a multi-dimensional integrable system where both continuous and discrete scattering data coexist without degeneration into complex plane contours [31–33].

This paper provides an overview of the IST for the KPII equation, focusing on the inverse problem for perturbed multi-line solitons. The aim is to demonstrate that, despite the introduction of new algebraic or analytic techniques to handle singular structures, the ISTs remain consistent across various background potentials, including the vacuum, 1-line solitons and multi-line solitons. Specifically, when either discrete or continuous scattering data vanish, the forward and inverse scattering transforms for perturbed line solitons reduce to those for rapidly decaying potentials or multi-line solitons.

The paper is organized as follows: In Section 2, we present the IST for the vacuum background, integrating Fourier theory, outlining the approaches to the KPII equation for different backgrounds and characterizing the scattering properties as $\lambda \to \infty$ for multi-line soliton backgrounds.

Section 3 discusses the IST for perturbed 1-line solitons without using Sato theory. We define the forward scattering transform, formulate the inverse problem as a Cauchy integral equation (CIE) with a $\mathcal{D}$-symmetry constraint and solve it using Hölder interior estimates and deformation methods. We elucidate the connection between the forward and inverse problems and emphasize key analytical tools in Sections 3.3.3 and 3.3.4.

In Section 4, we extend the IST for perturbed 1-solitons to perturbed multi-line solitons by applying Sato theory [2, 3, 19, 22–24] and the IST framework developed for the KP equation by Boiti *et al.* [4–6, 8–10, 21]. We present the complete theory, highlighting distinct features, and demonstrate that the TP condition is necessary. We also show that the differences between the IST for 1-solitons and multi-line solitons are primarily algebraic.

2. The IST for rapidly decaying potentials

2.1. Statement of results

Given a rapidly decaying initial data $u_0(x_1, x_2)$, the Cauchy problem of the KPII equation can be solved using IST [1, 14, 15, 20, 27]:

Theorem 2.1. (The Cauchy Problem) [27] *For initial data* $\sum_{|l|\le d+7} |\partial_{x_1}^{l_1}\partial_{x_2}^{l_2} u_0(x_1,x_2)|_{L^1\cap L^2} \ll 1$ *with* $d \ge 0$*, we can construct the* forward scattering transform

$$\mathcal{S} : u_0 \mapsto s_c(\lambda) \tag{2.1}$$

satisfying the algebraic and analytic constraints:

$$|\left[|\overline{\lambda} - \lambda|^{l_1} + |\overline{\lambda}^2 - \lambda^2|^{l_2}\right] s_c(\lambda)|_{L^\infty} \le C \textstyle\sum_{|h|=0}^{l} |\partial_x^h u_0|_{L^1\cap L^2}, \tag{2.2}$$

$$s_c(\lambda) = \overline{s_c(\overline{\lambda})}. \tag{2.3}$$

Moreover, the solution to the Cauchy problem for the KPII equation is given by

$$u(x) = -\frac{1}{\pi i}\partial_{x_1} \iint Tm\, d\overline{\zeta} \wedge d\zeta, \quad x = (x_1, x_2, x_3), \tag{2.4}$$

with

$$|(1 + |\xi|^k + |\eta|^k)\widehat{u}(\xi,\eta)|_{L^\infty} \le C|(1 + |\xi|^{k+2} + |\eta|^{k+2})s_c|_{L^\infty\cap L^2(d\xi d\eta)}, \quad |k| \le d+5. \tag{2.5}$$

Here $m(x,\lambda) = 1 + \mathcal{C}Tm(x,\lambda)$*,* $\mathcal{C}$ *is the Cauchy integral operator, and* T *is the continuous scattering operator:*

$$\begin{aligned} \mathcal{C}\phi(x,\lambda) &\equiv -\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\phi(x,\zeta)}{\zeta - \lambda} d\overline{\zeta} \wedge d\zeta, \\ T\phi(x,\lambda) &\equiv s_c(\lambda) e^{(\overline{\lambda}-\lambda)x_1 + (\overline{\lambda}^2 - \lambda^2)x_2 + (\overline{\lambda}^3 - \lambda^3)x_3} \phi(x,\overline{\lambda}). \end{aligned} \tag{2.6}$$

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Except in Sections 3.2.2 and 4.2.1, we define $x = (x_1, x_2, x_3)$ throughout the paper. We use the notation $\partial_x^l \equiv \partial_{x_1}^{l_1}\partial_{x_2}^{l_2}\partial_{x_3}^{l_3}$, where l_j are non-negative integers and $|l| = l_1+l_2+l_3$. The constant C represents a uniform constant that is independent of both x and λ. Theorem 2.1 follows from the direct and inverse scattering theories, as detailed in Theorems 2.2 and 2.4.

Theorem 2.2. (Direct Scattering Theory [27]) *Given* $\sum_{|l|\le d+7} |\partial_{x_1}^{l_1}\partial_{x_2}^{l_2}u_0(x_1,x_2)|_{L^1\cap L^2} \ll 1$ *for* $d \ge 0$, *the following holds:*

(1) There exists a unique eigenfunction $\Phi(x_1,x_2,\lambda) = e^{\lambda x_1+\lambda^2 x_2} m_0(x_1,x_2,\lambda)$ *of the* Lax equation

$$\begin{aligned} &(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda\partial_{x_1} + u_0(x_1,x_2))m_0(x_1,x_2,\lambda) = 0, \\ &\lim_{|x|\to\infty} m_0(x_1,x_2,\lambda) = 1, \quad |\partial_x^l\,[m_0 - 1]\,|_{L^\infty} \le C|\partial_x^l u_0|_{L^1\cap L^2}. \end{aligned} \tag{2.7}$$

(2) The forward scattering transform can be constructed as

$$\mathcal{S} : u_0 \mapsto s_c(\lambda) = \frac{\operatorname{sgn}(\lambda_I)}{2\pi i}\,[u_0(\cdot)m_0(\cdot,\lambda)]^\wedge\,(\frac{\overline{\lambda}-\lambda}{2\pi i}, \frac{\overline{\lambda}^2-\lambda^2}{2\pi i}), \tag{2.8}$$

satisfying

$$\partial_{\overline{\lambda}} m_0(x_1,x_2,\lambda) = s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2} m_0(x_1,x_2,\overline{\lambda}), \tag{2.9}$$

and the Cauchy integral equation

$$m_0(x_1,x_2,\lambda) = 1 + \mathcal{C}T_0 m_0(x_1,x_2,\lambda). \tag{2.10}$$

The scattering data satisfy the following algebraic and analytic constraints:

$$|\left[|\overline{\lambda}-\lambda|^{l_1} + |\overline{\lambda}^2-\lambda^2|^{l_2}\right] s_c(\lambda)|_{L^\infty} \le C\sum_{h=0}^{l} |\partial_x^h u_0|_{L^1\cap L^2}, \tag{2.11}$$

$$s_c(\lambda) = \overline{s_c(\overline{\lambda})}. \tag{2.12}$$

Here $\lambda = \lambda_R + i\lambda_I$, $\overline{\lambda} = \lambda_R - i\lambda_I$, T_0 is the continuous scattering operator at $x_3 = 0$, and $\widehat{\phi}$, $\check{\phi}$ denote the Fourier and the inverse Fourier transform, respectively:

$$\begin{aligned} \widehat{\phi}(\xi,\lambda) &\equiv \iint_{\mathbb{R}^2} e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\phi(x_1,x_2,\lambda)dx_1dx_2, \\ \check{\phi}(\xi,\lambda) &\equiv \iint_{\mathbb{R}^2} e^{+2\pi i(x_1\xi_1+x_2\xi_2)}\phi(x_1,x_2,\lambda)dx_1dx_2. \end{aligned} \tag{2.13}$$

Theorem 2.3. (Linearization Theorem) *If* $\Phi(x,\lambda) = e^{\lambda x_1+\lambda^2 x_2} m(x,\lambda)$ *satisfies the Lax pair (1.2) and*

$$\partial_{\overline{\lambda}} m(x,\lambda) = s_c(\lambda,x_3)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2} m(x,\overline{\lambda}),$$

then

$$s_c(\lambda,x_3) = e^{(\overline{\lambda}^3-\lambda^3)x_3} s_c(\lambda). \tag{2.14}$$

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Theorem 2.4. (Inverse Scattering Theory) [27] *For small scattering data $s_c(\lambda)$ decaying rapidly in $(\overline{\lambda}-\lambda,\overline{\lambda}^2-\lambda^2)$, the following holds:*

(1) The CIE has a unique solution:

$$m(x,\lambda)=1+\mathcal{C}Tm(x,\lambda), \tag{2.15}$$

$$|\partial_x^l\,[m-1]\,|_{L^\infty}\le C|\left(1+|\xi|^{l_1}+|\eta|^{l_2}\right)s_c(\lambda)|_{L^\infty\cap L^2(d\xi d\eta)}, \tag{2.16}$$

where $2\pi i\xi=\overline{\lambda}-\lambda$, $2\pi i\eta=\overline{\lambda}^2-\lambda^2$.

(2) The Lax equation holds:

$$\left(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u(x)\right)m(x,\lambda)=0, \tag{2.17}$$

$$u(x)=-\frac{1}{\pi i}\partial_{x_1}\iint Tm\,d\overline{\zeta}\wedge d\zeta, \tag{2.18}$$

$$|(1+|\xi|^k+|\eta|^k)\widehat{u}(\xi,\eta)|_{L^\infty}\le C|(1+|\xi|^{k+2}+|\eta|^{k+2})s_c(\lambda)|_{L^\infty\cap L^2(d\xi d\eta)}, \tag{2.19}$$

and the inverse scattering transform is defined as

$$\mathcal{S}^{-1}(s_c(\lambda))\equiv-\frac{1}{\pi i}\partial_{x_1}\iint Tm\,d\overline{\zeta}\wedge d\zeta; \tag{2.20}$$

(3) The KPII equation is fulfilled

$$(-4u_{x_3}+u_{x_1x_1x_1}+6uu_{x_1})_{x_1}+3u_{x_2x_2}=0. \tag{2.21}$$

2.2. *The strategy*

Detailed proof can be found in [27]. We highlight key features of the proof.

2.2.1. *Proof of Theorem 2.2*

(1) Using Fourier theory, we transform the Lax equation (2.7) into the integral equation:

$$m_0(x_1,x_2,\lambda)=1-\left[\frac{\widehat{u_0m_0}}{p_\lambda(\xi,\eta)}\right]^\vee=1-G_\lambda*u_0m_0, \tag{2.22}$$

where the Green function is defined by:

$$G_\lambda=\left[\frac{1}{p_\lambda}\right]^\vee,\ \ p_\lambda(\xi,\eta)=(2\pi i\xi+\lambda)^2-(2\pi i\eta+\lambda^2). \tag{2.23}$$

Hence, the unique solvability of the Lax equation and eigenfunction estimates follows from:

$$\left|\frac{1}{p_\lambda}\right|_{L^1(\Omega_\lambda,d\xi d\eta)}\le\frac{C}{(1+|\lambda_I|^2)^{1/2}},\ \left|\frac{1}{p_\lambda}\right|_{L^2(\Omega_\lambda^c,d\xi d\eta)}\le\frac{C}{(1+|\lambda_I|^2)^{1/4}}, \tag{2.24}$$

where $\Omega_\lambda=\{(\xi,\eta)\in\mathbb{R}^2\ :\ |p_\lambda(\xi,\eta)|<1\}$.

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(2) To define the scattering data, we compute the $\partial_{\bar{\lambda}}$-data of the eigenfunction m_0. This requires computing the $\partial_{\bar{\lambda}}(1/p_\lambda)$ and establishing the commutative relation between p_λ and the exponential function:

$$\partial_{\bar{\lambda}}\left[\frac{1}{p_\lambda}\right] = -\frac{\operatorname{sgn}(\lambda_I)}{2\pi i}\delta(\xi - \frac{\overline{\lambda}-\lambda}{2\pi i}, \eta - \frac{\overline{\lambda}^2-\lambda^2}{2\pi i}),$$
$$p_\lambda(D)f = e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}p_{\overline{\lambda}}(D)e^{-[(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2]}f.$$

From these two formulas, we obtain:

$$[\partial_{\bar{\lambda}}G_\lambda] * u_0m_0 = -s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2},$$
$$G_\lambda\, e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2} = e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}\, G_{\overline{\lambda}}.$$

Consequently,

$$\partial_{\bar{\lambda}}m_0(x_1,x_2,\lambda) == s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}m_0(x_1,x_2,\bar{\lambda}). \tag{2.25}$$

To prove the CIE for m_0, applying Liouville's theorem and the Lax equation, there exists $q(x_1,x_2)$ such that

$$m_0(x_1,x_2,\lambda) = q(x_1,x_2) + \mathcal{C}T_0m_0(x_1,x_2,\lambda), \tag{2.26}$$

$$u_0m_0 = -2\lambda\partial_{x_1}q - \partial_{x_1}^2 q + \partial_{x_2}q + \left(\partial_{x_2} - \partial_{x_1}^2 - 2\lambda\partial_{x_1}\right)\mathcal{C}T_0m_0. \tag{2.27}$$

Via a change of variables

$$\begin{aligned} &2\pi i\xi = \overline{\zeta} - \zeta, \quad 2\pi i\eta = \overline{\zeta}^2 - \zeta^2, \\ &\zeta = -i\pi\xi + \frac{\eta}{2\xi}, \quad d\overline{\zeta}\wedge d\zeta = \frac{i\pi}{|\xi|}d\xi d\eta, \end{aligned} \tag{2.28}$$

and from (2.23), (2.24), we obtain,

$$\begin{aligned} &|\mathcal{C}T_0\phi| \le C|\iint \frac{s_c(\zeta)e^{(\overline{\zeta}-\zeta)x_1+(\overline{\zeta}^2-\zeta^2)x_2}\phi}{\zeta-\lambda}d\overline{\zeta}\wedge d\zeta| \\ &\le C|\phi|_{L^\infty}\iint \frac{|s_c(\zeta(\xi,\eta))|}{|(2\pi\xi)^2 - 4\pi i\xi\lambda + 2\pi i\eta|}d\xi d\eta \\ &\le C|\phi|_{L^\infty}\{|s_c(\zeta)|_{L^2(d\xi d\eta)}\left|\frac{1}{p_\lambda}\right|_{L^2(\Omega_\lambda^c,d\xi d\eta)} + |s_c(\zeta)|_{L^\infty(d\xi d\eta)}\left|\frac{1}{p_\lambda}\right|_{L^1(\Omega_\lambda,d\xi d\eta)}\}. \end{aligned} \tag{2.29}$$

Similarly, if $|\left(|\overline{\lambda}-\lambda|^{l_1} + |\overline{\lambda}^2-\lambda^2|^{l_2}\right)s_c(\lambda)|_{L^2\cap L^\infty(d\xi d\eta)} < \infty$, $l_1 \le 2$, $l_2 \le 1$, from (2.24), one has $\left(\partial_{x_2} - \partial_{x_1}^2 - 2\lambda\partial_{x_1}\right)\mathcal{C}T_0m_0 \to o(|\lambda|)$ as $\lambda_I \to \infty$. Thus, from (2.27) and for $\lambda \gg 1$, we get $q = q(x_2)$. By choosing $x_1 \gg 1$ in (2.26), we find $q \equiv 1$, justifying the initial CIE (2.10).

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2.2.2. *Proof of Theorem 2.3*

Note that

$$\partial_{\overline{\lambda}}\Phi(x,\lambda) = s_c(\lambda,x_3)\Phi(x,\overline{\lambda}).$$

Denote

$$\mathcal{M}_\lambda = -\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} + \rho(\lambda),\ \ \rho(\lambda) = -\lambda^3,$$

we have

$$\begin{aligned}
0 =& \partial_{\overline{\lambda}}\left[\mathcal{M}_\lambda\Phi(x,\lambda)\right] = \mathcal{M}_\lambda\left[\partial_{\overline{\lambda}}\Phi(x,\lambda)\right] = \mathcal{M}_\lambda\left[s_c(\lambda,x_3)\Phi(x,\overline{\lambda})\right] \\
=& \Phi(x,\overline{\lambda})\left[-\partial_{x_3} + \rho(\lambda)\right]s_c(\lambda,x_3) + s_c(\lambda,x_3)\left[\mathcal{M}_\lambda - \rho(\lambda)\right]\Phi(x,\overline{\lambda}) \\
=& \Phi(x,\overline{\lambda})\left[-\partial_{x_3} + \rho(\lambda)\right]s_c(\lambda,x_3) + s_c(\lambda,x_3)\left[\mathcal{M}_{\overline{\lambda}} - \rho(\overline{\lambda})\right]\Phi(x,\overline{\lambda}) \\
=& \Phi(x,\overline{\lambda})\left[-\partial_{x_3} + \rho(\lambda) - \rho(\overline{\lambda})\right]s_c(\lambda,x_3).
\end{aligned} \tag{2.30}$$

2.2.3. *Proof of Theorem 2.4*

(1) Unique solvability of the CIE (2.15) and the estimate (2.16) follow from

$$\begin{aligned}
|\mathcal{C}T\phi| \le\ & C|\phi|_{L^\infty}\{|s_c(x_3,\zeta)|_{L^2(d\xi d\eta)}\left|\frac{1}{p_\lambda}\right|_{L^2(\Omega_\lambda^c,d\xi d\eta)} \\
&+|s_c(x_3,\zeta)|_{L^\infty(d\xi d\eta)}\left|\frac{1}{p_\lambda}\right|_{L^1(\Omega_\lambda,d\xi d\eta)}\}
\end{aligned} \tag{2.31}$$

which is proved by the same argument as (2.29).

(2) To prove the Lax equation (2.17), we introduce the shorthand notation for the heat operator

$$\begin{gathered}
-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda\partial_{x_1} = -\nabla_2 + \nabla_1^2, \\
\nabla_1 = \partial_{x_1} + \lambda,\ \nabla_2 = \partial_{x_2} + \lambda^2,\ \left[\nabla_j,\ T\right] = 0.
\end{gathered}$$

Applying the heat operator to both sides of the CIE (2.15), formally,

$$(-\nabla_2 + \nabla_1^2)m = \left[-\nabla_2 + \nabla_1^2, \mathcal{C}T\right]m + \mathcal{C}T(-\nabla_2 + \nabla_1^2)m, \tag{2.32}$$

and

$$\begin{aligned}
&\left[-\nabla_2 + \nabla_1^2, \mathcal{C}T\right]m = \left[-\nabla_2 + \nabla_1^2, \mathcal{C}\right]Tm = 2\left[\lambda, \mathcal{C}\right]\partial_{x_1}(Tm) \\
=&\frac{1}{\pi i}\partial_{x_1}\iint Tm\, d\overline{\zeta}\wedge d\zeta \equiv -u(x),
\end{aligned} \tag{2.33}$$

along with the unique solvability of the CIE, yields the Lax equation

$$(-\nabla_2 + \nabla_1^2)m = -(1-\mathcal{C}T)^{-1}u(x)1 = -u(x)(1-\mathcal{C}T)^{-1}1 = -u(x)m(x,\lambda). \tag{2.34}$$

To rigorously justify the argument, we focus on *a priori* estimates for:

$$\widehat{u} = -\widehat{m-1} * \widehat{u} - p_\lambda(\xi,\eta)\widehat{m-1}.$$

We will derive these estimates:

$$|\widehat{m-1}|_{L^1(d\xi d\eta)} \le C|s_c|_{L^2\cap L^\infty(d\xi d\eta)}, \tag{2.35}$$

$$|(1+|\xi|^k+|\eta|^k)p_\lambda(\xi,\eta)\widehat{m-1}|_{L^2\cap L^\infty(d\xi d\eta)} \le C|(1+|\xi|^{k+2}+|\eta|^{k+2})s_c|_{L^2\cap L^\infty(d\xi d\eta)}, \tag{2.36}$$

allowing us to apply Minkowski inequality to obtain:

$$|(1+|\xi|^k+|\eta|^k)\widehat{u}(x)|_{L^2\cap L^\infty(d\xi d\eta)} \le C|(1+|\xi|^{k+2}+|\eta|^{k+2})s_c|_{L^2\cap L^\infty(d\xi d\eta)}, \tag{2.37}$$

$$(-\nabla_2+\nabla_1^2)m \in L^\infty. \tag{2.38}$$

Write the CIE (2.15) as

$$\widehat{m-1} = [\mathcal{C}T(m-1)]^\wedge + [\mathcal{C}T1]^\wedge = [\mathcal{C}T(m-1)]^\wedge + \frac{2\pi i s_c(\zeta(\xi,\eta))}{p_\lambda(\xi,\eta)}. \tag{2.39}$$

Using $\left|\frac{2\pi i s_c(\zeta(\xi,\eta))}{p_\lambda(\xi,\eta)}\right|_{L^1} \le C|s_c|_{L^2\cap L^\infty(d\xi d\eta)}$ and

$$[\mathcal{C}Tf]^\wedge = -\frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{s_c(\zeta)\widehat{f}(\xi-\frac{\bar\zeta-\zeta}{2\pi i},\eta-\frac{\bar\zeta^2-\zeta^2}{2\pi i},\bar\zeta)}{\zeta-\lambda}d\bar\zeta\wedge d\zeta \equiv R_{s_c}\widehat{f},$$

which is a contraction of $\widehat{f}\in L^1(d\xi d\eta)$, we prove (2.35).

Next, we express (2.39) as:

$$p_\lambda\widehat{m-1} = p_\lambda R_{s_c}\widehat{m-1} + 2\pi i s_c(\zeta(\xi,\eta)) \equiv M_{s_c}\left(p_\zeta\widehat{m-1}\right) + 2\pi i s_c(\zeta(\xi,\eta)),$$
$$M_{s_c}f = \left(R_{s_c}f\right)(\xi,\eta,\lambda) - \left(R_{s_c}f\right)(\xi,\eta,\frac{\eta}{2\xi}-i\pi\xi),$$

and we prove

$$|(1+|\xi|^k+|\eta|^k)M_{s_c}f|_{L^2\cap L^\infty} \le C|(1+|\xi|^{k+2}+|\eta|^{k+2})s_c|_{L^2\cap L^\infty}|(1+|\xi|^k+|\eta|^k)f|_{L^2\cap L^\infty}.$$

This proves (2.36).

(3) To justify the KP equation (2.21), we verify the Lax pairs. If $|(|\bar\lambda-\lambda|^{l_1}+|\bar\lambda^2-\lambda^2|^{l_2})s_c(\lambda)|_{L^2\cap L^\infty(d\xi d\eta)}<\infty$, $l_1\le 5$, $l_2\le 2$, using the representation formula (2.18), we define $\Phi(x,\lambda)=e^{\lambda x_1+\lambda^2x_2}m(x,\lambda)$ and the evolution operator:

$$\mathcal{M} = -\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} - \lambda^3. \tag{2.40}$$

Then

$$\begin{aligned}\mathcal{M}\Phi(x,\lambda) =& e^{\lambda x_1+\lambda^2x_2}\left(\mathcal{M}+3\lambda\partial_{x_1}^2+3\lambda^2\partial_{x_1}+\lambda^3+\frac{3}{2}u\lambda\right)m(x,\lambda)\\ \equiv& e^{\lambda x_1+\lambda^2x_2}\mathfrak{M}m.\end{aligned}$$

Reversing the procedure to prove (2.30), we obtain:

$$\partial_{\overline{\lambda}} (\mathfrak{M}m)(x,\lambda) = s_c(\lambda) e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2+(\overline{\lambda}^3-\lambda^3)x_3} (\mathfrak{M}m)(x,\overline{\lambda}). \tag{2.41}$$

As $|\lambda| \to \infty$, $m(x,\lambda) \sim \sum_{j=0}^{\infty} \frac{M_j(x)}{\lambda^j}$. From the Lax equation (2.17), we obtain

$$\begin{gathered} 2\partial_{x_1} M_{j+1} = (\partial_{x_2} - \partial_{x_1}^2 - u) M_j, \\ M_0 = 1,\ M_1 = -\frac{1}{2}\partial_{x_1}^{-1} u,\ M_2 = -\frac{1}{4}\partial_{x_2}\partial_{x_1}^{-2} u + \frac{1}{4} u + \frac{1}{4}\partial_{x_1}^{-1}\left(u\partial_{x_1}^{-1} u\right),\ \cdots \end{gathered} \tag{2.42}$$

As a result, as $\lambda \to \infty$,

$$\begin{aligned} &\mathfrak{M}m \\ \to&\frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1} u_{x_2} + 3\lambda\partial_{x_1}^2 (1 + \frac{M_1}{\lambda}) + 3\lambda^2 \partial_{x_1} (1 + \frac{M_1}{\lambda} + \frac{M_2}{\lambda^2}) + \frac{3}{2}u\lambda \\ =&\frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1} u_{x_2} + \left(-\frac{3}{2}u_{x_1} + 3\partial_{x_1}[-\frac{1}{4}\partial_{x_2}\partial_{x_1}^{-2} u + \frac{u}{4} + \frac{1}{4}\partial_{x_1}^{-1}(u\partial_{x_1}^{-1} u)]\right) \\ &+\lambda\left(3\partial_{x_1} M_1 + \frac{3}{2}u\right) + \frac{3}{2}u(-\frac{1}{2}\partial_{x_1}^{-1} u) \\ =&0. \end{aligned} \tag{2.43}$$

Using the unique solvability of the CIE, we conclude that $\mathfrak{M}m(x,\lambda) = 0$, $\mathcal{M}\Phi(x,\lambda) = 0$, thus verifying the Lax pair and justifying the KPII equation.

3. The IST for perturbed 1-line solitons

3.1. 1-line solitons

The KPII equation (1.1) admits explicit solutions known as $\mathrm{Gr}(N,M)_{\geq 0}$ KP solitons, which are regular across the x_1x_2-plane with non-decaying localized peaks along specific line segments and rays for fixed time x_3. These solitons can be constructed using Sato theory as [2, 3, 19, 22–24]:

$$u_s(x) = 2\partial_{x_1}^2 \ln \tau(x), \tag{3.1}$$

where the τ-function is the Wronskian determinant

$$\begin{aligned} \tau(x) =& \left| \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{pmatrix} \begin{pmatrix} E_1 & \cdots & \kappa_1^{N-1} E_1 \\ E_2 & \cdots & \kappa_2^{N-1} E_2 \\ \vdots & \ddots & \vdots \\ E_M & \cdots & \kappa_M^{N-1} E_M \end{pmatrix} \right| \\ =& \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1,\cdots j_N}(A) E_{j_1,\cdots j_N}(x). \end{aligned} \tag{3.2}$$

Here $\kappa_1 < \cdots < \kappa_M$, $\kappa_j \neq 0$, $E_j(x) = \exp\theta_j(x) = \exp(\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3)$, $A = (a_{ij}) \in \mathrm{Gr}(N,M)_{\geq 0}$ represents a full rank $N \times M$ real matrices with non-negative minors, $\Delta_{j_1,\cdots j_N}(A) = \Delta_J(A)$ the $N \times N$ minor of the matrix A whose columns are labelled by the index set $J = \{j_1 < \cdots < j_N\} \subset \{1, \cdots, M\}$, and $E_J = E_{j_1,\cdots j_N}(x) = \Pi_{l<m}(\kappa_{j_m} - \kappa_{j_l}) \exp(\sum_{n=1}^{N} \theta_{j_n}(x))$. Moreover, $\mathrm{Gr}(N,M)_{>0}$ KP solitons means all

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minors $\Delta_{j_1,\cdots,j_N}$ are positive, namely, fulfilling the TP condition, form a dense subset of $\mathrm{Gr}(N,M)_{\geq 0}$ KP solitons. For example, the $\mathrm{Gr}(1,2)_{>0}$ KP solitons (or 1-line solitons) are given by:

$$u_s(x) = \frac{(\kappa_1-\kappa_2)^2}{2}\mathrm{sech}^2\frac{\theta_1(x)-\theta_2(x)-\ln a}{2}, \tag{3.3}$$

where $A=(1,a)$ and $a>0$.

A brief overview of Sato theory is provided in Section 4.2.1, and a formal inverse scattering transform (IST) applicable to multi-line solitons is shown in [28–30]. In this section, we present a rigorous IST for perturbed 1-line solitons without using the Sato theory.

Lemma 3.1. (IST for 1-line solitons) *Let $u_s(x)$ be a* $\mathrm{Gr}(1,2)_{>0}$ *KP soliton. The Sato eigenfunction φ and the Sato normalized eigenfunction χ, defined by*

$$\varphi(x,\lambda) = e^{\lambda x_1+\lambda^2 x_2}\frac{(1-\frac{\kappa_1}{\lambda})e^{\theta_1(x)}+(1-\frac{\kappa_2}{\lambda})ae^{\theta_2(x)}}{e^{\theta_1(x)}+ae^{\theta_2(x)}} \equiv e^{\lambda x_1+\lambda^2 x_2}\chi(x,\lambda) \tag{3.4}$$

satisfy the Lax equations for $\lambda\in\mathbb{C}\backslash\{0\}$,

$$\left(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u_s(x)\right)\chi(x,\lambda)=0; \tag{3.5}$$

and

$$\begin{gathered}\chi(x,\lambda)=1+\frac{\chi_{0,\mathrm{res}}(x)}{\lambda}, \qquad \lambda\in\mathbb{C}\backslash\{0\},\\ (e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}\chi(x,\kappa_1), e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_1^3x_3}\chi(x,\kappa_2))\mathcal{D}^\flat=0,\end{gathered} \tag{3.6}$$

with

$$\mathcal{D}^\flat = diag\,(\kappa_1,\kappa_2)\,A^T = \begin{pmatrix}\kappa_1\\ \kappa_2 a\end{pmatrix}. \tag{3.7}$$

The forward scattering transform *is defined by*

$$\mathcal{S}: u_s(x_1,x_2,0)\mapsto\{0,\kappa_1,\kappa_2,\mathcal{D}^\flat\}, \tag{3.8}$$

and the inverse scattering transform by

$$\mathcal{S}^{-1}(\{0,\kappa_1,\kappa_2,\mathcal{D}^\flat\}) = -2\partial_{x_1}\chi_{0,\mathrm{res}}(x). \tag{3.9}$$

Proof. The lemma is proved by using (3.3) and computing the λ^k-coefficients of (3.5). □

3.2. The direct problem for perturbed KP 1-solitons

3.2.1. Statement of results

Building upon Boiti *et al.*'s work [4–10], rigorous direct scattering theory for perturbed $\mathrm{Gr}(1,2)_{>0}$ KP solitons is carried out in [31, 32].

Theorem 3.2. (Direct Scattering Theory) *[31, 32] Given initial data*

$$u_0(x_1,x_2)=u_s(x_1,x_2,0)+v_0(x_1,x_2), \tag{3.10}$$

where $u_s(x)$ is a $\mathrm{Gr}(1,2)_{>0}$ KP soliton and $\sum_{|l|\le d+8}|(1+|x_1|+|x_2|)\partial_x^l v_0|_{L^1\cap L^\infty}\ll 1$, $d\ge 0$, we have:

(1) For $\lambda\in\mathbb{C}\backslash\{0,\kappa_1,\kappa_2\}$, there exists a unique solution to the Lax equation:

$$(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u_0(x_1,x_2))m_0(x_1,x_2,\lambda)=0, \tag{3.11}$$

$$\lim_{|x|\to\infty}m_0(x_1,x_2,\lambda)=\chi(x_1,x_2,0,\lambda). \tag{3.12}$$

(2) The forward scattering transform is defined by

$$\mathcal{S}(u_0)=(0,\kappa_1,\kappa_2,\mathcal{D},s_c(\lambda)) \tag{3.13}$$

where the scattering data satisfy the CIE and the $\mathcal{D}$-symmetry

$$m_0(x_1,x_2,\lambda)=1+\frac{m_{0,\mathrm{res}}(x_1,x_2)}{\lambda}+\mathcal{C}T_0m_0(x_1,x_2,\lambda), \tag{3.14}$$

$$(e^{\kappa_1x_1+\kappa_1^2x_2}m_0(x_1,x_2,\kappa_1^+),e^{\kappa_2x_1+\kappa_2^2x_2}m_0(x_1,x_2,\kappa_2^+))\mathcal{D}=0. \tag{3.15}$$

Here, $m_0\in W_0$, $\kappa_j^+=\kappa_j+0^+$, $\mathcal{C}$ is the Cauchy integral operator, T_0 is the continuous scattering operator at $x_3=0$ defined by (2.13), and $\mathrm{s_c}$ is the continuous scattering data arising from the $\overline{\partial}$-characterization

$$\begin{aligned}&\partial_{\overline{\lambda}}m_0(x_1,x_2,\lambda)=s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}m_0(x_1,x_2,\overline{\lambda}),\ \lambda\notin\mathbb{R},\\&s_c(\lambda)=\frac{\mathrm{sgn}(\lambda_I)}{2\pi i}\left[\xi(\cdot,0,\bar\lambda)v_0(\cdot)m_0(\cdot,\lambda)\right]^\wedge\left(\frac{\overline{\lambda}-\lambda}{2\pi i},\frac{\overline{\lambda}^2-\lambda^2}{2\pi i}\right),\end{aligned} \tag{3.16}$$

and $\xi(x,\lambda)$ being the normalized Sato adjoint eigenfunction (see (3.34), (3.35) for definition). Moreover, $\mathcal{D}$ can be computed by

$$\mathcal{D}=\mathcal{D}^\sharp\times\left(\mathcal{D}^\sharp_{11}\right)^{-1}\times\kappa_1=\begin{pmatrix}\kappa_1\\ \mathcal{D}_{21}\end{pmatrix}, \tag{3.17}$$

$$\mathcal{D}^\sharp=\begin{pmatrix}\mathcal{D}^\sharp_{11}\\ \mathcal{D}^\sharp_{21}\end{pmatrix}=\begin{pmatrix}\mathcal{D}^\flat_{11}+\dfrac{c_{11}\mathcal{D}^\flat_{11}}{1-c_{11}}+\dfrac{c_{12}\mathcal{D}^\flat_{21}}{1-c_{11}}\\ \mathcal{D}^\flat_{21}+\dfrac{c_{22}\mathcal{D}^\flat_{21}}{1-c_{22}}\end{pmatrix}, \tag{3.18}$$

$$\mathcal{D}^\flat=diag\,(\kappa_1,\kappa_2)\,A^T=\begin{pmatrix}\kappa_1\\ \kappa_2a\end{pmatrix}, \tag{3.19}$$

with $c_{jl}=-\int\Psi_j(x_1,x_2,0)v_0(x_1,x_2)\varphi_l(x_1,x_2,0)dx_1dx_2$, $\Psi_j(x)$, $\varphi_l(x)$ residues of the adjoint eigenfunction $\Psi(x,\lambda)$ at κ_j [32, (3.17))] and values of the Sato eigenfunction $\varphi(x,\lambda)$ at κ_l; $W_0=W_{(x_1,x_2,0)}$ is the eigenfunction space defined in Definition 3.3.

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Finally, the scattering data $\mathcal{S}(u_0)$ satisfies the algebraic and analytic constraints

$$s_c(\lambda) = \begin{cases} \frac{\frac{i}{2}\operatorname{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|} + \operatorname{sgn}(\lambda_I)h_j(\lambda), & \lambda \in D^\times_{\kappa_j}, \\ \operatorname{sgn}(\lambda_I)\hbar_0(\lambda), & \lambda \in D^\times_0, \end{cases} \tag{3.20}$$
$$\mathcal{D} = (\kappa_1, \mathcal{D}_{21})^T ,$$

and

$$\begin{aligned} &|(1-\textstyle\sum_{j=1}^2 \mathcal{E}_{\kappa_j})\sum_{|l|\le d+8}|\left(|\overline{\lambda}-\lambda|^{l_1} + |\overline{\lambda}^2 - \lambda^2|^{l_2}\right) s_c(\lambda)|_{L^\infty} \\ &+\textstyle\sum_{j=1}^2(|\gamma_j| + |h_j|_{L^\infty(D_{\kappa_j})}) + |\hbar_0|_{C^1(D_0)} + |\mathcal{D}-\mathcal{D}^\flat|_{L^\infty} \\ &\le C\textstyle\sum_{|l|\le d+8}|(1+|x_1|+|x_2|)\partial_x^l v_0|_{L^1\cap L^\infty}, \end{aligned} \tag{3.21}$$

$$s_c(\lambda) = \overline{s_c(\overline{\lambda})}, h_j(\lambda) = -\overline{h_j(\overline{\lambda})}, \hbar_0(\lambda) = -\overline{\hbar_0(\overline{\lambda})}. \tag{3.22}$$

Here $D_{z,a\delta} = \{\lambda = z + re^{i\alpha} : 0 \le r \le a\delta, |\alpha| \le \pi\}$, $D^\times_{z,a\delta} = D_{z,a\delta}\backslash\{z\}$, $1 \ge \delta = \frac{1}{2}\inf\{|z-z'| : z, z' \in \{0, \kappa_1, \kappa_2\}, z \ne z'\}$, $\mathcal{E}_{z,a\delta}(\lambda) \equiv 1$ on $D_{z,a\delta}$, $\mathcal{E}_{z,a\delta}(\lambda) \equiv 0$ elsewhere. We suppress the $a\delta$-dependence for simplicity if a = *1.*

Definition 3.3. *The eigenfunction space $W_0 = W_{x_1,x_2,0}$ consists of functions ϕ that satisfy the following conditions:*

(a) $\phi(x_1,x_2,\lambda) = \overline{\phi(x_1,x_2,\overline{\lambda})}$;
(b) $(1-\mathcal{E}_0)\phi(x_1,x_2,\lambda) \in L^\infty$;
(c) *For* $\lambda \in D^\times_0$, $\phi(x_1,x_2,\lambda) = \frac{\phi_{0,\mathrm{res}}(x_1,x_2)}{\lambda} + \phi_{0,r}(x_1,x_2,\lambda)$, $\phi_{0,\mathrm{res}}$, $\phi_{0,r} \in L^\infty(D_0)$;
(d) *For* $\lambda = \kappa_j + re^{i\alpha} \in D^\times_{\kappa_j}$, $\phi = \phi^\flat + \phi^\sharp$, $\phi^\flat = \sum_{l=0}^\infty \phi_l(X_1,X_2)(-\ln(1-\gamma_j|\alpha|))^l \in L^\infty(D_{\kappa_j})$, $\phi^\sharp \in C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}}) \cap L^\infty(D_{\kappa_j})$, *and* $\phi^\sharp(x_1,x_2,\kappa_j) = 0$.

Here, $\tilde\sigma = \max\{1, |X_1|, \sqrt{|X_2|}\}$ is the rescaling parameter, and X_k *is defined by the phase function coefficients:*

$$\begin{aligned} \wp(x_1,x_2,\lambda) =& i[(\overline{\lambda}-\lambda)x_1 + (\overline{\lambda}^2 - \lambda^2)x_2] = X_1 r\sin\alpha + X_2 r^2 \sin 2\alpha \equiv \wp(r,\alpha,X), \\ X_1 =& 2(x_1+2x_2 z),\ X_2 = 2x_2,\ \lambda = z + re^{i\alpha} \in D_z. \end{aligned} \tag{3.23}$$

Finally, $C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}}) = C(D_{\kappa_j,\frac{1}{\tilde\sigma}}) \cap H^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})$ and $H^\mu_{\tilde\sigma}(D_{z,\frac{1}{\tilde\sigma}})$ is the rescaled Hölder space for $z \in \mathbb{R}$, where the norm is given by:

$$|\phi|_{H^\mu_{\tilde\sigma}(D_{z,\frac{1}{\tilde\sigma}})} \equiv \sup_{\tilde r_1,\tilde r_2 \le 1, |\alpha_1|,|\alpha_2| \le \pi} \frac{|\phi(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X_1,X_2) - \phi(\frac{\tilde r_2}{\tilde\sigma},\alpha_2,X_1,X_2)|}{|\tilde r_1 e^{i\alpha_1} - \tilde r_2 e^{i\alpha_2}|^\mu} < \infty \tag{3.24}$$

for $\lambda_j = z + r_j e^{i\alpha_j} = z + \frac{\tilde r_j}{\tilde\sigma}e^{i\alpha_j} \in D_{z,\frac{1}{\tilde\sigma}}$ and $\phi(x,\lambda) \equiv \phi(r,\alpha,X_1,X_2)$.

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Theorem 3.4. (Linearization Theorem) [31, 32] *If* $\Phi = e^{\lambda x_1+\lambda^2 x_2} m(x,\lambda)$ *satisfies the Lax pair (1.2) and*

$$\partial_{\overline{\lambda}} m(x,\lambda) = s_c(\lambda, x_3) e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2} m(x,\overline{\lambda}), \tag{3.25}$$

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2} m(x,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2} m(x,\kappa_2^+))\mathcal{D}(x_3) = 0, \tag{3.26}$$

with $\mathcal{D}(x_3) = (\kappa_1, \mathcal{D}_{21}(x_3))^T$ *then*

$$s_c(\lambda, x_3) = e^{(\overline{\lambda}^3-\lambda^3)x_3} s_c(\lambda), \quad \mathcal{D}_{mn}(x_3) = e^{(\kappa_m^3-\kappa_n^3)x_3}\mathcal{D}_{mn}. \tag{3.27}$$

We make several remarks to conclude this subsection.

- Comparing (3.16)–(3.19), (3.34), (2.8) and (3.7), we show that when the discrete or continuous scattering data vanish, the forward scattering transform for perturbed 1-line solitons reduces to transforms for rapidly decaying potentials or 1-line solitons.
- For a perturbed 1-soliton, away from 0, κ_j, the continuous data s_c and eigenfunction m_0 are regular, similar to the rapidly decaying potential case. But at 0 and κ_j, the Cauchy integral operator $\mathcal{C}T_0 m_0$ is an oscillatory singular operator. Specifically, at κ_j, s_c has a ‘simple pole with a discontinuous residue’ and m_0 is multi-valued. To address this, we introduce the rescaled Hölder structure $C^{\mu}_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})$ (see Section 3.3).
- For the KPII equation, small $L^1(R^2)\cap L^\infty$ perturbations preserve the discrete scattering data κ_j, consistent with the fact that 1-line solitons form a discrete set in $L^p(R^2)$. In contrast, for the KdV equation, small $L^1(R)\cap L^\infty$ perturbations generically alter κ_j and even potentially increase the number of bound states.

3.2.2. *The strategy of the proof of Theorem 3.2*

Throughout Section 3.2.2, $x = (x_1, x_2, 0)$, $x' = (x_1', x_2', 0)$ for simplicity.

(1) The Lax equation is proved by transforming the Lax equation into an integral equation with the Green function defined by:

$$m_0(x,\lambda) = \chi(x,\lambda) - G * v_0 m_0, \tag{3.28}$$

$$G(x,x',\lambda) = G_c(x,x',\lambda) + G_d(x,x',\lambda), \tag{3.29}$$

$$G_c(x,x',\lambda) = -\frac{\operatorname{sgn}(x_2-x_2')}{2\pi} e^{\lambda(x_1'-x_1)+\lambda^2(x_2'-x_2)} \int_{\mathbb{R}} \theta((s^2-\lambda_I^2)(x_2-x_2')) \tag{3.30}$$

$$\times\varphi(x,\lambda_R+is)\psi(x',\lambda_R+is)\,ds,$$

$$\begin{aligned} G_d(x,x',\lambda) = &-\theta(x_2'-x_2)e^{\lambda(x_1'-x_1)+\lambda^2(x_2'-x_2)} \\ &\times(\theta(\lambda_R-\kappa_1)\varphi_1(x)\psi_1(x')+\theta(\lambda_R-\kappa_2)\varphi_2(x)\psi_2(x')), \end{aligned} \tag{3.31}$$

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and establishing the following estimates:

$$G * f(x,\lambda) \equiv \iint G(x,x',\lambda) f(x') dx',$$

$$|G_c(x,x',\lambda)| \leq C(1 + \frac{1}{\sqrt{|x_2 - x_2'|}}), \tag{3.32}$$

$$|G_d(x,x',\lambda)| \leq C, \tag{3.33}$$

$$\lim_{|x|\to\infty} G(x,x',\lambda) * f(x') = 0.$$

Here $\theta(s)$ is the Heaviside function, ψ, ξ are the Sato adjoint eigenfunction, normalized Sato adjoint eigenfunction [6, (2.12)], [12, Theorem 6.3.8. (6.3.13)]

$$\begin{aligned}\psi(x_1,x_2,x_3,\lambda) =& e^{-(\lambda x_1+\lambda^2 x_2)} \frac{\frac{e^{\theta_1(x_1,x_2,x_3)}}{(1-\frac{\kappa_1}{\lambda})} + \frac{a e^{\theta_2(x_1,x_2,x_3)}}{(1-\frac{\kappa_2}{\lambda})}}{e^{\theta_1(x_1,x_2,x_3)} + a e^{\theta_2(x_1,x_2,x_3)}} \\ \equiv& e^{-[\lambda x_1+\lambda^2 x_2]} \xi(x_1,x_2,x_3,\lambda),\end{aligned} \tag{3.34}$$

satisfying

$$\begin{aligned}&\left(\partial_{x_2} + \partial_{x_1}^2 + u_s(x_1,x_2,x_3)\right)\psi(x_1,x_2,x_3,\lambda) = 0,\\ &\left(\partial_{x_2} + \partial_{x_1}^2 - 2\lambda\partial_{x_1} + u_s(x_1,x_2,x_3)\right)\xi(x_1,x_2,x_3,\lambda) = 0.\end{aligned} \tag{3.35}$$

Finally,

$$\varphi_j(x) \equiv \varphi(x,\kappa_j) = e^{\kappa_j x_1 + \kappa_j^2 x_2}\chi_j(x), \ \ \psi_j(x) \equiv \mathrm{res}_{\lambda=\kappa_j}\psi(x,\lambda) = e^{-[\kappa_j x_1 + \kappa_j^2 x_2]}\xi_j(x).$$

In the following, we will explain the construction of the Green function and provide estimates.

- ► **Construction of the Green function (3.29)–(3.31):** [6, 10] Using Fourier inversion theorem, the residue theorem and the orthogonality

$$\sum_{j=1}^{2} \varphi_j(x)\psi_j(x') = 0, \tag{3.36}$$

we first derive the orthogonality relation

$$\begin{aligned}\delta(x-x') = \delta(x_2 - x_2')\{&\frac{1}{2\pi}\int_{\mathbb{R}} \varphi(x,\lambda_R + is)\psi(x',\lambda_R + is) ds \\ &- \sum_{j=1}^{2} \varphi_j(x)\psi_j(x')\theta(\lambda_R - \kappa_j)\}.\end{aligned} \tag{3.37}$$

Therefore, G defined by (3.29)–(3.31) satisfies

$$\left(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda\partial_{x_1} + u_s(x)\right) G(x,x',\lambda) = \delta(x-x')$$

by applying (3.37) and

$$\mathrm{sgn}(x_2 - x_2')\theta((s^2 - \lambda_I^2)(x_2 - x_2')) = \theta(x_2 - x_2')\chi_-(s) - \theta(x_2' - x_2)\chi_+(s),$$

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where $\chi_\pm(s)$ the characteristic function for $\{s|\,Re([\lambda+is]^2-\lambda^2)\gtrless 0\}$.

- ► **Estimates of the Green function (3.32), (3.33):** The proof for G_c only requires the totally non-negative (TNN) condition. For $\lambda\in D^c_{\kappa_1}\cap D^c_{\kappa_2}$, direct computation or properties of special functions give the estimate

$$|G_c(x,x',\lambda)|\le C(1+\frac{1}{\sqrt{|x_2-x_2'|}}).$$

For $\lambda\in D^\times_{\kappa_j}$, we define

$$G_c(x,x',\lambda)=-\frac{e^{i[\lambda_I(x_1'-x_1)+2\lambda_I\lambda_R(x_2'-x_2)]}}{2\pi}\left(I_j^{[1]}+I_j^{[2]}+I_j^{[3]}+I_j^{[4]}\right),\tag{3.38}$$

where

$$\begin{aligned}
I_j^{[1]}=:&\int_{-\delta}^{\delta}\operatorname{sgn}(x_2-x_2')\theta((s^2-\lambda_I^2)(x_2-x_2'))\chi(x,\lambda_R+is)\xi(x',\lambda_R+is)\\
&\times[e^{is[x_1-x_1'+2\lambda_R(x_2-x_2')]+(\lambda_I^2-s^2)(x_2-x_2')}-1]ds,\\
I_j^{[2]}=:&\int_{-\delta}^{\delta}\operatorname{sgn}(x_2-x_2')\theta((s^2-\lambda_I^2)(x_2-x_2'))\\
&\times[\chi(x,\lambda_R+is)\xi(x',\lambda_R+is)-\frac{\chi_j(x)\xi_j(x')}{\lambda_R+is-\kappa_j}]ds,\\
I_j^{[3]}=:&\int_{-\delta}^{\delta}\operatorname{sgn}(x_2-x_2')\theta((s^2-\lambda_I^2)(x_2-x_2'))\frac{\chi_j(x)\xi_j(x')}{\lambda_R+is-\kappa_j}ds,\\
I_j^{[4]}=:&\left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right)\operatorname{sgn}(x_2-x_2')\theta((s^2-\lambda_I^2)(x_2-x_2'))\chi(x,\lambda_R+is)\\
&\times\xi(x',\lambda_R+is)e^{(s^2-\lambda_I^2)(x_2'-x_2)-is[(x_1'-x_1)+2\lambda_R(x_2'-x_2)]}ds.
\end{aligned}\tag{3.39}$$

We prove that

$$|I_j^{[1]}|,\ |I_j^{[2]}|,\ |I_j^{[3]}|<C,\quad |I_j^{[4]}|\le C(1+\frac{1}{\sqrt{|x_2-x_2'|}}),$$

The uniform estimates for $|I_j^{[1]}|$ are derived using appropriate changes of variables and the residue theorem, while $|I_j^{[3]}|$ involves logarithmic functions, causing a discontinuity at κ_j.

For G_d, note that $\varphi_j(x)$ and $\psi_j(x')$ have 2-cells and 0-cells in their nominators, respectively:

$$\begin{aligned}
\varphi_1(x)&=\frac{1-a\frac{\kappa_2}{\kappa_1}e^{\theta_1(x)+\theta_2(x)}}{e^{\theta_1(x)}+ae^{\theta_2(x)}},\quad \varphi_2(x)=\frac{1-\frac{\kappa_1}{\kappa_2}e^{\theta_1(x)+\theta_2(x)}}{e^{\theta_1(x)}+ae^{\theta_2(x)}},\\
\psi_1(x')&=\frac{\kappa_1}{e^{\theta_1(x')}+ae^{\theta_2(x')}},\quad \psi_2(x')=\frac{a\kappa_2}{e^{\theta_1(x')}+ae^{\theta_2(x')}}.
\end{aligned}$$

Following the argument in [5], [4] to permute and exchange cells, we obtain the decomposition

$$G_d(x,x',\lambda)=G_d^1(x,x',\lambda)+G_d^2(x,x',\lambda),\tag{3.40}$$

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where

$$
\begin{aligned}
&G_d^1(x,x',\lambda)\\
&=(\kappa_2-\kappa_1)a\,e^{i\lambda_I[x_1'-x_1+2\lambda_R(x_2'-x_2)]}\,\theta(k_{12}(x_2'-x_2))\,\theta((\lambda_R-\kappa_1)(z_{12}-z_{12}'))\\
&\quad\times\operatorname{sgn}(z_{12}-z_{12}')\frac{e^{-k_{12}(x_2'-x_2)+(\lambda_R-\kappa_1)(z_{12}'-z_{12})}\,e^{\theta_1(x')+\theta_2(x)}}{\tau(x)\tau(x')},
\end{aligned}
\tag{3.41}
$$

and

$$
\begin{aligned}
&G_d^2(x,x',\lambda)\\
&=(\kappa_1-\kappa_2)a\,e^{i\lambda_I[x_1'-x_1+2\lambda_R(x_2'-x_2)]}\theta(k_{12}(x_2'-x_2))\,\theta((\lambda_R-\kappa_2)(z_{12}-z_{12}'))\\
&\quad\times\operatorname{sgn}(z_{12}-z_{12}')\frac{e^{-k_{12}(x_2'-x_2)+(\lambda_R-\kappa_2)(z_{12}'-z_{12})}\,e^{\theta_1(x)+\theta_2(x')}}{\tau(x)\tau(x')}.
\end{aligned}
\tag{3.42}
$$

Here $z_{mn}=x_1+(\kappa_m+\kappa_n)x_2$, $z_{mn}'=x_1'+(\kappa_m+\kappa_n)x_2'$, and $k_{mn}=\lambda_I^2-(\lambda_R-\kappa_m)(\lambda_R-\kappa_n)$ for $m,\,n\in\{1,\,2\}$. Now all exponentials in the numerators are bounded or dominated by the tau functions in the denominators due to the TP condition, thus proving the result.

(2) • ►The continuous scattering data s_c is derived in a similar way to that for rapidly decaying potentials. Specifically, we first compute $\partial_{\bar\lambda}G(x,x',\lambda)$ and verify the commutative relation between the Green function and the exponential functions:

$$
\begin{aligned}
\partial_{\bar\lambda}G(x,x',\lambda)=-\frac{\operatorname{sgn}(\lambda_I)}{2\pi i}e^{(\bar\lambda-\lambda)(x_1-x_1')+(\bar\lambda^2-\lambda^2)(x_2-x_2')}\chi(x,\bar\lambda)\xi(x',\bar\lambda),\\
G_\lambda\,e^{(\bar\lambda-\lambda)x_1+(\bar\lambda^2-\lambda^2)x_2}=e^{(\bar\lambda-\lambda)x_1+(\bar\lambda^2-\lambda^2)x_2}\,G_{\bar\lambda}.
\end{aligned}
\tag{3.43}
$$

As a result,

$$
\partial_{\bar\lambda}m_0(x,\lambda)=s_c(\lambda)e^{(\bar\lambda-\lambda)x_1+(\bar\lambda^2-\lambda^2)x_2}m_0(x,\bar\lambda),
\tag{3.44}
$$

with $s_c(\lambda)=\frac{\operatorname{sgn}(\lambda_I)}{2\pi i}\left[\xi(\cdot,\bar\lambda)v_0(\cdot)m_0(\cdot,\lambda)\right]^\wedge(\frac{\bar\lambda-\lambda}{2\pi i},\frac{\bar\lambda^2-\lambda^2}{2\pi i})$.

We analyse the analytic properties of the continuous scattering data s_c at ∞, κ_j and 0:

∗ **Away from $\boldsymbol{\kappa_j}$**, the Fourier theory gives

$$
|(1-\sum_{j=1}^2\mathcal{E}_{\kappa_j}(\lambda))(|\bar\lambda-\lambda|^{l_1}+|\bar\lambda^2-\lambda^2|^{l_2})s_c(\lambda)|_{L^\infty}\le C\sum_{h=0}^{l}|\partial_x^hv_0|_{L^1\cap L^\infty}.
\tag{3.45}
$$

∗ **Near $\boldsymbol{\kappa_j}$**, we derive the following asymptotics for the Green's function:

$$
G(x,x',\lambda)=\mathfrak{G}_j(x,x')+\frac{1}{\pi}\chi_j(x)\xi_j(x')|\alpha|+\omega_j(x,x',\lambda),
\tag{3.46}
$$

$$
|\mathfrak{G}_j|_{C(D_{\kappa_j})}\le C(1+\frac{1}{\sqrt{|x_2-x_2'|}}),\ \omega_j(x,x',\kappa_j)=0,
\tag{3.47}
$$

$$
|\omega_j|_{L^\infty(D_{\kappa_j})\cap C^\mu_{\bar\sigma}(D_{\kappa_j},\frac{1}{\bar\sigma})}\le C(1+\frac{1+|x'|}{\sqrt{|x_2-x_2'|}}).
\tag{3.48}
$$

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Here to derive (3.48), we have used, for $\lambda_j = z + r_j e^{i\alpha_j} = z + \frac{\tilde r_j}{\tilde\sigma} e^{i\alpha_j} \in D_{z,\frac{1}{\tilde\sigma}}$, $z \in \{0, \kappa_1, \kappa_2\}$,

$$\begin{aligned}
&|e^{(\overline{\lambda}-\lambda)(x_1-x_1')+(\overline{\lambda}^2-\lambda^2)(x_2-x_2')} - 1|_{H^{\mu}_{\tilde\sigma}(D_{z,\frac{1}{\tilde\sigma}})}\\
=&\sup_{\tilde r_1,\tilde r_2\le 1,|\alpha_1|,|\alpha_2|\le\pi}\\
&\frac{|e^{(\overline{\lambda}_1-\lambda_1)(x_1-x_1')+(\overline{\lambda}_1^2-\lambda_1^2)(x_2-x_2')} - e^{(\overline{\lambda}_2-\lambda_2)(x_1-x_1')+(\overline{\lambda}_2^2-\lambda_2^2)(x_2-x_2')}|}{|\tilde r_1 e^{i\alpha_1}-\tilde r_2 e^{i\alpha_2}|^{\mu}}\\
\le&\sup_{\tilde r_1,\tilde r_2\le 1,|\alpha_1|,|\alpha_2|\le\pi}\\
&\frac{|e^{i([X_1-X_1']\frac{\tilde r_1}{\tilde\sigma}\sin\alpha_1+[X_2-X_2'](\frac{\tilde r_1}{\tilde\sigma})^2\sin 2\alpha_1)} - e^{i([X_1-X_1']\frac{\tilde r_2}{\tilde\sigma}\sin\alpha_2+[X_2-X_2'](\frac{\tilde r_2}{\tilde\sigma})^2\sin 2\alpha_2)}|}{|\tilde r_1 e^{i\alpha_1}-\tilde r_2 e^{i\alpha_2}|}\\
\le& C(1+|x'|).
\end{aligned}$$

Plugging (3.46) and (3.47) into (3.28), we obtain

$$m_0(x,\kappa_j+0^+e^{i\alpha}) = \frac{\Theta_j(x)}{1-\gamma_j|\alpha|}, \tag{3.49}$$

with

$$\begin{aligned}
\Theta_j(x) =& [1+\mathfrak{G}_j(x,x')*v_0(x')]^{-1}\chi_j(x'),\\
\gamma_j =& -\frac{1}{\pi}\iint \xi_j(x)v_0(x)\Theta_j(x)dx,
\end{aligned} \tag{3.50}$$

and

$$|m_0(x,\lambda)-m_0(x,\kappa_j+0^+e^{i\alpha})|_{C^{\mu}_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})\cap L^\infty(D_{\kappa_j})} < C|(1+|x|)v_0|_{L^1\cap L^\infty}, \tag{3.51}$$

$$\left|\frac{m_0(x,\lambda)-m_0(x,\kappa_j+0^+e^{i\alpha}))}{\lambda-\kappa_j}\right|_{L^\infty(D_{\kappa_j})} < C(1+|x|)|(1+|x|)v_0|_{L^1\cap L^\infty}. \tag{3.52}$$

Combining (3.34) and (3.49)–(3.52), we obtain:

$$\begin{aligned}
s_c(\lambda) =& \frac{\mathrm{sgn}(\lambda_I)}{2\pi i}\iint e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}\left(\frac{\xi_j(x)}{\overline{\lambda}-\kappa_j}+h.o.t.\right)\\
&\times v_0(x)\left(\frac{\Theta_j(x)}{1-\gamma_j|\alpha|}+h.o.t.\right)dx = \frac{\frac{i}{2}\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|}+\mathrm{sgn}(\lambda_I)h_j(\lambda)
\end{aligned}$$

with $|h_j|_{L^\infty(D_{\kappa_j})} < |(1+|x|)v_0|_{L^1\cap L^\infty}$.

$*$ **For** $\lambda\in D_0^\times$, similarly, using $|m_{0,r}|_{C^1(D_0)} < C(1+|x|)\sum_{|k|=0}^1|\partial_x^k(1+|x|)v_0|_{L^1\cap L^\infty}$, we find:

$$\begin{aligned}
s_c(\lambda) =& \frac{\mathrm{sgn}(\lambda_I)}{2\pi i}\iint e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}(0\cdot\lambda+h.o.t.)\\
&\times v_0(x)\left(\frac{m_{0,\mathrm{res}}(x)}{\lambda}+h.o.t.\right)dx = \mathrm{sgn}(\lambda_I)\hbar_0(\lambda)
\end{aligned}$$

with $|\hbar_0|_{C^1(D_0)} < |(1+|x|)v_0|_{L^1\cap L^\infty}$.

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- ► Building on the integral equation of m_0 and the estimates regarding the Green function G, we can identify the following properties of m_0:
  (a) $m_0(x,\lambda) = \overline{m_0(x,\overline{\lambda})}$.
  (b) $(1-\mathcal{E}_0)m_0(x,\lambda) \in L^\infty$.
  (c) For $\lambda \in D_0^\times$, $m_0(x_1,x_2,\lambda) = \frac{m_{0,\mathrm{res}}(x_1,x_2)}{\lambda} + m_{0,r}(x_1,x_2,\lambda)$, with $m_{0,\mathrm{res}}, m_{0,r} \in L^\infty(D_0)$.
  (d) For $\lambda = \kappa_j + re^{i\alpha} \in D_{\kappa_j}^\times$,

$$m_0 = m_0^\flat + m_0^\sharp, \qquad m_0^\flat = \frac{\Theta_j(x)}{1-\gamma_j|\alpha|}. \tag{3.53}$$

  Therefore, $m_0^\sharp \in C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}}) \cap L^\infty(D_{\kappa_j})$, $m_0^\sharp(x_1,x_2,\kappa_j) = 0$, and

$$m_0^\flat = \sum_{l=0}^{\infty} m_{0,l}(X_1,X_2)(-\ln(1-\gamma_j|\alpha|))^l \in L^\infty(D_{\kappa_j}).$$

  These observations confirm that m_0 is in the eigenfunction space W_0.
- ► To prove the CIE for m_0, we follow the same method as used for rapidly decaying potentials. Using $T_0 m_0 \in L^1$, we can apply Liouville's theorem to show that there exists $g(x)$ such that

$$m_0(x,\lambda) = g(x) + \frac{m_{0,\mathrm{res}}(x)}{\lambda} + \mathcal{C}T_0 m_0(x,\lambda). \tag{3.54}$$

  To prove $g \equiv 1$, we apply the Lax operator to both sides of (3.54) and utilize the Lax equation, yielding

$$\begin{aligned} u(x)m_0(x,\lambda) = &\left(\partial_{x_2} - \partial_{x_1}^2 - 2\lambda\partial_{x_1}\right)[g(x) + \frac{m_{0,\mathrm{res}}(x)}{\lambda}] \\ &+ \left(\partial_{x_2} - \partial_{x_1}^2 - 2\lambda\partial_{x_1}\right)\mathcal{C}T_0 m_0. \end{aligned} \tag{3.55}$$

  Then it reduces to demonstrate that these CI's are uniformly $o(|\lambda|)$.

  To this aim, we decompose

$$\begin{aligned} \mathcal{C}T_0 m_0 = &\iint_{D_0\cup D_{\kappa_1}\cup D_{\kappa_2}} \frac{s_c(\zeta)e^{(\overline{\zeta}-\zeta)x_1+(\overline{\zeta}^2-\zeta^2)x_2}m_0}{\zeta-\lambda} d\overline{\zeta}\wedge d\zeta \\ &+ \iint_{\mathbb{C}\backslash(D_0\cup D_{\kappa_1}\cup D_{\kappa_2})} \frac{s_c(\zeta)e^{(\overline{\zeta}-\zeta)x_1+(\overline{\zeta}^2-\zeta^2)x_2}m_0}{\zeta-\lambda} d\overline{\zeta}\wedge d\zeta \\ \equiv& P_1 + P_2. \end{aligned} \tag{3.56}$$

  The estimate for P_1 is standard. The estimate for P_2 corresponds to the CI near infinity, and this follows from the arguments presented by Wickerhauser.
- ► [6] To prove the $\mathcal{D}$-symmetry (3.15), we introduce the total Green function $\mathcal{K}$, which is the fundamental solution of the full Lax operator:

$$\begin{aligned} \overrightarrow{\mathcal{L}_{v_0}}\mathcal{K} = \mathcal{K}\overleftarrow{\mathcal{L}_{v_0}} = \delta(x-x'), \\ \mathcal{L}_{v_0} = \mathcal{L} + v_0, \quad \mathcal{L} = -\partial_{x_2} + \partial_{x_1}^2 + u_s(x) \end{aligned} \tag{3.57}$$

with $\overrightarrow{\mathcal{L}}$ the operator $\mathcal{L}$ applying to the x variable of $\mathcal{K}$ and $\overleftarrow{\mathcal{L}}$ the operator applying to the x' variable of $\mathcal{K}$. The total Green function $\mathcal{K}$ can be solved using these integral equations.

$$\begin{aligned} \mathcal{K}(x,x',\lambda) &= \mathcal{G} - \mathcal{G} * v_0\, \mathcal{K}, \\ \mathcal{K}(x,x',\lambda) &= \mathcal{G} - \mathcal{K} * v_0\, \mathcal{G}, \\ \mathcal{G}(x,x',\lambda) &= e^{\lambda(x_1-x_1')+\lambda^2(x_2-x_2')} G(x,x',\lambda). \end{aligned} \tag{3.58}$$

Therefore, the eigenfunction Φ and adjoint eigenfunction Ψ can be written as

$$\begin{aligned} \Phi(x,\lambda) =&\mathcal{K}(x,x',\lambda) *_{x'} \overleftarrow{\mathcal{L}}\varphi(x',\lambda) \equiv \mathcal{K} * \overleftarrow{\mathcal{L}}\varphi, \\ \Psi(x',\lambda) =&\psi(x,\lambda) *_x \overrightarrow{\mathcal{L}}\mathcal{K}(x,x',\lambda) \equiv \psi * \overrightarrow{\mathcal{L}}\mathcal{K}, \end{aligned} \tag{3.59}$$

with φ and ψ the Sato eigenfunction and the Sato adjoint eigenfunction (see (3.4), (3.5), (3.34), (3.35)). Furthermore, letting $\mathcal{G}_j = \lim_{\lambda\to\kappa_j^+} \mathcal{G}$, $\mathcal{K}_j = \lim_{\lambda\to\kappa_j^+} \mathcal{K}$, and successively using (3.59) and (3.58) [6, 32], we can prove:

$$(\varphi_1(x),\varphi_2(x))\mathcal{D}^\flat = 0, \tag{3.60}$$

$$\mathcal{G}_{j-1} = \mathcal{G}_j + \varphi_j(x)\psi_j(x'), \tag{3.61}$$

$$\mathcal{K}_{j-1} = \mathcal{K}_j + \frac{\Phi_j(x)\Psi_j(x')}{1-c_{jj}}. \tag{3.62}$$

Using these formulas, we establish:

$$\sum_{j=1}^{2} \frac{\Phi_j(x)\Psi_j(x')}{1-c_j} = 0, \tag{3.63}$$

$$\mathcal{K}_l = \mathcal{K}_i + \sum_{j=l+1}^{i+2} \frac{\Phi_j(x)\Psi_j(x')}{1-c_j}. \tag{3.64}$$

Here $c_j = c_{jj}$ and the mod 2-condition is adopted.

Applying $\overleftarrow{\mathcal{L}}\varphi_i$ to (3.64) from the right and using (3.59), we obtain

$$\mathcal{K}_l * \overleftarrow{\mathcal{L}}\varphi_i = \Phi_i + \sum_{j=l+1}^{i+2} \frac{\Phi_j(x)c_{ji}}{1-c_j}. \tag{3.65}$$

Multiplying (3.65) by $\mathcal{D}^\flat_{im}$, summing up and using the symmetry (3.60), we derive

$$\sum_{i=1}^{2} \Phi_i \mathcal{D}^\flat_{im} + \sum_{i=1}^{2}\sum_{j=l+1}^{i+2} \frac{\Phi_j(x)c_{ji}\mathcal{D}^\flat_{im}}{1-c_j} = 0. \tag{3.66}$$

Taking $l=2$ in (3.66) and using (3.63), we prove

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2} m_0(x_1,x_2,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2} m_0(x_1,x_2,\kappa_2^+))\mathcal{D}^\sharp = 0. \tag{3.67}$$

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Multiplying $\left(\mathcal{D}_{11}^{\sharp}\right)^{-1}\kappa_1$ from the right to both sides, we justify

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2} m_0(x_1,x_2,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2} m_0(x_1,x_2,\kappa_2^+))\mathcal{D} = 0.$$

3.2.3. The strategy of the proof of Theorem 3.4

Note that [6, 32]

$$\partial_{\overline{\lambda}}\Phi(x,\lambda) = s_c(\lambda,x_3)\Phi(x,\overline{\lambda}),$$
$$-\kappa_1\Phi(x,\kappa_1) = \mathcal{D}_{21}(x_3)\Phi(x,\kappa_2).$$

Denote

$$\mathcal{M}_\lambda = -\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} + \rho(\lambda), \ \ \rho(\lambda) = -\lambda^3.$$

The linearity of the continuous scattering data s_c can be proved in the same way as that for rapidly decaying potentials. Similarly,

$$\begin{aligned} 0 =& -\mathcal{M}_{\kappa_1}\left[\kappa_1\Phi(x,\kappa_1)\right] = \mathcal{M}_{\kappa_1}\left[\mathcal{D}_{21}(x_3)\Phi(x,\kappa_2)\right] \\ =&\Phi(x,\kappa_2)\left[-\partial_{x_3}+\rho(\kappa_1)\right]\mathcal{D}_{21}(x_3) + \mathcal{D}_{21}(x_3)\left[\mathcal{M}_{\kappa_2}-\rho(\kappa_1)\right]\Phi(x,\kappa_2) \\ =&\Phi(x,\kappa_2)\left[-\partial_{x_3}+\rho(\kappa_1)\right]\mathcal{D}_{21}(x_3) + \mathcal{D}_{21}(x_3)\left[\mathcal{M}_{\kappa_2}-\rho(\kappa_2)\right]\Phi(x,\kappa_2) \\ =&\Phi(x,\kappa_2)\left[-\partial_{x_3}+\rho(\kappa_1)-\rho(\kappa_2)\right]\mathcal{D}_{21}(x_3). \end{aligned} \tag{3.68}$$

3.3. The inverse problem for perturbed KP 1-solitons

In this subsection, we will explore the inverse problem for perturbed 1-line solitons, without relying on the Sato theory.

3.3.1. Statement of results

We begin with the definition of admissible scattering data:

Definition 3.5. *Let* $0<\epsilon_0\ll 1$*,* $d\geq 0$ *and* $\mathrm{u_s}$ *be a* $\mathrm{Gr}(1,2)_{>0}$ *KP soliton defined by* $\{\kappa_j\}, A=(1,a)$*. A scattering data* $\mathcal{S}=(\{0\},\{\kappa_1,\kappa_2\},\mathcal{D},s_c(\lambda))$ *is called* d*-admissible if*

$$s_c(\lambda) = \begin{cases} \frac{\frac{i}{2}\,\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|} + \mathrm{sgn}(\lambda_I)h_j(\lambda), & \lambda\in D_{\kappa_j}^{\times}, \\ \mathrm{sgn}(\lambda_I)\hbar_0(\lambda), & \lambda\in D_0^{\times}, \end{cases} \tag{3.69}$$

$$\mathcal{D} = (\kappa_1,\mathcal{D}_{21})^T, \tag{3.70}$$

and

$$\begin{aligned} \epsilon_0 \geq& (1-\textstyle\sum_{j=1}^2\mathcal{E}_{\kappa_j})\sum_{|l|\leq d+8}|\left(|\overline{\lambda}-\lambda|^{l_1}+|\overline{\lambda}^2-\lambda^2|^{l_2}\right)s_c(\lambda)|_{L^\infty} \\ &+\textstyle\sum_{j=1}^2(|\gamma_j|+|h_j|_{L^\infty(D_{\kappa_j})}) + |\hbar_0|_{C^1(D_0)} + |\mathcal{D}-\mathcal{D}^\flat|_{L^\infty}, \\ s_c(\lambda) =& \overline{s_c(\overline{\lambda})},\ h_j(\lambda) = -\overline{h_j(\overline{\lambda})},\ \hbar_0(\lambda) = -\overline{\hbar_0(\overline{\lambda})}, \\ \mathcal{D}^\flat =& \,diag\,(\kappa_1,\kappa_2)\,A^T = (\kappa_1,\kappa_2 a)^T, \kappa_j\neq 0. \end{aligned}$$

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Define T as the continuous scattering operator

$$T\phi(x,\lambda) \equiv s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2+(\overline{\lambda}^3-\lambda^3)x_3}\phi(x,\overline{\lambda}). \tag{3.71}$$

Definition 3.6. *The eigenfunction space $W = W_x$ consists of ϕ satisfying*

(a) $\phi(x,\lambda) = \overline{\phi(x,\overline{\lambda})}$;
(b) $(1-\mathcal{E}_0)\phi(x,\lambda) \in L^\infty$;
(c) for $\lambda \in D_0^\times$, $\phi(x,\lambda) = \frac{\phi_{0,\mathrm{res}}(x)}{\lambda} + \phi_{0,r}(x,\lambda)$, $\phi_{0,\mathrm{res}}$, $\phi_{0,r} \in L^\infty(D_0)$*;*
(d) for $\lambda = \kappa_j + re^{i\alpha} \in D_{\kappa_j}^\times$, $\phi = \phi^\flat + \phi^\sharp$, $\phi^\flat = \sum_{l=0}^\infty \phi_l(X)(-\ln(1-\gamma_j|\alpha|))^l \in L^\infty(D_{\kappa_j})$, $\phi^\sharp \in C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}}) \cap L^\infty(D_{\kappa_j})$, $\phi^\sharp(x,\kappa_j) = 0$.

Here

$$\tilde\sigma = \max\{1, |X_1|, \sqrt{|X_2|}, \sqrt[3]{|X_3|}\}, \tag{3.72}$$

is the rescaling parameter with $\mathrm{X_k}$ *the phase function coefficients:*

$$\begin{aligned}\wp(x,\lambda) =& i[(\overline{\lambda}-\lambda)x_1 + (\overline{\lambda}^2-\lambda^2)x_2 + (\overline{\lambda}^3-\lambda^3)x_3] \qquad \lambda = z + re^{i\alpha} \in D_z\\ =& X_1 r\sin\alpha + X_2 r^2\sin 2\alpha + X_3 r^3 \sin 3\alpha \equiv \wp(r,\alpha,X),\\ X_1(x,z) =& 2(x_1 + 2x_2 z + 3x_3 z^2),\ X_2(x,z) = 2(x_2 + 3x_3 z),\ X_3(x,z) = 2x_3.\end{aligned} \tag{3.73}$$

Finally, for $\phi \in W$, define

$$\begin{aligned}|\phi|_W \equiv& |(1-\mathcal{E}_0)\phi|_{L^\infty} + (|\phi_{0,\mathrm{res}}|_{L^\infty} + |\phi_{0,r}|_{L^\infty(D_0)})\\ &+ \sum_{j=1}^{2}(|\phi^\flat|_{L^\infty(D_{\kappa_j})} + |\phi^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})\cap L^\infty(D_{\kappa_j})}).\end{aligned} \tag{3.74}$$

The theorem of the inverse problem is stated as follows:

Theorem 3.7. (The Inverse scattering Theory) [33] *There exists a positive constant $\epsilon_0 \ll 1$, such that for any* d*-admissible scattering data $\mathcal{S} = (\{0\}, \kappa_1, \kappa_2, \mathcal{D}, s_c(\lambda))$ defined by a* $\mathrm{Gr}(1,2)_{>0}$ *KP soliton* $\mathrm{u_s}$ *corresponding to data $\{\kappa_j\}, A = (1,a)$,*

(1) the system of the CIE and the $\mathcal{D}$-symmetry,

$$m(x,\lambda) = 1 + \frac{m_{0,\mathrm{res}}(x)}{\lambda} + \mathcal{C}Tm,\ \lambda \neq 0, \tag{3.75}$$

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3}m(x,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2+\kappa_2^3 x_3}m(x,\kappa_2^+))\mathcal{D} = 0, \tag{3.76}$$

is uniquely solved in W *satisfying*

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} |\partial_x^l\,[m(x,\lambda) - \chi(x,\lambda)]\,|_W \le C\epsilon_0. \tag{3.77}$$

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(2) Moreover,

$$\left(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u(x)\right)m(x,\lambda)=0, \tag{3.78}$$

$$u(x)\equiv-2\partial_{x_1}m_{0,\mathrm{res}}(x)-\frac{1}{\pi i}\partial_{x_1}\iint Tm\,d\overline{\zeta}\wedge d\zeta, \tag{3.79}$$

$$\sum_{0\le l_1+2l_2+3l_3\le d+4}|\partial_x^l\,[u(x)-u_s(x)]\,|_{L^\infty}\le C\epsilon_0. \tag{3.80}$$

We define the inverse scattering transform by

$$\mathcal{S}^{-1}(\{z_n,\kappa_j,\mathcal{D},s_c(\lambda)\})=-\frac{1}{\pi i}\partial_{x_1}\iint Tm\,d\overline{\zeta}\wedge d\zeta-2\partial_{x_1}m_{0,\mathrm{res}}(x); \tag{3.81}$$

(3) $u:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ *solves the KPII equation*

$$(-4u_{x_3}+u_{x_1x_1x_1}+6uu_{x_1})_{x_1}+3u_{x_2x_2}=0. \tag{3.82}$$

Using the direct and inverse scattering theories (Theorems 3.2 and 3.7), we can solve the Cauchy problem for the KPII on 1-line soliton backgrounds (see Corollary 4.6).

3.3.2. The strategy of the proof of Theorem 3.7

(1) We will demonstrate the existence, uniqueness and estimates of this system by taking the limit of the iteration sequence

$$\phi^{(k)}(x,\lambda)=1+\frac{\phi_{0,\mathrm{res}}^{(k)}(x)}{\lambda}+\mathcal{C}T\phi^{(k-1)}(x,\lambda),\quad k>0, \tag{3.83}$$

$$(e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}\phi^{(k)}(x,\kappa_1^+),e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}\phi^{(k)}(x,\kappa_2^+))\mathcal{D}=0, \tag{3.84}$$

$$\phi^{(0)}(x,\lambda)=\chi(x,\lambda) \tag{3.85}$$

in the eigenfunction space W.

Evaluating the CIE (3.83) at κ_1^+,κ_2^+ and using the $\mathcal{D}$-symmetry (3.84), one obtains a linear system of $2+1$ variables $\phi^{(k)}(x,\kappa_1^+)$, $\phi^{(k)}(x,\kappa_2^+)$, and $\phi_{0,\mathrm{res}}^{(k)}(x)$. Hence the iteration turns into

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$$\begin{aligned}
\phi^{(k)}(x,\lambda) =& 1+\frac{\phi^{(k)}_{0,\mathrm{res}}(x)}{\lambda}+\mathcal{C}T\phi^{(k-1)}(x,\lambda),\quad k>0,\\
\phi^{(k)}_{0,\mathrm{res}}(x) =& -\frac{\mathcal{D}_{11}e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}+\mathcal{D}_{21}e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}}{\frac{\mathcal{D}_{11}}{\kappa_1}e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}+\frac{\mathcal{D}_{21}}{\kappa_2}e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}}\\
&-\frac{\mathcal{D}_{11}e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}}{\frac{\mathcal{D}_{11}}{\kappa_1}e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}+\frac{\mathcal{D}_{21}}{\kappa_2}e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}}\mathcal{C}T\phi^{(k-1)}(x,\kappa_1^+)\\
&-\frac{\mathcal{D}_{21}e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}}{\frac{\mathcal{D}_{11}}{\kappa_1}e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}+\frac{\mathcal{D}_{21}}{\kappa_2}e^{\kappa_2x_1+\kappa_2^2x_2+\kappa_2^3x_3}}\mathcal{C}T\phi^{(k-1)}(x,\kappa_2^+),\\
\phi^{(0)}(x,\lambda) =& \chi(x,\lambda).
\end{aligned}$$

By the TP condition, the convergence of the iteration sequence reduces to deriving uniform estimates of CIO near ∞, κ_j and 0.

- ► **Estimates on D_∞:** Away from 0, κ_j, the continuous data s_c and the eigenfunction m_0 are regular, similar to the case of rapidly decaying potentials. Thus, we can derive estimates using Wickerhauser's arguments.
- ► **Estimates on D_{κ_j}:** To derive estimates on D_{κ_j}, we take advantage of several factors:

(1) The boundedness of h_j and estimates of the Cauchy integral operator:

$$\begin{aligned}
&|h_j|_{L^\infty(D_{\kappa_j})}<|(1+|x|)v_0|_{L^1\cap L^\infty},\\
&|\mathcal{C}\mathcal{E}_z\phi|_{L^\infty}\le C|\phi|_{L^p(D_z)},\quad |\mathcal{C}\mathcal{E}_z\phi|_{H^\nu(D_z)}\le C|\phi|_{L^p(D_z)},
\end{aligned}\tag{3.86}$$

allow us to replace CIO estimates with kernel s_c by those with kernel $\widetilde{\gamma}_j=\frac{\frac{i}{2}\,\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|}$, the leading singular term.

(2) The special form of $\widetilde{\gamma}_j$ allows us to apply Stokes' theorem to integrate the leading singularity:

$$\begin{aligned}
\widetilde{\gamma}_j(\lambda)&=\frac{\frac{i}{2}\,\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|}=-\partial_{\overline{\lambda}}\ln(1-\gamma_j|\alpha|)\\
\mathcal{C}\widetilde{\gamma}_j\mathcal{E}_{\kappa_j}[-\ln(1-\gamma_j|\beta|)]^l&=\frac{[-\ln(1-\gamma_j|\alpha|)]^{l+1}}{l+1}\\
&\quad-\frac{1}{2\pi i}\oint_{|\zeta-\kappa_j|=\delta}\frac{\frac{1}{l+1}[-\ln(1-\gamma_j|\beta|)]^{l+1}}{\zeta-\lambda}d\zeta.
\end{aligned}\tag{3.87}$$

(3) The scaling invariant properties:

$$\begin{aligned}
&|\mathcal{C}_\lambda\widetilde{\gamma}_je^{-i\wp(x,\zeta)}\mathcal{E}_{\kappa_j}f(\kappa_j+se^{i\beta})|_{C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})\cap L^\infty(D_{\kappa_j})}\\
&\quad=|\mathcal{C}_{\tilde\lambda}\widetilde{\gamma}_je^{-i\wp(\frac{\tilde s}{\tilde\sigma},\beta,X)}\mathcal{E}_{\kappa_j,\tilde\sigma\delta}f(\kappa_j+\frac{\tilde s}{\tilde\sigma}e^{i\beta})|_{C^\mu(D_{\kappa_j,1})\cap L^\infty(D_{\kappa_j,\tilde\sigma\delta})},
\end{aligned}\tag{3.88}$$

where the dilating polar coordinates near $z=\kappa_j$ is defined by

$$\begin{aligned}
\lambda=z+re^{i\alpha}=z+\frac{\tilde r}{\tilde\sigma}e^{i\alpha},\quad \zeta=z+se^{i\beta}=z+\frac{\tilde s}{\tilde\sigma}e^{i\beta},\\
\tilde\lambda=z+\tilde re^{i\alpha},\quad \tilde\zeta=z+\tilde se^{i\beta},
\end{aligned}$$

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and $r,\ s \le \delta,\ \tilde{r},\ \tilde{s} \le \tilde{\sigma}\delta,\ |\alpha|, |\beta| \le \pi$.

To achieve this, we decompose

$$\mathcal{C}_\lambda \widetilde{\gamma}_j e^{-i\wp(x,\zeta)} \mathcal{E}_{\kappa_j} f \equiv I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.89}$$

with

$$I_1 = -\frac{\theta(1-\tilde{r})}{2\pi i} \iint_{\tilde{s}<2} \frac{\widetilde{\gamma}_j(\tilde{s},\beta) f^\flat(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta}-\tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}, \tag{3.90}$$

$$I_2 = -\frac{\theta(1-\tilde{r})}{2\pi i} \iint_{\tilde{s}<2} \frac{\widetilde{\gamma}_j(\tilde{s},\beta)[e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}-1] f^\flat(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta}-\tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}, \tag{3.91}$$

$$I_3 = -\frac{\theta(1-\tilde{r})}{2\pi i} \iint_{\tilde{s}<2} \frac{\widetilde{\gamma}_j(\tilde{s},\beta) e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f^\sharp(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta}-\tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}, \tag{3.92}$$

$$I_4 = -\frac{\theta(1-\tilde{r})}{2\pi i} \iint_{2<\tilde{s}<\tilde{\sigma}\delta} \frac{\widetilde{\gamma}_j(\tilde{s},\beta) e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta}-\tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}, \tag{3.93}$$

$$I_5 = -\frac{\theta(\tilde{r}-1)}{2\pi i} \iint_{\tilde{s}<\tilde{\sigma}\delta} \frac{\widetilde{\gamma}_j(\tilde{s},\beta) e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta}-\tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}. \tag{3.94}$$

For I_1, I_2 and I_3, integrals over uniformly compact domains, we will apply Stokes' theorem (3.87) and Hölder interior estimates [13] to derive:

$$|I_1^\flat|_{L^\infty},\ |I_1^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})},\ |I_2|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})},\ |I_3|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})}.$$

For the estimates of

$$|I_4|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})}, \qquad |I_5|_{L^\infty(D_{\kappa_j})},$$

slowly decaying integrals over non-uniformly compact domains, we first express them as iterated integrals in polar coordinates

$$\begin{aligned} I_4 &= -\frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma_j|\beta|)] \int_{2<\tilde{s}<\tilde{\sigma}\delta} \frac{e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{s}-\tilde{r}e^{i(\alpha-\beta)}} d\tilde{s}, \\ I_5 &= -\frac{\theta(\tilde{r}-1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma_j|\beta|)] \int_0^{\tilde{\sigma}\delta} \frac{e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{s}-\tilde{r}e^{i(\alpha-\beta)}} d\tilde{s}. \end{aligned}$$

Then, estimates will be derived by applying holomorphic extension properties in $\tilde{s}$, the deformation method and stationary point analysis of $\wp$. The main ideas for the deformation are as follows: near stationary points $\tilde{s}_*$, we deform

$$\tilde{s} \in \mathbb{R} \quad \longrightarrow \quad \tilde{s}e^{i\tau} \in \mathbb{C} \tag{3.95}$$

such that the integral domain of $\tilde{s}$ deforms into a union of line segments Γ's and arcs S's, satisfying the following conditions:

$$\begin{aligned}
&(a)\ \ on\ \Gamma' s,\ \Re\mathfrak{e}(-i\wp(\frac{\tilde{s}e^{i\tau}}{\tilde{\sigma}},\beta,X)) \le -\frac{1}{C}|\sin(k\beta)(\tilde{s}-\tilde{s}_*)^k|\\
&(b)\ \ on\ \Gamma' s,\ |\tilde{\zeta}-\tilde{\lambda}| \ge \frac{1}{C}\max\{|\tilde{s}-\tilde{s}_*|,|\tilde{r}e^{i\alpha}-\tilde{s}_*e^{i\beta}|\},\\
&(c)\ \ on\ S' s,\ \Re\mathfrak{e}(-i\wp(\frac{\tilde{s}e^{i\tau}}{\tilde{\sigma}},\beta,X)) \le 0.
\end{aligned} \tag{3.96}$$

Therefore,

- $*$ using improper integrals, if $\tilde{\sigma} = \sqrt[k]{|X_k|}$, $k = 3,2$, estimates can be established when $\tilde{s}$ is away from the stationary point;
- $*$ When both $\tilde{s}$ and $\tilde{r}$ are close to the stationary point, we will show that I_5 near the stationary point is no longer a singular integral, allowing us to obtain estimates (see Proposition 3.14 for $\tilde{\sigma} = \sqrt[k]{|X_k|}$, $k = 3,2$).

The estimate for $\tilde{\sigma} = |X_1|$ is more complicated because $e^{\Re\mathfrak{e}(-i\wp(\frac{\tilde{s}e^{i\tau}}{\tilde{\sigma}},\beta,X))}$ either decays non-uniformly in X or leads to a $\frac{1}{|\sin\beta|}$ singularity which is not suitable for an improper integral. To address these difficulties, we will either utilize the scaling invariant properties of the Hilbert transform or look for a finer decomposition (see Proposition 3.17 and remarks before Definition 3.15).

More details regarding the key estimates for I_3, I_4 and I_5 will be provided in Sections 3.3.3 and 3.3.4, as they represent significant additional analytic features of the inverse problem for perturbed multi-line solitons.

- ► **Estimates on D_0 :** To leverage the methods for estimates on D_{κ_j}, we adopt a similar decomposition of the CI near 0, that is, a combination of uniformly compact domain integrals and non-uniformly compact domain integrals:

$$\begin{aligned}
&\mathcal{C}T\mathcal{E}_0\phi\\
&= -\frac{1}{2\pi i}\iint_{D_{0,\tilde{\sigma}\delta}} \frac{\mathrm{sgn}(\beta)\hbar_0(\frac{\tilde{s}}{\tilde{\sigma}},\beta)e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}}\wedge d\tilde{\zeta} + \mathcal{C}_\lambda T E_0\phi_{0,r}\\
&\equiv II_1 + II_2 + II_3 + II_4 + II_5,
\end{aligned}$$

where

$$\begin{aligned}
II_1 &= -\frac{\theta(1-\tilde{r})}{2\pi i}\iint_{\tilde{s}<2} \frac{\mathrm{sgn}(\beta)\hbar_0(\frac{\tilde{s}}{\tilde{\sigma}},\beta)\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}}\wedge d\tilde{\zeta},\\
II_2 &= -\frac{\theta(1-\tilde{r})}{2\pi i}\iint_{\tilde{s}<2} \frac{\mathrm{sgn}(\beta)\hbar_0(\frac{\tilde{s}}{\tilde{\sigma}},\beta)[e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}-1]\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}}\wedge d\tilde{\zeta},\\
II_3 &= \mathcal{C}_\lambda E_0 T\phi_{0,r},\\
II_4 &= -\frac{\theta(1-\tilde{r})}{2\pi i}\iint_{2<\tilde{s}<\tilde{\sigma}\delta} \frac{\mathrm{sgn}(\beta)\hbar_0(\frac{\tilde{s}}{\tilde{\sigma}},\beta)e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}}\wedge d\tilde{\zeta},\\
II_5 &= -\frac{\theta(\tilde{r}-1)}{2\pi i}\iint_{\tilde{s}<\tilde{\sigma}\delta} \frac{\mathrm{sgn}(\beta)\hbar_0(\frac{\tilde{s}}{\tilde{\sigma}},\beta)e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\bar{\tilde{\zeta}}} d\bar{\tilde{\zeta}}\wedge d\tilde{\zeta}.
\end{aligned}$$

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The estimate for II_1, the leading singular term, can be derived using $|\hbar_0|_{C^1(D_0)} < |v_0|_{L^1\cap L^\infty}$, the mean value theorem, Cauchy integral estimates and the Hilbert transform theory [13].

II_2 and II_3 are estimated using (3.86).

Writing $\hbar_0(\zeta) = \hbar_0(0) + [\hbar_0(\zeta) - \hbar_0(0)]$ and decompose $II_4 = II_{41} + II_{42}$, $II_5 = II_{51} + II_{52}$ with

$$\begin{aligned} II_{41} &= -\frac{\theta(1-\tilde{r})}{2\pi i}\iint_{2<\tilde{s}<\tilde{\sigma}\delta} \frac{\operatorname{sgn}(\beta)\hbar_0(0)e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\overline{\tilde{\zeta}}}d\overline{\tilde{\zeta}}\wedge d\tilde{\zeta}, \\ II_{51} &= -\frac{\theta(\tilde{r}-1)}{2\pi i}\iint_{\tilde{s}<\tilde{\sigma}\delta} \frac{\operatorname{sgn}(\beta)\hbar_0(0)e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)}\phi_{0,\mathrm{res}}(x)}{(\tilde{\zeta}-\tilde{\lambda})\overline{\tilde{\zeta}}}d\overline{\tilde{\zeta}}\wedge d\tilde{\zeta}. \end{aligned}$$

Thanks to $\tilde{s}$-meromorphic properties and adapting argument for estimating I_4, I_5 (see Section 3.3.3) for II_{41}, II_{51}, we obtain

$$|II_{41}|_{L^\infty(D_0)},\ |II_{51}|_{L^\infty(D_0)} \le C\epsilon_0|\phi_{0,\mathrm{res}}|_{L^\infty}.$$

For the remaining terms, by (3.86) and $|\hbar_0|_{C^1(D_0)} < |v_0|_{L^1\cap L^\infty}$,

$$|II_{42}|_{L^\infty(D_0)},\ |II_{52}|_{L^\infty(D_0)} \le C\epsilon_0|\phi_{0,\mathrm{res}}|_{L^\infty}.$$

(2) To derive the Lax equation, we introduce the shorthand notation:

$$\begin{gathered} -\partial_{x_2} + \partial_{x_1}^2 + 2\lambda\partial_{x_1} = -\nabla_2 + \nabla_1^2, \\ \nabla_1 = \partial_{x_1} + \lambda,\ \nabla_2 = \partial_{x_2} + \lambda^2,\ \ J\phi = \frac{\phi_{0,\mathrm{res}}(x)}{\lambda}. \end{gathered}$$

By applying the heat operator to the system of the Cauchy Integral Equation (CIE) and the D-symmetry, formally,

$$(-\nabla_2 + \nabla_1^2)m = \left[-\nabla_2 + \nabla_1^2, J + \mathcal{C}T\right]m + (J + \mathcal{C}T)(-\nabla_2 + \nabla_1^2)m, \tag{3.97}$$

$$(e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3}\left[(-\nabla_2 + \nabla_1^2)m\right](x, \kappa_1^+), e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3}\left[(-\nabla_2 + \nabla_1^2)m\right](x, \kappa_2^+))\mathcal{D} = 0, \tag{3.98}$$

and

$$\left[-\nabla_2 + \nabla_1^2, J + \mathcal{C}T\right]m = +\frac{1}{\pi i}\partial_{x_1}\iint Tm\, d\overline{\zeta}\wedge d\zeta + 2\partial_{x_1}m_{0,\mathrm{res}}(x) \equiv -u(x).$$

Hence, formally, the unique of the CIE and the $\mathcal{D}$ symmetry constraint implies

$$(-\nabla_2 + \nabla_1^2)m = -(1 - J - \mathcal{C}T)^{-1}u(x)1 = -\,u(x)(1 - J - \mathcal{C}T)^{-1}1 = -u(x)m(x,\lambda). \tag{3.99}$$

Therefore, (3.78)–(3.80) are verified.

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A rigorous argument is carried out by introducing

$$\phi^{(k)} = 1 + J\phi^{(k)} + \mathcal{C}T\phi^{(k-1)},$$
$$(e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3}\left[(-\nabla_2+\nabla_1^2)\phi^{(k)}\right](x,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2+\kappa_2^3 x_3}\left[(-\nabla_2+\nabla_1^2)\phi^{(k)}\right](x,\kappa_2^+))\mathcal{D} = 0, \tag{3.100}$$

and using the estimates of the CIO's and the iteration method to establish:

$$\begin{aligned}
\left[-\nabla_2+\nabla_1^2, J\right]\phi^{(k)} &\ isindependentof\ \lambda,\\
\left[-\nabla_2+\nabla_1^2, \mathcal{C}T\right]\phi^{(k-1)} &\ isindependentof\ \lambda,\\
\left[-\nabla_2+\nabla_1^2, J\right]\phi^{(k)} &\to \left[-\nabla_2+\nabla_1^2, J\right] m = 2\partial_{x_1} m_{0,\mathrm{res}}(x),\\
\left[-\nabla_2+\nabla_1^2, \mathcal{C}T\right]\phi^{(k-1)} &\to \left[-\nabla_2+\nabla_1^2, \mathcal{C}T\right] m = \frac{1}{\pi i}\partial_{x_1}\iint Tm\, d\overline{\zeta}\wedge d\zeta,\\
J(-\nabla_2+\nabla_1^2)\phi^{(k)} &\to J(-\nabla_2+\nabla_1^2)m \quad in\ W,\\
\mathcal{C}T(-\nabla_2+\nabla_1^2)\phi^{(k)} &\to \mathcal{C}T(-\nabla_2+\nabla_1^2)m \quad in\ W,\\
\sum_{0\le l_1+2l_2+3l_3\le d+4} &|\partial_x^l([-\nabla_2+\nabla_1^2, J]m+u_s)|_{L^\infty} \le C\epsilon_0,\\
|\left[-\nabla_2+\nabla_1^2, \mathcal{C}T\right]\phi^{(k-1)} &- \left[-\nabla_2+\nabla_1^2, \mathcal{C}T\right]\chi|_{L^\infty} \le (C\epsilon_0)^k.
\end{aligned}$$

(3) **The KP equation:** The KP equation will be derived by justifying the Lax pair. By the representation formula (3.80) and $\Phi(x,\lambda) = e^{\lambda x_1+\lambda^2 x_2} m(x,\lambda)$, we define the evolution operators

$$\begin{aligned}
\mathcal{M} =& -\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} - \lambda^3,\\
\mathcal{M}\Phi(x,\lambda) =& e^{\lambda x_1+\lambda^2 x_2}\left(\mathcal{M} + 3\lambda\partial_{x_1}^2 + 3\lambda^2\partial_{x_1} + \lambda^3 + \frac{3}{2}u\lambda\right) m(x,\lambda)\\
\equiv& e^{\lambda x_1+\lambda^2 x_2}\left(\mathfrak{M}m\right)(x,\lambda),
\end{aligned}$$

We reverse the procedure in the linearization theorem Theorem 3.4 to prove

$$\partial_{\overline{\lambda}}\left(\mathfrak{M}m\right)(x,\lambda) = s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2+(\overline{\lambda}^3-\lambda^3)x_3}\left(\mathfrak{M}m\right)(x,\overline{\lambda}), \tag{3.101}$$

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3}\mathfrak{M}m(x,\kappa_1^+), e^{\kappa_2 x_1+\kappa_2^2 x_2+\kappa_2^3 x_3}\mathfrak{M}m(x,\kappa_2^+))\mathcal{D} = 0. \tag{3.102}$$

As $|\lambda|\to\infty$, letting $m \sim \sum_{j=0}^{\infty}\frac{M_j(x)}{\lambda^j}$, from the Lax equation (3.78),

$$\begin{aligned}
2\partial_{x_1}M_{j+1} &= (\partial_{x_2} - \partial_{x_1}^2 - u)M_j,\\
M_0 = 1,\ M_1 = -\frac{1}{2}\partial_{x_1}^{-1}u,\ M_2 &= -\frac{1}{4}\partial_{x_2}\partial_{x_1}^{-1}u + \frac{1}{4}u + \frac{1}{4}\partial_{x_1}^{-1}\left(u\partial_{x_1}^{-1}u\right), \cdots
\end{aligned} \tag{3.103}$$

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As a result, as $\lambda \to \infty$,

$$\begin{aligned}
&\mathfrak{M}m \\
\to&\frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} + 3\lambda\partial_{x_1}^2(1+\frac{M_1}{\lambda}) + 3\lambda^2\partial_{x_1}(1+\frac{M_1}{\lambda}+\frac{M_2}{\lambda^2}) + \frac{3}{2}u\lambda \\
=&\frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} + \left(-\frac{3}{2}u_{x_1} + 3\partial_{x_1}(-\frac{1}{4}\partial_{x_2}\partial_{x_1}^{-2}u + \frac{u}{4} + \frac{1}{4}\partial_{x_1}^{-1}[u\partial_{x_1}^{-1}u])\right) \\
+&\lambda\left(3\partial_{x_1}M_1 + \frac{3}{2}u\right) + \frac{3}{2}u(-\frac{1}{2}\partial_{x_1}^{-1}u) = 0.
\end{aligned} \tag{3.104}$$

Therefore, $\mathfrak{M}m \in W$. Together with (3.101), (3.102), and unique solvability of the system of the CIE and $\mathcal{D}$ symmetry, yields $\mathfrak{M}m(x,\lambda) = 0$ and the Lax pair.

3.3.3. Highlight: estimates for I_3

The scaled Hölder estimate for

$$I_3 = -\frac{\theta(1-\tilde r)}{2\pi i}\iint_{\tilde s<2}\frac{\widetilde{\gamma}_j(\tilde s,\beta)e^{-i\wp(\frac{\tilde s}{\tilde\sigma},\beta,X)}f^\sharp(\frac{\tilde s}{\tilde\sigma},-\beta,X)}{\tilde\zeta-\tilde\lambda}d\bar{\tilde\zeta}\wedge d\tilde\zeta$$

is reminiscent of the Hölder estimate of the Beltrami's equation [25, Theorem 1.32], which involves mainly estimates of

$$I_3'g = \iint_D \frac{g(\zeta)}{(\zeta-\lambda)^2}d\bar\zeta\wedge d\zeta, \qquad |g|_{C^\mu} < \infty.$$

Both leading singular terms of I_3 and I_3' can be integrated by Stokes' theorem.

We only give the proof for the case $\kappa_j = \kappa_1$ since the proof for $j = 2$ is the same.

Proposition 3.8.

$$|I_3|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})} \le C\epsilon_0|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}. \tag{3.105}$$

Proof. From $f^\sharp \in C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})$ and $f^\sharp(x,\kappa_1) = 0$,

$$|\widetilde{\gamma}_1(\tilde s,\beta)f^\sharp(\frac{\tilde s}{\tilde\sigma},\beta,X)|_{L^\infty(D_{\kappa_1})} \le C\epsilon_0|f^\sharp|_{H^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}\tilde s^{\mu-1}.$$

Therefore, an improper integral yields

$$|I_3|_{L^\infty(D_{\kappa_1})} \le C\epsilon_0|f^\sharp|_{H^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}. \tag{3.106}$$

To derive the $H^\mu_{\tilde\sigma}$-estimate of I_3, let $\tilde\lambda_j = \kappa_1 + \tilde r_j e^{i\alpha_j}$, $\tilde r_j \le 1$, $j = 1, 2$, define

$$\varphi_{f^\sharp}(x,\zeta) = e^{-i\wp(\frac{\tilde s}{\tilde\sigma},\beta,X)}f^\sharp(x,\bar\zeta), \tag{3.107}$$

and decompose

$$
\begin{aligned}
& I_3(x,\lambda_1)-I_3(x,\lambda_2) \\
=& -\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde s<2}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
& -\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde s<2}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_2}{\tilde\sigma},\alpha_2,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
& +\frac{\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{4\pi i}\iint_{\tilde s<2}\widetilde\gamma_1(\tilde\zeta)[\frac{1}{\tilde\zeta-\tilde\lambda_2}-\frac{1}{\tilde\zeta-\tilde\lambda_1}]d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
& +\frac{\varphi_{f^\sharp}(\frac{\tilde r_2}{\tilde\sigma},\alpha_2,X)}{4\pi i}\iint_{\tilde s<2}\widetilde\gamma_1(\tilde\zeta)[\frac{1}{\tilde\zeta-\tilde\lambda_2}-\frac{1}{\tilde\zeta-\tilde\lambda_1}]d\bar{\tilde\zeta}\wedge d\tilde\zeta.
\end{aligned}
\tag{3.108}
$$

In view of $f^\sharp\in C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})$ and $f^\sharp(x,\kappa_1)=0$, we have

$$
|\varphi_{f^\sharp}(\frac{\tilde r}{\tilde\sigma},\alpha,X)|_{L^\infty(D_{\kappa_1})}\le C|f^\sharp|_{H^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}\tilde r^\mu. \tag{3.109}
$$

Therefore, estimates for the last two terms can be derived using Stokes' theorem to integrate the integrals.

We focus on the proof of the first term on the RHS of (3.108), as the proof of the second right-hand term is identical. Applying (3.86), it suffices to derive the estimate for all λ_1,λ_2 with $\tilde D\subset\{\tilde s\le 2\}$ being a disk centred at $\tilde\lambda_1$ with radius l and $l=2|\tilde\lambda_2-\tilde\lambda_1|$. Write

$$
\begin{aligned}
& -\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde s\le 2}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
=& -\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde D}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
& -\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\{\tilde s\le 2\}/\tilde D}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta.
\end{aligned}
\tag{3.110}
$$

Let $\tilde D_0=\{\zeta:|\zeta-\tilde\lambda_1|<\frac{3l}{2}\}$.

- ► If $\tilde\zeta\in\{\tilde s\le 2\}/\tilde D$ and $\kappa_1\in\tilde D_0$, then

$$
\frac{1}{C}\le|\frac{\tilde\zeta-\tilde\lambda_1}{\tilde\zeta-\tilde\lambda_2}|,|\frac{\tilde\zeta-\kappa_1}{\tilde\zeta-\tilde\lambda_1}|,|\frac{\tilde\zeta-\kappa_1}{\tilde\zeta-\tilde\lambda_2}|\le C.
$$

  In this case, using $f^\sharp\in C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})$ and [25, Chapter 1,§ 6.1],

$$
\begin{aligned}
& |-\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\{\tilde s\le 2\}/\tilde D}\widetilde\gamma_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta| \\
\le & C\epsilon_0|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|\iint_{\{\tilde s\le 2\}/\tilde D}\frac{1}{|\tilde\zeta-\lambda_2||\tilde\zeta-\tilde\lambda_1|^{2-\mu}}d\bar{\tilde\zeta}\wedge d\tilde\zeta \\
\le & C\epsilon_0|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|^\mu.
\end{aligned}
\tag{3.111}
$$

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- ► If $\tilde{\zeta} \in \{\tilde{s} \leq 2\}/\tilde{D}$ and $\kappa_1 \notin \tilde{D}_0$ then

$$\frac{1}{C} \leq |\frac{\tilde{\zeta} - \tilde{\lambda}_1}{\tilde{\zeta} - \tilde{\lambda}_2}| \leq C, \quad |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq \frac{1}{C} \min\{|\tilde{\lambda}_1 - \kappa_1|, |\tilde{\lambda}_2 - \kappa_1|\}.$$

In this case, using $f^\sharp \in C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})$ and [25, Chapter 1,§ 6.1],

$$\begin{aligned} &| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\{\tilde{s} \leq 2\}/\tilde{D}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{s}}{\tilde{\sigma}}, \beta, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_1}{\tilde{\sigma}}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ \leq & C\epsilon_0 |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} |\tilde{\lambda}_1 - \tilde{\lambda}_2| \iint_{\{\tilde{s} \leq 2\}/\tilde{D}} \frac{1}{|\tilde{\zeta} - \kappa_1||\tilde{\zeta} - \tilde{\lambda}_1|^{2-\mu}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta} \\ \leq & C\epsilon_0 |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu. \end{aligned} \tag{3.112}$$

Therefore, the second term on the RHS of (3.110) is done.

Let $\tilde{L}(\zeta) = 0$ be the line perpendicular to $\overline{\lambda_1 \lambda_2}$ passing through $\frac{1}{2}(\lambda_1 + \lambda_2)$. Set

$$\tilde{D}_{\tilde{\lambda}_1, \pm} = \tilde{D} \cap \{\zeta : \tilde{L}(\zeta)\tilde{L}(\lambda_1) \gtrless 0\}.$$

Therefore, thanks to $f^\sharp \in C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})$, setting $\eta = \frac{\tilde{\zeta} - \tilde{\lambda}_1}{|\tilde{\lambda}_1 - \tilde{\lambda}_2|}$, $\frac{\tilde{\zeta} - \kappa_1}{|\tilde{\lambda}_1 - \tilde{\lambda}_2|} = \eta - r_0 e^{i\alpha_0}$, and using [25, Chapter 1,§ 6.1],

$$\begin{aligned} &|\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, +}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{s}}{\tilde{\sigma}}, \beta, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_1}{\tilde{\sigma}}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ \leq & C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2| |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} | \iint_{\tilde{D}_{\tilde{\lambda}_1, +}} \frac{1}{|\tilde{\zeta} - \kappa_1||\tilde{\zeta} - \tilde{\lambda}_1|^{1-\mu}|\tilde{\zeta} - \tilde{\lambda}_2|} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ \leq & C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} | \iint_{\{|\eta| \leq 2\} \cap \tilde{D}_{\tilde{\lambda}_1, +}} \frac{1}{|\eta - r_0 e^{i\alpha_0}||\eta|^{1-\mu}|\eta - e^{i\alpha'}|} d\eta_R d\eta_I| \\ \leq & C\epsilon_0 |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} | \iint_{\{|\eta| \leq 2\} \cap \tilde{D}_{\tilde{\lambda}_1, +}} \frac{1}{|\eta - r_0 e^{i\alpha_0}||\eta|^{1-\mu}} d\eta_R d\eta_I| \\ \leq & C\epsilon_0 |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu. \end{aligned} \tag{3.113}$$

By analogy,

$$\begin{aligned} &|\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, -}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{s}}{\tilde{\sigma}}, \beta, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_1}{\tilde{\sigma}}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ \leq & |\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, -}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{s}}{\tilde{\sigma}}, \beta, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_2}{\tilde{\sigma}}, \alpha_2, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ + & |\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, -}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{r}_2}{\tilde{\sigma}}, \alpha_2, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_1}{\tilde{\sigma}}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}| \\ \leq & C\epsilon_0 |f^\sharp|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1, \frac{1}{\tilde{\sigma}}})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu \\ + & |\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, -}} \widetilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\frac{\tilde{r}_2}{\tilde{\sigma}}, \alpha_2, X) - \varphi_{f^\sharp}(\frac{\tilde{r}_1}{\tilde{\sigma}}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta}|. \end{aligned}$$

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Applying $f^\sharp \in C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})$, Stokes' theorem and $|\tilde\zeta-\tilde\lambda_1|, |\tilde\zeta-\tilde\lambda_2| \sim |\tilde\lambda_1-\tilde\lambda_2|$ on the boundary of $\tilde D_{\tilde\lambda_1,-}$ (assured by $|\tilde\lambda|\le 1$, $\tilde D\subset\{\tilde s<2\}$),

$$\begin{aligned}
&|\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde D_{\tilde\lambda_1,-}}\widetilde{\gamma}_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde r_2}{\tilde\sigma},\alpha_2,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta|\\
\le& C|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|^{1+\mu}\iint_{\tilde D_{\tilde\lambda_1,-}}\widetilde{\gamma}_1(\tilde\zeta)\frac{1}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta|\\
=& C|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|^{1+\mu}\iint_{\tilde D_{\tilde\lambda_1,-}}\frac{\partial_{\bar{\tilde\zeta}}\left[\ln(1-\gamma|\beta|)\frac{1}{\tilde\zeta-\tilde\lambda_1}\right]}{\tilde\zeta-\tilde\lambda_2}d\bar{\tilde\zeta}\wedge d\tilde\zeta|\\
\le& C\epsilon_0|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|^{\mu}.
\end{aligned}$$

Therefore the first term on the RHS of (3.110) is done. Thus

$$\begin{aligned}
&|\frac{\tilde\lambda_1-\tilde\lambda_2}{4\pi i}\iint_{\tilde s\le 2}\widetilde{\gamma}_1(\tilde\zeta)\frac{\varphi_{f^\sharp}(\frac{\tilde s}{\tilde\sigma},\beta,X)-\varphi_{f^\sharp}(\frac{\tilde r_1}{\tilde\sigma},\alpha_1,X)}{(\tilde\zeta-\tilde\lambda_1)(\tilde\zeta-\tilde\lambda_2)}d\bar{\tilde\zeta}\wedge d\tilde\zeta|\\
&\qquad\le C\epsilon_0|f^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}|\tilde\lambda_1-\tilde\lambda_2|^{\mu},
\end{aligned}\tag{3.114}$$

for $|\tilde\lambda_j-\kappa_1|\le 1$, $j=1,2$. Hence (3.105) can be justified. □

3.3.4. Highlight: estimates for I_4, I_5

Without loss of generality and for simplicity, we assume

$$\kappa_j=\kappa_1,\quad |\lambda-\kappa_1|\le\frac{\delta}{2},\quad X_3>0,\quad X_1,X_2\ge 0,\tag{3.115}$$

in this section. For $\widehat\sigma\in\{X_1,\sqrt{X_2},\sqrt[3]{X_3}\}$, consider the $\widehat\sigma$-scaled coordinates

$$\zeta=\kappa_1+se^{i\beta}=\kappa_1+\frac{\widehat s}{\widehat\sigma}e^{i\beta}\quad\in D_{\kappa_1},$$

and the deformation

$$\widehat s\in\mathbb{R}\longrightarrow\widehat s e^{i\tau}\in\mathbb{C}.$$

Observe that

$$\begin{aligned}
\Re e(-i\wp(\frac{\widehat s e^{i\tau}}{\widehat\sigma},\beta,X))=&\frac{X_3}{\widehat\sigma^3}\sin 3\tau\sin 3\beta\widehat s^3+\frac{X_2}{\widehat\sigma^2}\sin 2\tau\sin 2\beta\widehat s^2+\frac{X_1}{\widehat\sigma}\sin\tau\sin\beta\widehat s\\
\equiv&\frac{X_3}{\widehat\sigma^3}\sin 3\tau\sin 3\beta\widehat s(\widehat s-\rho_+)(\widehat s-\rho_-),\\
\partial_{\widehat s}\wp(\frac{\widehat s}{\widehat\sigma},\beta,X)=&3\frac{X_3}{\widehat\sigma^3}\sin 3\beta\widehat s^2+2\frac{X_2}{\widehat\sigma^2}\sin 2\beta\widehat s+\frac{X_1}{\widehat\sigma}\sin\beta\\
\equiv&3\frac{X_3}{\widehat\sigma^3}\sin 3\beta(\widehat s-\widehat s_+)(\widehat s-\widehat s_-).
\end{aligned}$$

Hence as $|\tau|\ll 1$, $\rho_\pm\sim\widehat s_\pm$. This is the primary motivation behind our definition of stationary points.

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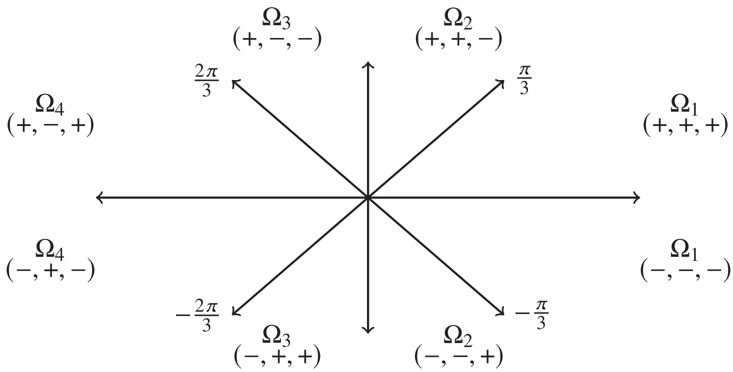

Figure 1. *Signatures of* $(\sin\beta, \sin 2\beta, \sin 3\beta)$ *for* $X_1, X_2, X_3 > 0$.

Definition 3.9. *The stationary points are defined to be*

$$\widehat{s}_\pm = \frac{-1 \pm \sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma} X_2}\frac{\sin 3\beta}{\sin 2\beta}}, \quad \Delta = 3\frac{X_1 X_3}{X_2^2}\frac{\sin\beta \sin 3\beta}{\sin^2 2\beta}, \tag{3.116}$$

which satisfy

$$\partial_{\widehat{s}}\wp(\frac{\widehat{s}_\pm}{\widehat{\sigma}}, \beta, X) = 3\frac{X_3}{\widehat{\sigma}^2}\sin 3\beta \widehat{s}_\pm^2 + 2\frac{X_2}{\widehat{\sigma}^2}\sin 2\beta \widehat{s}_\pm + \frac{X_1}{\widehat{\sigma}}\sin\beta = 0.$$

Denote

$$\begin{aligned} &\Omega_1 = \{0 \le |\beta| \le \tfrac{\pi}{3}\}, \quad \Omega_2 = \{\tfrac{\pi}{3} \le |\beta| \le \tfrac{\pi}{2}\}, \quad \Omega_3 = \{\tfrac{\pi}{2} \le |\beta| \le \tfrac{2\pi}{3}\}, \\ &\Omega_4 = \{\tfrac{2\pi}{3} \le |\beta| \le \pi\}, \end{aligned} \tag{3.117}$$

one has Figure 1 for signatures of $\sin(k\beta)$ *on* Ω_j*. Moreover, according to the determinant* Δ*, we have Table 1 to classify points in* (β, X) *into 5 Types and their subtypes. We outline the properties of stationary points for each type or subtype. Note that*

$$|\widehat{s}_+ - \widehat{s}_-| = |\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma} X_2}\frac{\sin 3\beta}{\sin 2\beta}}|. \tag{3.118}$$

In the following, we will define essential stationary points $\widehat{s}_{j,*}$ and decompose $[0, \widehat{\sigma}]$ into intervals $\mho_j$ around $\widehat{s}_{j,*}$. The deformation will be defined on $\mho_j$.

Definition 3.10. *We define the essential stationary points* $\widehat{s}_{j,*} = \widehat{s}_{j,*}(\widehat{\sigma}, \beta, X)$ *by*

$$\begin{aligned} \widehat{s}_{0,*} =& 0, \\ \widehat{s}_{1,*} =& \begin{cases} \frac{\widehat{s}_+ + \widehat{s}_-}{2} \gtrless 0, & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} \in \{\sqrt{X_2}, \sqrt[3]{X_3}\}),\ \mathfrak{A}'', \\ \inf \widehat{s}_\pm > 0, & \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\ \sup \widehat{s}_\pm > 0, & \textit{Type } \mathfrak{D}, \mathfrak{E}, \\ -, & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} = X_1),\ \mathfrak{B}', \mathfrak{C}', \end{cases} \\ \widehat{s}_{2,*} =& \begin{cases} \sup \widehat{s}_\pm, & \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\ -, & \textit{others}, \end{cases} \end{aligned} \tag{3.119}$$

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Table 1. *Properties of* $\widehat{s}_\pm$ *and* Δ *for Type* $\mathfrak{A}, \cdots, \mathfrak{E}$ *when* $X_1 > 0, X_2, X_3 \geq 0$

Type	**Subtype**	**Range of Δ**	**Properties of** $\widehat{s}_\pm$	β**-domain (order of** $\widehat{s}_\pm$**)**
$\mathfrak{A}(\beta, X)$	$\mathfrak{A}'$	$(2, \infty)$	$*\,\widehat{s}_\pm$ complex roots	$\Omega_1(\frac{\widehat{s}_+ + \widehat{s}_-}{2} \leq 0)$
	$\mathfrak{A}''$	$(1, 2)$		$\Omega_4(\frac{\widehat{s}_+ + \widehat{s}_-}{2} \geq 0)$
$\mathfrak{B}(\beta, X)$	$\mathfrak{B}'$	$*\,(\frac{1}{2}, 1)$	adjacent real roots w.	$\Omega_1(\widehat{s}_- \leq \widehat{s}_+ \leq 0)$
	$\mathfrak{B}''$			$\Omega_4(\widehat{s}_- \geq \widehat{s}_+ \geq 0)$
$\mathfrak{C}(\beta, X)$	$\mathfrak{C}'$	$*\,(0, \frac{1}{2})$	$\widehat{s}_+ = \frac{-\Delta}{6\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.},$	$\Omega_1(\widehat{s}_- \leq \widehat{s}_+ \leq 0)$
	$\mathfrak{C}''$		$\widehat{s}_- = \frac{-2}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.}$	$\Omega_4(\widehat{s}_- \geq \widehat{s}_+ \geq 0)$
$\mathfrak{D}(\beta, X)$		$*\,(-\frac{1}{2}, 0)$	$\widehat{s}_+ = \frac{-\Delta}{6\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.},$	$\Omega_2(\widehat{s}_- \geq 0 \geq \widehat{s}_+)$
			$\widehat{s}_- = \frac{-2}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.}$	$\Omega_3(\widehat{s}_+ \geq 0 \geq \widehat{s}_-)$
$\mathfrak{E}(\beta, X)$		$*\,(-\infty, -\frac{1}{2})$	$*\,\widehat{s}_\pm$ real roots	$\Omega_2(\widehat{s}_- \geq 0 \geq \widehat{s}_+)$
				$\Omega_3(\widehat{s}_+ \geq 0 \geq \widehat{s}_-)$

where – means no definition. Given $0 < \epsilon_1 < \frac{\pi}{2k} \ll 1$*, define neighborhood* $\mho_j(\widehat{\sigma}, \beta, X)$ *of essential critical points* $\widehat{s}_{j,*}$ *by*

$$
\begin{aligned}
\mho_0 &= \begin{cases} [0, \frac{1}{2}], & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} \in \{\sqrt{X_2}, \sqrt[3]{X_3}\}), \mathfrak{A}'', \\ [0, \frac{1}{2\cos\epsilon_1}\widehat{s}_{1,*}], & \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \mathfrak{D}, \mathfrak{E}, \\ [0, \widehat{\sigma}\delta] = [0, \frac{1}{2}] \cup [\frac{1}{2}, \widehat{\sigma}\delta] = \mho_{0,<} \cup \mho_{0,>}, & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} = X_1),\ \mathfrak{B}', \mathfrak{C}', \end{cases} \\
\mho_1 &= \begin{cases} [\frac{1}{2}, \widehat{s}_{1,*}] \cup [\widehat{s}_{1,*}, \widehat{\sigma}\delta] \equiv \mho_{1,<} \cup \mho_{1,>}, & \textit{Type } \{[\mathfrak{A}' \wedge (\widehat{\sigma} \in \{\sqrt{X_2}, \sqrt[3]{X_3}\})] \vee \mathfrak{A}''\} \wedge (\widehat{s}_{1,*} > 0), \\ [\frac{1}{2}, \widehat{\sigma}\delta] \equiv \mho_{1,>}, & \textit{Type } \{[\mathfrak{A}' \wedge (\widehat{\sigma} \in \{\sqrt{X_2}, \sqrt[3]{X_3}\})] \vee \mathfrak{A}''\} \wedge (\widehat{s}_{1,*} < 0), \\ [(1 - \frac{1}{2\cos\epsilon_1})\widehat{s}_{1,*}, \widehat{s}_{1,*}] \cup [\widehat{s}_{1,*}, \widehat{s}_{1,*} + \frac{\widehat{s}_{2,*} - \widehat{s}_{1,*}}{2\cos\epsilon_1}] \equiv \mho_{1,<} \cup \mho_{1,>}, & \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\ [(1 - \frac{1}{2\cos\epsilon_1})\widehat{s}_{1,*}, \widehat{s}_{1,*}] \cup [\widehat{s}_{1,*}, \widehat{\sigma}\delta] \equiv \mho_{1,<} \cup \mho_{1,>}, & \textit{Type } \mathfrak{D}, \mathfrak{E}, \\ \phi, & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} = X_1),\ \mathfrak{B}', \mathfrak{C}', \end{cases} \\
\mho_2 &= \begin{cases} [\widehat{s}_{2,*} - \frac{\widehat{s}_{2,*} - \widehat{s}_{2,*}}{2\cos\epsilon_1}, \widehat{s}_{2,*}] \cup [\widehat{s}_{2,*}, \widehat{\sigma}\delta] \equiv \mho_{2,<} \cup \mho_{2,>}, & \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\ \phi & \textit{otherwise}, \end{cases}
\end{aligned} \tag{3.120}
$$

Write

$$
\begin{gathered}
\lambda = \kappa_1 + \frac{\widehat{r}e^{i\alpha}}{\widehat{\sigma}} = \kappa_1 + \frac{\widehat{s}_{j,*}e^{i\beta} + \widehat{r}_j e^{i\alpha_j}}{\widehat{\sigma}}, \\
\widehat{r}_j = \widehat{r}_j(\widehat{\sigma}, \beta, X, \lambda),\ \alpha_j = \alpha_j(\widehat{\sigma}, \beta, X, \lambda),\ j = 0, 1, 2.
\end{gathered} \tag{3.121}
$$

We define the deformation defined by

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$$
\begin{aligned}
\zeta &= \kappa_1 + s e^{i\beta} = \kappa_1 + \frac{\widehat{s} e^{i\beta}}{\widehat{\sigma}} \\
\widehat{s} \mapsto \xi_j &\equiv \widehat{s}_{j,*} + \widehat{s}_j e^{i\tau_j},\ \widehat{s} \equiv \widehat{s}_{j,*} \pm \widehat{s}_j \in \mho_j,\ |\tau_j| \leqslant \frac{\pi}{2},\ \widehat{s}_j \geq 0,\ j = 0, 1, 2,
\end{aligned}
\tag{3.122}
$$

with

$$
\left\{
\begin{array}{llll}
\mp\pi \lessgtr \tau_1 \lessgtr \mp\pi \pm \epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<} \neq \phi,\ \textit{Type } \mathfrak{A}, \\
\mp\epsilon_1 \lessgtr \tau_1 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>} \neq \phi,\ \textit{Type } \mathfrak{A}, \\
\mp\pi \lessgtr \tau_1 \lessgtr \mp\pi \pm \epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\pm\epsilon_1 \gtrless \tau_1 \gtrless 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\pm\pi \gtrless \tau_1 \gtrless \pm\pi \mp \epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\mp\epsilon_1 \lessgtr \tau_1 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, \textit{Type } \mathfrak{D}, \mathfrak{E}, \mathfrak{B}', \mathfrak{C}', \\
\mp\pi \lessgtr \tau_1 \lessgtr \mp\pi \pm \frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<} \neq \phi,\ \textit{Type } \mathfrak{A}, \\
\mp\frac{\epsilon_1}{4} \lessgtr \tau_1 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>} \neq \phi,\ \textit{Type } \mathfrak{A}, \\
\mp\pi \lessgtr \tau_1 \lessgtr \mp\pi \pm \frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\pm\frac{\epsilon_1}{4} \gtrless \tau_1 \gtrless 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\pm\pi \gtrless \tau_1 \gtrless \pm\pi \mp \frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_1 - \beta| - \pi| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\mp\frac{\epsilon_1}{4} \lessgtr \tau_1 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_1 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, \textit{Type } \mathfrak{D}, \mathfrak{E}, \mathfrak{B}', \mathfrak{C}',
\end{array}
\right.
$$

$$
\left\{
\begin{array}{llll}
\pm\pi \gtrless \tau_2 \gtrless \pm\pi \mp \epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_2 - \beta| - \pi| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2<}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\mp\epsilon_1 \lessgtr \tau_2 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_2 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2>}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\pm\pi \gtrless \tau_2 \gtrless \pm\pi \mp \frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & ||\alpha_2 - \beta| - \pi| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2<}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\mp\frac{\epsilon_1}{4} \lessgtr \tau_2 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_2 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2>}, \textit{Type } \mathfrak{B}'', \mathfrak{C}'',
\end{array}
\right.
$$

$$
\left\{
\begin{array}{llll}
\tau_0 \equiv 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in [0, \frac{1}{2}] \subset \mho_0, \textit{Type } \mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \\
\tau_0 \equiv 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in [0, \frac{1}{2}] \subset \mho_0, \textit{Type } \mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \\
\pm\epsilon_1 \gtrless \tau_0 \gtrless 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\pm\frac{\epsilon_1}{4} \gtrless \tau_0 \gtrless 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\mp\epsilon_1 \lessgtr \tau_0 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{0,>}, \textit{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \\
\mp\frac{\epsilon_1}{4} \lessgtr \tau_0 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{0,>}, \textit{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \\
\mp\epsilon_1 \lessgtr \tau_0 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\mp\frac{\epsilon_1}{4} \lessgtr \tau_0 \lessgtr 0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{B}'', \mathfrak{C}'',
\end{array}
\right.
$$

and

$$
\tau_{0,\dagger} = \left\{
\begin{array}{llll}
0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in [0, \frac{1}{2}] \subset \mho_0, \textit{Type } \mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \\
0, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in [0, \frac{1}{2}] \subset \mho_0, \textit{Type } \mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \\
\pm\epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0,\ \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\pm\frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0,\ \textit{Type } \mathfrak{D}, \mathfrak{E}, \\
\mp\epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{0,>}, \textit{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \\
\mp\frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{0,>}, \textit{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \\
\mp\epsilon_1, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \leq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\mp\frac{\epsilon_1}{4}, & \textit{for } \sin 3\beta \gtrless 0, & |\alpha_0 - \beta| \geq \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_0, \textit{Type } \mathfrak{B}'', \mathfrak{C}'',
\end{array}
\right.
$$

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$$\tau_{1,\dagger} = \begin{cases} \mp\pi \pm \epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<} \ne \phi,\ Type\ \mathfrak{A}, \\ \mp\epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>} \ne \phi,\ Type\ \mathfrak{A}, \\ \mp\pi \pm \epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \pm\epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \pm\pi \mp \epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, Type\ \mathfrak{D}, \mathfrak{E}, \\ \mp\epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, Type\ \mathfrak{D}, \mathfrak{E}, \\ \mp\pi \pm \frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<},\ Type\ \mathfrak{A}, \\ \mp\frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>},\ Type\ \mathfrak{A}, \\ \mp\pi \pm \frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \pm\frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \pm\pi \mp \frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_1-\beta|-\pi| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1<}, Type\ \mathfrak{D}, \mathfrak{E}, \\ \mp\frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_1-\beta| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{1>}, Type\ \mathfrak{D}, \mathfrak{E}, \end{cases}$$

$$\tau_{2,\dagger} = \begin{cases} \pm\pi \mp \epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_2-\beta|-\pi| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2<}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \mp\epsilon_1, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_2-\beta| \le \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2>}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \pm\pi \mp \frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ ||\alpha_2-\beta|-\pi| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2<}, Type\ \mathfrak{B}'', \mathfrak{C}'', \\ \mp\frac{\epsilon_1}{4}, & \text{for } \sin 3\beta \gtrless 0,\ |\alpha_2-\beta| \ge \frac{\epsilon_1}{2}, & \widehat{s} \in \mho_{2>}, Type\ \mathfrak{B}'', \mathfrak{C}'', \end{cases}$$

We confirm Goals (b) and (c) in the following lemma:

Lemma 3.11. *Define the deformation* $\widehat{s} \mapsto \xi_j = \widehat{s}_j e^{i\tau_j} + \widehat{s}_{j,*}$ *on* $\mho_j$ *by Definition 3.10. We have, for* $j = 0, 1, 2,$

$$|\widehat{s}_j e^{i\tau_{j,\dagger}} - \widehat{r}_j e^{i(\alpha_j-\beta)}| \ge \frac{1}{C}\max\{\widehat{r}_j, \widehat{s}_j\}, \quad if\ \ \tau_{j,\dagger} \ne 0, \tag{3.125}$$

and

$$\mathfrak{Re}(-i\wp(\frac{\xi_j}{\widehat{\sigma}}, \beta, X)) \le 0. \tag{3.126}$$

Proof.

- ► **Proof of (3.125):** According to the definition of $\tau_{j,\dagger}$ in Definition 3.10, if $\tau_{j,\dagger} \ne 0$ then

$$|\alpha_j - \beta - \tau_{j,\dagger}| \ge \frac{\epsilon_1}{4}, \quad j = 0, 1, 2. \tag{3.127}$$

  As a result, (3.125) is justified.
- ► **Proof of (3.126):** In view of Definition 3.10, Table 2 and Figure 1,
  - for Type $\mathfrak{A}$:
    * On $\mho_0$, given our assumption, we focus on *Type* $\mathfrak{A}'$ when $\widehat{\sigma} = X_1$. We observe that both terms of $\mathfrak{Re}(-i\wp(\frac{\widehat{s}e^{i\tau}}{\widehat{\sigma}}, \beta, X))$ share the same signature by the conditions $\Delta \ge 2$, $\epsilon_1 \ll 1$, and $\tau \in \Omega_1$, $\beta \in \Omega_1 \cup \Omega_4$.
    * On $\mho_1$, both terms of $\mathfrak{Re}(-i\wp(\frac{\xi_j}{\widehat{\sigma}}, \beta, X))$ share the same signatures by the conditions $\Delta \ge 1$, and $\tau_1, \beta \in \Omega_1 \cup \Omega_4$.
  - For Type $\mathfrak{B}'', \mathfrak{C}''$:

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Table 2. $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}}{\widehat{\sigma}},\beta,X))$ *for deformation defined by Definition 3.10*

Case	Type $\mathfrak{A}$
$\widehat{s}\in\mho_0$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}e^{i\tau}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau\sin 3\beta\widehat{s}(\widehat{s}-\frac{3\sin 2\tau}{2\sin 3\tau}\widehat{s}_{1,*})^2+\frac{X_1}{\widehat{\sigma}}\sin\tau\sin\beta\widehat{s}(1-\frac{3\sin^2 2\tau}{4\sin\tau\sin 3\tau}\frac{1}{\Delta})$
$\widehat{s}\in\mho_1$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}_1e^{i\tau_1}+\widehat{s}_{1,*}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau_1\sin 3\beta\widehat{s}_1^3+\frac{X_1}{\widehat{\sigma}}\sin\tau_1\sin\beta\widehat{s}_1(1-\frac{1}{\Delta})$
Case	Type $\mathfrak{B}$, $\mathfrak{C}$
$\widehat{s}\in\mho_0$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}e^{i\tau}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau\sin 3\beta\widehat{s}(\widehat{s}-\frac{-1+\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})(\widehat{s}-\frac{-1-\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})$
$\widehat{s}\in\mho_1$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}_1e^{i\tau_1}+\widehat{s}_{1,*}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau_1\sin 3\beta\widehat{s}_1^2(\widehat{s}_1+\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}})$
$\widehat{s}\in\mho_2$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}_2e^{i\tau_2}+\widehat{s}_{2,*}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau_2\sin 3\beta\widehat{s}_2^2(\widehat{s}_2-\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_2}{3\sin 2\tau_2}})$
Case	Type $\mathfrak{D}$, $\mathfrak{E}$
$\widehat{s}\in\mho_0$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}e^{i\tau}}{\widehat{\sigma}},\beta,X))$
	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau\sin 3\beta\widehat{s}(\widehat{s}-\frac{-1+\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})(\widehat{s}-\frac{-1-\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})$
$\widehat{s}\in\mho_1$	► $\Re\mathfrak{e}(-i\wp(\frac{\widehat{s}_1e^{i\tau_1}+\widehat{s}_{1,*}}{\widehat{\sigma}},\beta,X))$
$\widehat{s}_{1,*}=\widehat{s}_\pm$	$=\frac{X_3}{\widehat{\sigma}^3}\sin 3\tau_1\sin 3\beta\widehat{s}_1^2(\widehat{s}_1\pm\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}})$

* on $\mho_0$, from $\widehat{s}\le\frac{1}{2\cos\epsilon_1}\widehat{s}_{1,*}$ and $\epsilon_1\ll 1$, we prove (3.126) by means of

$$(\widehat{s}-\frac{-1+\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})(\widehat{s}-\frac{-1-\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}})\ge\frac{1}{8}\widehat{s}_{1,*}^2. \tag{3.128}$$

* On $\mho_1$, from Figure 1, $\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{\sin 2\tau_1}\gtrless 0$ on $\mho_{1,\lessgtr}$. It reduces to proving (3.126) on $\mho_{1,>}$. From the definition of $\mho_{1,>}$ and (3.118), we have

$$\widehat{s}_1\le\frac{\widehat{s}_{2,*}-\widehat{s}_{1,*}}{2\cos\epsilon_1}=\frac{1}{\cos\epsilon_1}\left|\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}}\right|. \tag{3.129}$$

Therefore,

$$\widehat{s}_1+\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}}\le\frac{1}{2}\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\widehat{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}}\le 0,\quad \widehat{s}_1\in\mho_{1,>}. \tag{3.130}$$

As a result, (3.126) follows.

* On $\mho_2$, from Figure 1, $-\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_2}{\sin 2\tau_2} \lessgtr 0$ on $\mho_{2,\lessgtr}$. It reduces to proving (3.126) on $\mho_{2,<}$. From the definition of $\mho_{2,<}$, we have

$$\widehat{s}_2 \le \frac{\widehat{s}_{2,*} - \widehat{s}_{1,*}}{2\cos\epsilon_1} = \frac{1}{\cos\epsilon_1}\Big|\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}}\Big|. \tag{3.131}$$

Therefore,

$$\widehat{s}_2 - \frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_2}{3\sin 2\tau_2}} \le -\frac{1}{2}\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_2}{3\sin 2\tau_2}} \le 0, \quad \widehat{s}_2 \in \mho_{2,<}. \tag{3.132}$$

Hence (3.126) is proved.

- For Type $\mathfrak{B}', \mathfrak{C}'$: it is sufficient to consider (3.126) for $\widehat{s} \in \mho_{0,>}$. Hence (3.126) is proved by $\widehat{s}_\pm \le 0$ and $\widehat{s} > \frac{1}{2}$.
- For Type $\mathfrak{D}, \mathfrak{E}$:
  * on $\mho_0$, we prove (3.126) by means of $\widehat{s} \le \frac{1}{2\cos\epsilon_1}\widehat{s}_{1,*}$, $\epsilon_1 \ll 1$, and

$$\begin{aligned}&\Big(\widehat{s} - \frac{-1+\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}}\Big)\Big(\widehat{s} - \frac{-1-\sqrt{1-\frac{4}{3}\frac{\sin\tau\sin 3\tau}{\sin^2 2\tau}\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{2\sin 3\tau}{3\sin 2\tau}}\Big)\\ &\le -\frac{1}{4}\widehat{s}\widehat{s}_{1,*} \le -\frac{1}{2}\widehat{s}^2.\end{aligned} \tag{3.133}$$

  * on $\mho_1$, if $\widehat{s}_{1,*} = \widehat{s}_+$, by means of Table 1 and Figure 1, $\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{\sin 2\tau_1} \lessgtr 0$ on $\mho_{1,\lessgtr}$. It reduces to proving (3.126) on $\mho_{1,<}$. From the definition of $\mho_{1,<}$, (3.118) and two roots are opposite signs, we have

$$\widehat{s}_1 + \frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}} \le +\frac{1}{2}\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}} \le -\frac{1}{4}\widehat{s}_1, \tag{3.134}$$

and (3.126) is proved.

On $\mho_1$, if $\widehat{s}_{1,*} = \widehat{s}_-$, by means of Table 1 and Figure 1, $-\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{\sin 2\tau_1} \lessgtr 0$ on $\mho_{1,\lessgtr}$. It reduces to proving (3.126) on $\mho_{1,<}$. From the definition of $\mho_{1,<}$, (3.118), and two roots are opposite signs, we have

$$\widehat{s}_1 - \frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}} \le -\frac{1}{2}\frac{\sqrt{1-\Delta}}{3\frac{X_3}{\bar{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}\frac{\sin 3\tau_1}{3\sin 2\tau_1}} \le -\frac{1}{4}\widehat{s}_1, \tag{3.135}$$

and (3.126) is proved.

□

Lemma 3.12. *Let* $\tilde{\sigma} = \max\{1, X_1, \sqrt{X_2}, \sqrt[3]{X_3}\}$ *and* $X_j \geq 0$ *defined by (3.73).*

(i) For Type $\mathfrak{B}''$, $\mathfrak{C}''$, $\mathfrak{D}$, $\mathfrak{E}$ *on* $\mho_j$, $j = 0, 1, 2$, *and Type* $\mathfrak{B}'$, $\mathfrak{C}'$ *on* $\mho_{0,>}$,

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_j e^{i\tau_{j,\dagger}} + \widehat{s}_{j,*}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{|\sin 3\beta|}{C}\widehat{s}_j^3, \qquad \widehat{\sigma} = \sqrt[3]{X_3}; \tag{3.136}$$

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_j e^{i\tau_{j,\dagger}} + \widehat{s}_{j,*}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{|\sin 2\beta|}{C}\widehat{s}_j^2, \qquad \widehat{\sigma} = \sqrt{X_2}. \tag{3.137}$$

(ii) For Type $\mathfrak{A}$ *on* $\mho_1$,

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_j e^{i\tau_{j,\dagger}} + \widehat{s}_{j,*}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{|\sin 3\beta|}{C}\widehat{s}_j^3, \qquad \widehat{\sigma} = \sqrt[3]{X_3}; \tag{3.138}$$

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_j e^{i\tau_{j,\dagger}} + \widehat{s}_{j,*}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{|\sin 2\beta|}{C}\widehat{s}_j^3, \qquad \widehat{\sigma} = \sqrt{X_2} = \tilde{\sigma}. \tag{3.139}$$

(iii) For

- *Type* $\mathfrak{B}''$, $\mathfrak{C}''$, $\mathfrak{D}$, $\mathfrak{E}$ *on* $\mho_0$;
- *Type* $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$ *on* $\mho_{0,>}$,

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_0 e^{i\tau_{0,\dagger}}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{1}{C}|\sin\beta|\widehat{s}, \qquad \widehat{\sigma} = X_1. \tag{3.140}$$

Proof.

- ► **Proof of (3.136) and (3.138):** In view of $\widehat{\sigma} = \sqrt[3]{X_3}$, Definition 3.10, Table 2 and Figure 1, by refining arguments in proving Goal (c), we can derive satisfactory estimates.
- ► **Proof of (3.137):** In view of $\widehat{\sigma} = \sqrt{X_2}$, Definition 3.10, Table 2 and Figure 1, for Type $\mathfrak{B}'', \mathfrak{C}'', \mathfrak{D}, \mathfrak{E}$ on $\mho_j$, $j = 0, 1, 2$; and for Type $\mathfrak{B}', \mathfrak{C}'$ on $\mho_{0,>}$, by refining arguments in proving Goal (c), we can derive satisfactory estimates.
- ► **Proof of (3.139):** Note (3.116) implies

$$3\frac{X_3}{X_2^{3/2}}\sin 3\beta = \frac{X_2^{1/2}}{X_1}\frac{\sin 2\beta}{\sin\beta}\Delta\sin 2\beta. \tag{3.141}$$

  In this case, $\widehat{\sigma} = \sqrt{X_2} = \max\{X_1, \sqrt{X_2}, \sqrt[3]{X_3}\}$ for Type $\mathfrak{A}$ on $\mho_1$. Together with (3.141), Table 1, $\Delta > \frac{1}{2}$, yields

$$\mathfrak{Re}(-i\wp(\frac{\widehat{s}_j e^{i\tau_{j,\dagger}} + \widehat{s}_{j,*}}{\widehat{\sigma}}, \beta, X)) \leq -\frac{1}{C}|\sin 2\beta|\widehat{s}_j^3. \tag{3.142}$$

  Hence (3.139) is justified.
- ► **Proof of (3.140):** In this case, $\widehat{\sigma} = X_1$,
  - For *Type* $\mathfrak{A}'$ on $\mho_0$, use $\Delta \geq 2$, $\widehat{\sigma} = X_1$, $\epsilon_1 \ll 1$.

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- For *Type* $\mathfrak{B}''$, $\mathfrak{C}''$, $\mathfrak{D}$, $\mathfrak{E}$ on $\mho_0$, and for *Type* $\mathfrak{B}'$, $\mathfrak{C}'$, on $\mho_{0,>}$, we use

$$3\frac{X_3}{X_1^3}\sin 3\beta = \left(\frac{X_2}{X_1^2}\frac{\sin 2\beta}{\sin\beta}\right)^2\sin\beta\Delta, \quad 3\frac{X_3}{X_1X_2}\frac{\sin 3\beta}{\sin 2\beta} = \frac{X_2}{X_1^2}\frac{\sin 2\beta}{\sin\beta}\Delta. \tag{3.143}$$

Hence (3.140) follows from

$$\begin{cases} (3.122), (3.141), \frac{1}{2}\le\Delta\le 1, (3.116), Table 3.2 & Type\ \mathfrak{B}, \\ (3.116), Table 3.2, (3.114), |(\widehat{s}-\widehat{s}_+)(\widehat{s}-\widehat{s}_-)| \ge \frac{1}{C}\frac{1}{(\frac{X_2}{X_1^2}\frac{\sin 2\beta}{\sin\beta})^2|\Delta|}, & Type\ \mathfrak{C}, \mathfrak{D}, \\ (3.116), Table 3.2, (3.141), |(\widehat{s}-\widehat{s}_+)(\widehat{s}-\widehat{s}_-)| \ge \frac{1}{C}\frac{1}{(\frac{X_2}{X_1^2}\frac{\sin 2\beta}{\sin\beta})^2|\Delta|}, & Type\ \mathfrak{E}. \end{cases} \tag{3.144}$$

□

Definition 3.13. *Let* $\tilde{\sigma} = \max\{1, X_1, \sqrt{X_2}, \sqrt[3]{X_3}\}$ *with* $\mathrm{X_j}$ *defined by (3.73) and scaled coordinates*

$$\zeta = \kappa_1 + \frac{\tilde{s}}{\tilde{\sigma}}e^{i\beta} \quad \in D_{\kappa_1}$$

by replacing $\widehat{\sigma}, \widehat{s}, \widehat{s}_j, \widehat{s}_{j,*}, \widehat{\lambda}, \widehat{\lambda}_{j,*}$ *by* $\tilde{\sigma}, \tilde{s}, \tilde{s}_j, \tilde{s}_{j,*}, \tilde{\lambda}, \tilde{\lambda}_{j,*}$ *in Definition 3.10. We decompose* $X_1, X_2 \ge 0, X_3 > 0$ *into following three cases*

$$(F1)\ \tilde{\sigma} = X_1, \qquad (F2)\ \tilde{\sigma} = \sqrt{X_2}, \qquad (F3)\ \tilde{\sigma} = \sqrt[3]{X_3}. \tag{3.145}$$

Thus, we have achieved Goals (a), (b) and (c) for Cases $(F3)$ and $(F2)$. We will now demonstrate the estimates for I_4 and I_5.

Proposition 3.14. *For Case* $(F3)$, $(F2)$ *and* $f \in L^\infty(D_{\kappa_j})$ *is* $\tilde{s}$*-holomorphic,*

$$|I_4|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})} \le C\epsilon_0|f|_{L^\infty(D_{\kappa_j})}, \tag{3.146}$$

$$|I_5|_{L^\infty(D_{\kappa_j})} \le C\epsilon_0|f|_{L^\infty(D_{\kappa_j})}. \tag{3.147}$$

Proof.

- **Estimates for** $|I_4|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1,\frac{1}{\tilde{\sigma}}})}$**:** Using the $\tilde{s}$-holomorphic property of *f*, and a residue theorem,

$$\begin{aligned} I_4 = &-\frac{\theta(1-\tilde{r})}{2\pi i}\int_{-\pi}^{\pi} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\{\left(\int_{S_<}+\int_{\Gamma_{40}}\right)\frac{e^{-i\wp(\frac{\xi_0}{\tilde{\sigma}},\beta,X)}f(\frac{\xi_0}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_0e^{i\tau_0}-\tilde{r}_0e^{i(\alpha_0-\beta)}}d\xi_0 \\ &+\int_{\Gamma_{41}}\frac{e^{-i\wp(\frac{\xi_1}{\tilde{\sigma}},\beta,X)}f(\frac{\xi_1}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_1e^{i\tau_1}-\tilde{r}_1e^{i(\alpha_1-\beta)}}d\xi_1 + \int_{\Gamma_{42}}\frac{e^{-i\wp(\frac{\xi_2}{\tilde{\sigma}},\beta,X)}f(\frac{\xi_2}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_1e^{i\tau_2}-\tilde{r}_1e^{i(\alpha_2-\beta)}}d\xi_2 \\ &+\int_{S_>}\frac{e^{-i\wp(\frac{\xi_h}{\tilde{\sigma}},\beta,X)}f(\frac{\xi_h}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_he^{i\tau_h}-\tilde{r}_he^{i(\alpha_h-\beta)}}d\xi_h\}, \end{aligned} \tag{3.148}$$

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with

$$\begin{aligned} S_<(\tilde\sigma,\beta,X,\lambda) =&\{\xi_0 : \tilde s = 2\},\\ \Gamma_{40}(\tilde\sigma,\beta,X,\lambda) =&\{\xi_0 : \tilde s \in (2,\tilde\sigma\delta)\cap \mho_0,\ \tau_0=\tau_{0,\dagger}\},\\ \Gamma_{41}(\tilde\sigma,\beta,X,\lambda) =&\{\xi_1 : \tilde s \in (2,\tilde\sigma\delta)\cap \mho_1,\ \tau_1=\tau_{1,\dagger}\},\\ \Gamma_{42}(\tilde\sigma,\beta,X,\lambda) =&\{\xi_2 : \tilde s \in (2,\tilde\sigma\delta)\cap \mho_2,\ \tau_2=\tau_{2,\dagger}\},\\ S_>(\tilde\sigma,\beta,X,\lambda) =&\{\xi_h : h=\sup_{\mho_j\neq\phi} j,\ \tilde s=\tilde\sigma\delta\}, \end{aligned} \tag{3.149}$$

and ξ_j, τ_j, $\tau_{j,\dagger}$, $\mho_j = \mho_j(\tilde\sigma,\beta,X)$ defined by Definition 3.10.
In view of (3.115), $\tilde r<1$, (3.125), (3.126) and (3.149),

$$\begin{aligned} &|\frac{\theta(1-\tilde r)}{2\pi i}\int_{-\pi}^{\pi} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\\ \times&(\int_{S_<}+\int_{S_>})\frac{e^{-i\wp(\frac{\tilde s}{\tilde\sigma}e^{i\tau},\beta,X)}f(\frac{\tilde s}{\tilde\sigma}e^{i\tau},-\beta,X)}{\tilde s e^{i\tau}-\tilde r e^{i(\alpha-\beta)}}d\tilde s e^{i\tau}|_{C^{\mu}_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})} \le C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}. \end{aligned} \tag{3.150}$$

Applying (3.136) and (3.138) for Case $(F3)$, or (3.137) and (3.139) for $(F2)$, (3.125), $\tilde r<1$, (3.126), (3.149), and improper integrals,

$$\begin{aligned} &\sum_{j=0}^{2}|\frac{\theta(1-\tilde r)}{2\pi i}\int_{-\pi}^{\pi} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\\ \times&\int_{\Gamma_{4j}}\frac{e^{-i\wp(\frac{\tilde s_j}{\tilde\sigma}e^{i\tau_j},\beta,X)}f(\frac{\tilde s_j}{\tilde\sigma}e^{i\tau_j},-\beta,X)}{\tilde s_j e^{i\tau_j}-\tilde r_j e^{i(\alpha_j-\beta)}}d\tilde s e^{i\tau_j}|_{C^{\mu}_{\tilde\sigma}(D_{\kappa_1,\frac{1}{\tilde\sigma}})}\\ \le& C\sum_{n=1}^{2}\sum_{j=0}^{2}|\frac{\theta(1-\tilde r)}{2\pi i}\int_{-\pi}^{\pi} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]e^{-i(n-1)\beta}\\ \times&\int_{\Gamma_{4j}}\frac{e^{-i\wp(\frac{\tilde s_j}{\tilde\sigma}e^{i\tau_j},\beta,X)}f(\frac{\tilde s_j}{\tilde\sigma}e^{i\tau_j},-\beta,X)}{(\tilde s_j e^{i\tau_j}-\tilde r_j e^{i(\alpha_j-\beta)})^n}d\tilde s_j e^{i\tau_j}|_{L^\infty(D_{\kappa_1})}\\ \le&\begin{cases} C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}\int_{-\pi}^{\pi}d\beta\int e^{-\frac{1}{C}\tilde s_j^3|\sin 3\tau_\dagger \sin 3\beta|}d\tilde s_j & if \tilde\sigma=\sqrt[3]{X_3},\\ C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}\int_{-\pi}^{\pi}d\beta\int e^{-\frac{1}{C}\tilde s_j^2|\sin 2\tau_\dagger \sin 2\beta|}d\tilde s_j & if \tilde\sigma=\sqrt{X_2},\end{cases}\\ \le&\begin{cases} C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}\int_{-\pi}^{\pi}d\beta\frac{1}{\sqrt[3]{|\sin 3\beta|}}\int_0^\infty e^{-t^3|\sin 3\tau_\dagger|}dt & if \tilde\sigma=\sqrt[3]{X_3},\\ C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}\int_{-\pi}^{\pi}d\beta\frac{1}{\sqrt{|\sin 2\beta|}}\int_0^\infty e^{-t^2|\sin 2\tau_\dagger|}dt & if \tilde\sigma=\sqrt{X_2},\end{cases}\\ \le& C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}. \end{aligned} \tag{3.151}$$

Combining (3.148), (3.150) and (3.151), we derive (3.146).

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- **Estimates for $|I_5|_{L^\infty(D_{\kappa_1})}$:** Using the $\tilde{s}$-holomorphic property of f, and the residue theorem,

$$I_5 = -\frac{\theta(\tilde{r}-1)}{2\pi i}\int_{-\pi}^{\pi} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\{\int_{\Gamma_{50}} \frac{e^{-i\wp(\frac{\xi_0}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_0}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_0 e^{i\tau_0} - \tilde{r}_0 e^{i(\alpha_0-\beta)}} d\xi_0 \tag{3.152}$$
$$+\int_{\Gamma_{51}} \frac{e^{-i\wp(\frac{\xi_1}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_1}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_1 e^{i\tau_1} - \tilde{r}_1 e^{i(\alpha_1-\beta)}} d\xi_1 + \int_{\Gamma_{52}} \frac{e^{-i\wp(\frac{\xi_2}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_2}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_2 e^{i\tau_2} - \tilde{r}_2 e^{i(\alpha_2-\beta)}} d\xi_2$$
$$+\int_{S_>} \frac{e^{-i\wp(\frac{\xi_h}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_h}{\tilde{\sigma}},-\beta,X)}{\tilde{s}_h e^{i\tau_h} - \tilde{r}_h e^{i(\alpha_h-\beta)}} d\xi_h\}$$
$$-\theta(\tilde{r}-1)\theta(\tilde{r}_1-\frac{1}{4})\theta(\tilde{r}_2-\frac{1}{4})\int_{\beta\in\Delta(\lambda)} d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\,\mathrm{sgn}(\beta)$$
$$\times e^{-i\wp(\frac{\tilde{r}e^{i(\alpha-\beta)}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{r}e^{i(\alpha-\beta)}}{\tilde{\sigma}},-\beta,X),$$

where

$$\Delta(\lambda) \equiv \{\beta : \begin{cases} |\alpha_0-\beta| < \frac{\epsilon_1}{2}, & (\alpha_0-\beta)\beta < 0, \quad \tilde{r}\in \mho_0, \\ ||\alpha_1-\beta|-\pi| < \frac{\epsilon_1}{2}, & (\alpha_1-\beta)\beta < 0, \quad \tilde{r}\in \mho_{1<}, \\ |\alpha_1-\beta| < \frac{\epsilon_1}{2}, & (\alpha_1-\beta)\beta < 0, \quad \tilde{r}\in \mho_{1>}, \\ ||\alpha_2-\beta|-\pi| < \frac{\epsilon_1}{2}, & (\alpha_2-\beta)\beta < 0, \quad \tilde{r}\in \mho_{2<}, \\ |\alpha_2-\beta| < \frac{\epsilon_1}{2}, & (\alpha_2-\beta)\beta < 0, \quad \tilde{r}\in \mho_{2>} \end{cases} \}, \tag{3.153}$$

and $S_>$ defined by (3.149), $\Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda)$, $j = 0, 1, 2$ defined by

$$\begin{aligned} \Gamma_{50} =& \{\xi_0 : \tilde{s}\in \mho_0,\ \tau_0 = \tau_{0,\dagger}\} \cup S_{50}, \\ \Gamma_{51} =& \Gamma_{51,out} \cup S_{51} \cup \Gamma_{51,in}, \\ \Gamma_{52} =& \Gamma_{52,out} \cup S_{52} \cup \Gamma_{52,in}, \end{aligned} \tag{3.154}$$

with

$$S_{50} = \begin{cases} \{\xi_1 : \tilde{s}_0 = \frac{1}{2}^+\} & \textit{Type } \mathfrak{A}' \wedge (\widehat{\sigma} = X_1),\ \mathfrak{B}', \mathfrak{C}', \\ \phi, & \textit{otherwise}, \end{cases}$$
$$\Gamma_{51,in} = \begin{cases} \phi, & \tilde{r}_1 > \frac{1}{4}, \\ \{\xi_1 : \tilde{s}\in \mho_1,\ \tau_1 = 0 on \mho_{1>}, & \tilde{r}_1 < \frac{1}{4}, \\ \qquad \tau_1 = \pi\ on \mho_{1<},\ \tilde{s}_1 < 1/2\}, & \end{cases}$$
$$\Gamma_{51,out} = \begin{cases} \{\xi_1 : \tilde{s}\in \mho_1,\ \tau_1 = \tau_{1,\dagger}\}, & \tilde{r}_1 > \frac{1}{4}, \\ \{\xi_1 : \tilde{s}\in \mho_1,\ \tau_1 = \tau_{1,\dagger},\ \tilde{s}_1 > 1/2\}, & \tilde{r}_1 < \frac{1}{4}, \end{cases}$$
$$S_{51} = \begin{cases} \phi, & \tilde{r}_1 > \frac{1}{4}, \\ \{\xi_1 : \tilde{s}_1 = 1/2\} & \tilde{r}_1 < \frac{1}{4}, \end{cases}$$
$$\Gamma_{52,in} = \begin{cases} \phi, & \tilde{r}_2 > \frac{1}{4}, \\ \{\xi_2 : \tilde{s}\in \mho_2,\ \ \tau_2 = 0 on \mho_{2>}, & \tilde{r}_2 < \frac{1}{4}, \\ \qquad \tau_2 = \pi on \mho_{2<},\ \tilde{s}_2 < 1/2\}, & \end{cases}$$
$$\Gamma_{52,out} = \begin{cases} \{\xi_2 : \tilde{s}\in \mho_2,\ \tau_2 = \tau_{2,\dagger}\}, & \tilde{r}_2 > \frac{1}{4}, \\ \{\xi_2 : \tilde{s}\in \mho_2,\ \tau_2 = \tau_{2,\dagger},\ \tilde{s}_2 > 1/2\}, & \tilde{r}_2 < \frac{1}{4}, \end{cases}$$
$$S_{52} = \begin{cases} \phi, & \tilde{r}_2 > \frac{1}{4}, \\ \{\xi_2 : \tilde{s}_2 = 1/2\} & \tilde{r}_2 < \frac{1}{4}, \end{cases}$$

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and α_j, ξ_j, $\tau_{j,\dagger}$, $\mho_j = \mho_j(\tilde{\sigma}, \beta, X)$ defined by Definition 3.15.

Using (3.86), $\tilde{r} > 1$, and the same argument as that for I_4,

$$|I_5|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + \sum_{j=1}^{2} |\frac{\theta(\tilde{r}-1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \times \int_{\Gamma_{5j,in}} \frac{e^{-i\wp(\frac{\xi_j}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_j}{\tilde{\sigma}}, -\beta, X)}{\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j-\beta)}} d\xi_j |_{L^\infty(D_{\kappa_1})}. \tag{3.156}$$

Namely, we have to pay extra attention when both $\tilde{\zeta}$ and $\tilde{\lambda}$ are close to one of the essential stationary points $\tilde{s}_{j,*}(\tilde{\sigma}, \beta, X)$, say $\tilde{s}_{1,*}(\tilde{\sigma}, \beta, X)$, without loss of generality, because the other case can be done by analogy. In this situation, $\Gamma_{52,in} = \phi$. For the estimates on $\Gamma_{51,in}$, in view of (3.154), we have $|\kappa_1 - \tilde{\zeta}| \geq 1/4$ for $\tilde{s} \in \Gamma_{51,in}$. Hence I_5 for $\tilde{s} \in \Gamma_{51,in}$ is no longer a singular integral and we can apply (3.86). Namely,

$$\begin{aligned} &|\theta(\tilde{r}-1) \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \int_{\Gamma_{51,in}} \frac{e^{-i\wp(\frac{\xi_1}{\tilde{\sigma}},\beta,X)} f(\frac{\xi_1}{\tilde{\sigma}}, -\beta, X)}{\tilde{s}_1 e^{i\tau_1} - \tilde{r}_1 e^{i(\alpha_1-\beta)}} d\xi_1 |_{L^\infty(D_{\kappa_1})} \\ \leq &C| \iint_{\tilde{s}\in\Gamma_{51,in}} \frac{\widetilde{\gamma}_1(\tilde{s},\beta) e^{-i\wp(\frac{\tilde{s}}{\tilde{\sigma}},\beta,X)} f(\frac{\tilde{s}}{\tilde{\sigma}}, -\beta, X)}{\tilde{\zeta} - \tilde{\lambda}} d\bar{\tilde{\zeta}} \wedge d\tilde{\zeta} |_{L^\infty(D_{\kappa_1})} \\ \leq &C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \end{aligned} \tag{3.157}$$

Consequently, (3.147) is established. □

Estimates for Case $(F1)$ are complex. Below, we outline difficulties and our approach:

- For $\mho_j$ $(j \geq 1)$: Lemma 3.12 shows no uniform estimates exist for $\Re\mathfrak{e}(-i\wp(\frac{\tilde{s}_j e^{i\tau_{j,\dagger}} + \tilde{s}_{j,*}}{X_1}, \beta, X))$. We utilize the scaling invariance of the Hilbert transform and estimates (3.137) and (3.138), where $\sqrt{X_2}$ and $\sqrt[3]{X_3}$ are not equal to $\tilde{\sigma} = X_1$, to derive estimates for Γ_{4j} or Γ_{5j}, $j \geq 1$. Additionally, the Cauchy integral near renormalized critical points, as outlined in Lemma 3.16, needs to avoid singularities for the proper application of (3.86).
- For $\mho_0$: The scaling argument does not work here since 0 is a singular point. We have to take advantage of estimate (3.140)! If we directly apply the arguments from Case $(F3)$ or $(F2)$ in Proposition 3.14, the difficulty lies in the Jacobian $\frac{1}{|\sin\beta|} d\beta$. Since it is not suitable for improper integrals. To address this for small $|\beta|$, we use a finer decomposition with the $\mathfrak{J}_1$–$\mathfrak{J}_5$ approach in (3.169)–(3.173) to extract extra $|\sin\beta|$ decay on Γ_{40} or Γ_{50}.

Definition 3.15. *For Case $(F1)$, introduce new scaled σ_j-parameters on $\mho_j(\beta, X)$, $j = 0, 1, 2$,*

$$\begin{cases} \sigma_0 = \tilde{\sigma}, \ \sigma_1 = \sigma_2 = \sqrt[3]{|X_3|}, & \textit{for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{E}, \\ \sigma_0 = \tilde{\sigma}, \ \sigma_1 = \sigma_2 = \sqrt[2]{|X_2|}, & \textit{for Type } \mathfrak{C}, \mathfrak{D}, \end{cases} \tag{3.158}$$

and scaled σ_j-coordinates on $\mho_j(\tilde{\sigma}, \beta, X)$

$$\lambda = \kappa_1 + \frac{\tilde{r}}{\tilde{\sigma}}e^{i\alpha} = \kappa_1 + \frac{s_{j,*}e^{i\beta} + r_j e^{i\alpha_j}}{\sigma_j}, \tag{3.159}$$
$$s_{j,*} \equiv \tilde{s}_{j,*}\frac{\sigma_j}{\tilde{\sigma}},\ r_j = \tilde{r}_j\frac{\sigma_j}{\tilde{\sigma}} \ge 0,\ \alpha_j = \alpha_j(\beta, X, \lambda),$$
$$\frac{\tilde{s}}{\tilde{\sigma}} \mapsto \frac{\vartheta_j}{\sigma_j} \equiv \frac{s_{j,*} + s_j e^{i\tau_j}}{\sigma_j},\ \ \tilde{s} \equiv (s_{j,*} \pm s_j)\frac{\tilde{\sigma}}{\sigma_j} \in \mho_j(\tilde{\sigma}, \beta, X)$$

where $\mho_j(\tilde{\sigma}, \beta, X)$, $\tilde{s}_{j,*}$, *and* $\tilde{r}_j$ *are defined by Definition 3.13.*

Lemma 3.16. *For Case* $(F1)$, *introduce the new scaled* σ_j*-coordinates defined by Definition 3.15,*

$$\inf_\beta s_{1,*} = c_0, \quad 0 < c_0 < 1. \tag{3.160}$$

Proof.

- ► **Proof for** *Type* $\mathfrak{A}'', \mathfrak{B}'', \mathfrak{E}$**:** from Table 1, (3.158), and

$$|s_+ s_-| = |\frac{X_1}{X_3^{1/3}}\frac{\sin\beta}{\sin 3\beta}| \ge \frac{1}{3}, \qquad s_\pm = \frac{-1 \pm \sqrt{1-\Delta}}{3\frac{X_3^{2/3}}{X_2}\frac{\sin 3\beta}{\sin 2\beta}},$$

  we derive $|s_+| \sim |s_-|$ and then (3.160) for *Type* $\mathfrak{A}'', \mathfrak{B}'', \mathfrak{E}$.
- ► **Proof for** *Type* $\mathfrak{C}'', \mathfrak{D}$**:** from (3.116), Table 1 and (3.158),

$$|s_+| = |\frac{-\Delta}{6\frac{X_3}{\tilde{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.}| = |\frac{-3\frac{X_1X_3}{X_2^2}\frac{\sin\beta\sin 3\beta}{\sin^2 2\beta}}{6\frac{X_3}{X_2^{3/2}}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.}| \ge \frac{1}{C},$$
$$|s_-| = |\frac{-2}{3\frac{X_3}{\tilde{\sigma}X_2}\frac{\sin 3\beta}{\sin 2\beta}} + \text{l.o.t.}| \ge \frac{1}{C}|s_+| \ge \frac{1}{C}.$$

  Hence (3.160) is proved for Type $\mathfrak{C}''$ and $\mathfrak{D}$.

□

Proposition 3.17. *For Case* $(F1)$, *and* $f \in L^\infty(D_{\kappa_j})$ *is* $\tilde{s}$*-holomorphic,*

$$|I_4|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_j,\frac{1}{\tilde{\sigma}}})} \le C\epsilon_0 |f|_{L^\infty(D_{\kappa_j})}, \tag{3.161}$$

$$|I_5|_{L^\infty(D_{\kappa_j})} \le C\epsilon_0 |f|_{L^\infty(D_{\kappa_j})}. \tag{3.162}$$

Proof. Thanks to Lemma 3.16, by following the same argument (3.148)–(3.150), (3.152)–(3.154), we obtain

$$|I_4|_{C^\mu_{\tilde{\sigma}}(D_{\kappa_1,\frac{1}{\tilde{\sigma}}})} \le C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + \sum_{j=0}^{2} |I_{4j}|_{L^\infty(D_{\kappa_1})}, \tag{3.163}$$

$$|I_5|_{L^\infty(D_{\kappa_1})} \le C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + \sum_{j=0}^{2} |I_{5j}|_{L^\infty(D_{\kappa_1})}, \tag{3.164}$$

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with

$$I_{4j} = \sum_{h=1,2} \frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] e^{-i(n-1)\beta} \int_{\Gamma_{4j}} \frac{e^{-i\wp(\frac{\tilde{s}_j}{\tilde{\sigma}} e^{i\tau_j},\beta,X)} f(\frac{\tilde{s}_j}{\tilde{\sigma}} e^{i\tau_j}, -\beta, X)}{(\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j-\beta)})^h} d\tilde{s}_j e^{i\tau_j},$$

$$I_{5j} = \frac{\theta(\tilde{r}-1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \int_{\Gamma_{5j,out}} \frac{e^{-i\wp(\frac{\tilde{s}_j}{\tilde{\sigma}} e^{i\tau_j},\beta,X)} f(\frac{\tilde{s}_j}{\tilde{\sigma}} e^{i\tau_j}, -\beta, X)}{\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j-\beta)}} d\tilde{s}_j e^{i\tau_j}.$$

- ► **Proof for** $I_{41}, I_{42}, I_{51}, I_{52}$**:** For $j \geq 1$, using the scaling invariant of the Hilbert transform and

- for $\Gamma_{4j} \cap \{s_j < 1\}$: applying Lemma 3.16, (3.86);
- for $\Gamma_{4j} \cap \{s_j > 1\}$: thanks to the scaling invariant of the Hilbert transform, applying (3.138) on $\mho_1$, $\mho_2$ for *Type* $\mathfrak{A}''$, $\mathfrak{B}''$ $\mathfrak{E}$, and (3.139) on $\mho_1$, $\mho_2$ for *Type* $\mathfrak{C}''$, $\mathfrak{D}$,

we have

$$\begin{aligned}
&|I_{4j}|_{C^1(D_{\kappa_1,\frac{1}{\sigma}})} \\
\leq &\sum_{h=0}^{1} \{|\frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \\
&\times \int_{\Gamma_{4j}\cap\{s_j<1\}} \frac{e^{-i\wp(\frac{s_j}{\sigma} e^{i\tau_j},\beta,X)} \widetilde{\chi}(\frac{s_j}{\sigma} e^{i\tau_j}, -\beta, X)}{(\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j-\beta)})^h (s_j e^{i\tau_j} - r_j e^{i(\alpha_j-\beta)})} ds_j e^{i\tau_j}|_{L^\infty(D_{\kappa_1})} \\
+ &\sum_{h=0}^{1} \{|\frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \\
&\times \int_{\Gamma_{4j}\cap\{s_j>1\}} \frac{e^{-i\wp(\frac{s_j}{\sigma} e^{i\tau_j},\beta,X)} \widetilde{\chi}(\frac{s_j}{\sigma} e^{i\tau_j}, -\beta, X)}{(\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j-\beta)})^h (s_j e^{i\tau_j} - r_j e^{i(\alpha_j-\beta)})} ds_j e^{i\tau_j}|_{L^\infty(D_{\kappa_1})}\} \\
\leq &C\epsilon_0 |\widetilde{\chi}|_{L^\infty(D_{\kappa_1})}.
\end{aligned} \tag{3.165}$$

In an entirely similar way, namely, for $j \geq 1$,

- for $\Gamma_{5j} \cap \{s_j < 1\}$: applying Lemma 3.16, (3.86);
- for $\Gamma_{5j} \cap \{s_j > 1\}$: thanks to the scaling invariant of the Hilbert transform, applying (3.138) on $\mho_1$, $\mho_2$ for *Type* $\mathfrak{A}''$, $\mathfrak{B}''$ $\mathfrak{E}$, and (3.139) on $\mho_1$, $\mho_2$ for *Type* $\mathfrak{C}''$, $\mathfrak{D}$,

we have

$$|I_{5j}|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \tag{3.166}$$

As a result, the proof for I_{4j}, I_{5j} for $j \geq 1$ is done.

- ► **Proof for** I_{40}, I_{50}**:** Decompose

$$I_{40} = -\frac{\theta(1-\tilde{r})}{2\pi i} \int_0^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \int_{\Gamma_{40}} \mathfrak{I}_<, \tag{3.167}$$

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$$\mathfrak{J}_< = \begin{cases} \mathfrak{J}_1+\mathfrak{J}_2+\mathfrak{J}_3+\mathfrak{J}_4+\mathfrak{J}_5, & if 0<\beta<\epsilon_1/8,\\ 0, & if -\epsilon_1/8<\beta<0,\\ \frac{e^{-i\wp(\frac{\vartheta_0}{\sigma_0},\beta,X)}f(\frac{\vartheta_0}{\sigma_0},-\beta,X)}{s_0e^{i\tau_0}-r_0e^{i(\alpha_0-\beta)}}d\vartheta_0, & if|\beta|>\epsilon_1/8; \end{cases} \tag{3.168}$$

and $\mathfrak{J}_1,\cdots,\mathfrak{J}_5$ defined by

$$\mathfrak{J}_1 = \theta(\frac{1}{|\sin\beta|}-|\tilde{s}-\tilde{r}|)\frac{[e^{-i\wp(\frac{\tilde{s}e^{i\tau}}{X_1},\beta,X)}-1]f(\frac{\tilde{s}e^{i\tau}}{X_1},-\beta,X)}{\tilde{s}e^{i\tau}-\tilde{r}e^{i(\alpha-\beta)}}d\tilde{s}e^{i\tau}, \tag{3.169}$$

$$\mathfrak{J}_2 = \theta(\frac{1}{|\sin\beta|}-|\tilde{s}-\tilde{r}|)\frac{[1-e^{-i\wp(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)}]f(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)}{\tilde{s}e^{-i\tau}-\tilde{r}e^{i(\alpha-\beta)}}d\tilde{s}e^{-i\tau}, \tag{3.170}$$

$$\mathfrak{J}_3 = \theta(\frac{1}{|\sin\beta|}-|\tilde{s}-\tilde{r}|)e^{-i\wp(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)}f(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X) \tag{3.171}$$

$$\times[\frac{1}{\tilde{s}e^{-i\tau}-\tilde{r}e^{i(\alpha-\beta)}}-\frac{1}{\tilde{s}e^{-i\tau}-\tilde{r}e^{i(\alpha+\beta)}}]d\tilde{s}e^{-i\tau},$$

$$\mathfrak{J}_4 = \theta(\frac{1}{|\sin\beta|}-|\tilde{s}-\tilde{r}|)e^{-i\wp(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)}\frac{f(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)-f(\frac{\tilde{s}e^{-i\tau}}{X_1},+\beta,X)}{\tilde{s}e^{-i\tau}-\tilde{r}e^{i(\alpha+\beta)}}d\tilde{s}e^{-i\tau}, \tag{3.172}$$

$$\mathfrak{J}_5 = \theta(|\tilde{s}-\tilde{r}|-\frac{1}{|\sin\beta|})(e^{-i\wp(\frac{\tilde{s}e^{i\tau}}{X_1},\beta,X)}\frac{f(\frac{\tilde{s}e^{i\tau}}{X_1},-\beta,X)}{\tilde{s}e^{i\tau}-\tilde{r}e^{i(\alpha-\beta)}}d\tilde{s}e^{i\tau} \tag{3.173}$$

$$-e^{-i\wp(\frac{\tilde{s}e^{-i\tau}}{X_1},-\beta,X)}\frac{f(\frac{\tilde{s}e^{-i\tau}}{X_1},+\beta,X)}{\tilde{s}e^{-i\tau}-\tilde{r}e^{i(\alpha+\beta)}}d\tilde{s}e^{-i\tau}),$$

with τ defined by Definition 3.10 for $\beta\in[0,\pi]$.

Let's explain our strategy before providing detailed estimates for $\mathfrak{J}_1,\cdots,\mathfrak{J}_5$.

- For $\mathfrak{J}_1$-$\mathfrak{J}_4$, where $|\tilde{s}-\tilde{r}|<\frac{1}{|\sin\beta|}$, we can extract additional $|\sin\beta|$-decay from the numerators of the integrands by utilizing the difference terms. Moreover, through a change of variables,

$$\tilde{s}\mapsto t=\tilde{s}|\sin\beta|, \tag{3.174}$$

  the t-domain is compact. This allows us to derive effective estimates.
- For $\mathfrak{J}_5$, where $|\tilde{s}-\tilde{r}|>\frac{1}{|\sin\beta|}$. We work on the same change of variables, leverage the scaling invariance of the Hilbert transform to cancel the Jacobian. The cut-off function ensures that the kernel of the t-Hilbert transform remains bounded, while the numerator exhibits an exponential decay in t. Consequently, we can derive estimates.

Precisely,

- $\mathfrak{J}_1,\mathfrak{J}_2$: From the mean value theorem and (3.140),

$$|e^{-i\wp(\frac{\tilde{s}e^{\pm i\tau}}{X_1},\pm\beta,X)}-1|\le C|\sin\beta|, \quad for\ \tilde{s}\in\mho_0(\beta,X),\ |\beta|<\frac{\epsilon_1}{8}. \tag{3.175}$$

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- $\mathfrak{J}_3$: From $|\beta| < \epsilon_1/8$,

$$
\begin{aligned}
&|\frac{1}{\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha-\beta)}} - \frac{1}{\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha+\beta)}}| \\
=&|\frac{\tilde{r}e^{i\alpha}2\sin\beta}{(\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha-\beta)})(\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha+\beta)})}| \le C|\sin\beta|.
\end{aligned}
\tag{3.176}
$$

- $\mathfrak{J}_4$: From $\tilde{s}$-holomorphic properties of f,

$$
\begin{aligned}
&|\frac{f(\frac{\tilde{s}}{\tilde{\sigma}}e^{-i\tau_\dagger}, -\beta, X) - f(\frac{\tilde{s}}{\tilde{\sigma}}e^{-i\tau_\dagger}, +\beta, X)}{\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha+\beta)}}| \\
\le& C|f|_{L^\infty(D_{\kappa_1})}\frac{\tilde{s}|\sin\beta|}{|\tilde{s}e^{-i\tau_\dagger} - \tilde{r}e^{i(\alpha+\beta)}|} \le C|f|_{L^\infty(D_{\kappa_1})}|\sin\beta|.
\end{aligned}
\tag{3.177}
$$

Applying (3.169)–(3.172), (3.175)–(3.177) and the change of variables (3.174), for $j = 1, \dots, 4$,

$$
\begin{aligned}
&| - \frac{\theta(1-\tilde{r})}{2\pi i}\int_0^\pi d\beta[\partial_\beta \ln(1-\gamma|\beta|)] + \int_{\Gamma_{40}} \mathfrak{J}_j|_{L^\infty(D_{\kappa_1})} \\
\le& C|f|_{L^\infty(D_{\kappa_1})}|\int_0^\pi d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\int_{2|sin\beta|}^{X_1\delta|\sin\beta|}\theta(1 - |t - (\tilde{r}|\sin\beta|)\,|)dt|_{L^\infty(D_{\kappa_1})} \\
\le& C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\end{aligned}
\tag{3.178}
$$

On the other hand, using (3.173), $\epsilon_1 > 0$, the rescaling (3.174), the scaling invariant property of the Hilbert transform, one obtains

$$
\begin{aligned}
&| - \frac{\theta(1-\tilde{r})}{2\pi i}\int_0^\pi d\beta[\partial_\beta \ln(1-\gamma|\beta|)]\int_{\Gamma_{40}} \mathfrak{J}_5|_{L^\infty(D_{\kappa_1})} \\
\le& C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}\int_{2|sin\beta|}^{X_1\delta|\sin\beta|}\theta(|t-\tilde{r}|\sin\beta|\,| - 1)\frac{e^{-t\sin\frac{\epsilon_1}{4}}}{|t-\tilde{r}|\sin\beta|\,|}dt|_{L^\infty(D_{\kappa_1})} \\
\le& C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}.
\end{aligned}
\tag{3.179}
$$

From (3.178) and (3.179), $|I_{40}|_{L^\infty(D_{\kappa_1})} \le C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}$. In an entirely similar way, $|I_{50}|_{L^\infty(D_{\kappa_1})} \le C\epsilon_0|f|_{L^\infty(D_{\kappa_1})}$. Combining with (3.163)–(3.166), the proof for (3.161), (3.162) is completed. □

4. The IST for perturbed multi-line solitons

Using Sato's theory [2, 3, 19, 22–24] and Boiti *et al.*'s direct scattering theory of the KP equation [4–6, 8–10, 21], we extend the IST method for perturbed 1-solitons to perturbed $\mathrm{Gr}(N,M)_{>0}$ KP solitons. We present the complete theory for these solitons, focusing on the distinct features which simultaneously demonstrate the necessity of the TP condition and clarify that the differences between IST for perturbed 1-solitons and multi-line solitons are primarily algebraic.

4.1. Statement of results

Definition 4.1. *Given $0 < \epsilon_0 \ll 1$, $d \ge 0$, and a $\mathrm{Gr}(N,M)_{>0}$ KP soliton $u_s(x)$ defined by $\{\kappa_j\}, A$, a scattering data $\mathcal{S} = (\{z_n\}, \{\kappa_j\}, \mathcal{D}, s_c(\lambda))$ is called* d*-admissible if*

$$
s_c(\lambda) = \begin{cases} \frac{\frac{i}{2}\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|} + \mathrm{sgn}(\lambda_I)h_j(\lambda), & \lambda \in D^\times_{\kappa_j}, \\ \mathrm{sgn}(\lambda_I)\hbar_n(\lambda), & \lambda \in D^\times_{z_n}, \end{cases}
\tag{4.1}
$$

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$$\mathcal{D}=\begin{pmatrix} \kappa_1^N & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \kappa_N^N\\ \mathcal{D}_{N+11} & \cdots & \mathcal{D}_{N+1N}\\ \vdots & \ddots & \vdots\\ \mathcal{D}_{M1} & \cdots & \mathcal{D}_{MN}\end{pmatrix}, \tag{4.2}$$

and

$$\det\left(\frac{1}{\kappa_k - z_h}\right)_{1\le k,h\le N}\neq 0,\ z_1=0,\{z_n,\kappa_j\}\text{distinct real}, \tag{4.3}$$

$$\epsilon_0 \ge (1-\sum_{j=1}^M \mathcal{E}_{\kappa_j})\sum_{|l|\le d+8}|\left(|\overline{\lambda}-\lambda|^{l_1}+|\overline{\lambda}^2-\lambda^2|^{l_2}\right)s_c(\lambda)|_{L^\infty} \tag{4.4}$$

$$+\sum_{j=1}^M(|\gamma_j|+|h_j|_{L^\infty(D_{\kappa_j})})+\sum_{n=1}^N|\hbar_n|_{C^1(D_{z_n})}$$
$$+|diag\,(q_1,\cdots,q_M)^{-1}\times\mathcal{D}\times diag\,(q_1,\cdots,q_N)A_N^T-\mathcal{D}^\flat|_{L^\infty},$$

$$s_c(\lambda)=\overline{s_c(\overline{\lambda})},h_j(\lambda)=-\overline{h_j(\overline{\lambda})},\hbar_n(\lambda)=-\overline{\hbar_n(\overline{\lambda})}, \tag{4.5}$$

$$\mathcal{D}^\flat = diag\,(\kappa_1^N,\cdots,\kappa_M^N)\,A^T,\ A_N=(a_{kl})_{1\le k,l\le N},\ \ q_j=\frac{\Pi_{2\le n\le N}(\kappa_j-z_n)}{(\kappa_j-z_1)^{N-1}}. \tag{4.6}$$

Define T as the continuous scattering operator

$$T\phi(x,\lambda)\equiv s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2+(\overline{\lambda}^3-\lambda^3)x_3}\phi(x,\overline{\lambda}). \tag{4.7}$$

Definition 4.2. *Given $\{z_n,\kappa_j\}$, $1\le n\le N$, $1\le j\le M$, the eigenfunction space $W=W_x$ consists of ϕ satisfying*

(a) $\phi(x,\lambda)=\overline{\phi(x,\overline{\lambda})}$;
(b) $(1-\sum_{n=1}^N\mathcal{E}_{z_n})\phi(x,\lambda)\in L^\infty$;
(c) for $\lambda\in D_{z_n}^\times$, $\phi(x,\lambda)=\frac{\phi_{z_n,\mathrm{res}}(x)}{\lambda-z_n}+\phi_{z_n,r}(x,\lambda)$, $\phi_{z_n,\mathrm{res}}$, $\phi_{z_n,r}\in L^\infty(D_{z_n})$;
(d) for $\lambda=\kappa_j+re^{i\alpha}\in D_{\kappa_j}^\times$, $\phi=\phi^\flat+\phi^\sharp$, $\phi^\flat=\sum_{l=0}^\infty\phi_l(X)(-\ln(1-\gamma_j|\alpha|))^l\in L^\infty(D_{\kappa_j})$, $\phi^\sharp\in C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})\cap L^\infty(D_{\kappa_j})$, $\phi^\sharp(x,\kappa_j)=0$.

Here the rescaling parameter $\tilde\sigma$ and rescaled Hölder spaces $C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})$ are defined as in Definition 3.6. Finally, for $\phi\in W$,

$$|\phi|_W\equiv|(1-\sum_{n=1}^N\mathcal{E}_{z_n})\phi|_{L^\infty}+\sum_{n=1}^N(|\phi_{z_n,\mathrm{res}}|_{L^\infty}+|\phi_{z_n,r}|_{L^\infty(D_{z_n})}) \tag{4.8}$$
$$+\sum_{j=1}^M(|\phi^\flat|_{L^\infty(D_{\kappa_j})}+|\phi^\sharp|_{C^\mu_{\tilde\sigma}(D_{\kappa_j,\frac{1}{\tilde\sigma}})\cap L^\infty(D_{\kappa_j})}).$$

We now introduce the Sato eigenfunction φ and the Sato adjoint eigenfunction ψ for a $Gr(N, M)_{\geq 0}$ KP soliton:

$$\begin{aligned}\varphi(x,\lambda) =& e^{\lambda x_1+\lambda^2 x_2}\frac{\sum_{1\leq j_1<\cdots<j_N\leq M}\Delta_{j_1,\cdots,j_N}(A)(1-\frac{\kappa_{j_1}}{\lambda})\cdots(1-\frac{\kappa_{j_N}}{\lambda})E_{j_1,\cdots,j_N}(x)}{\tau(x)}\\ \equiv& e^{\lambda x_1+\lambda^2 x_2}\chi(x,\lambda),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\psi(x,\lambda) =& e^{-(\lambda x_1+\lambda^2 x_2)}\frac{\sum_{1\leq j_1<\cdots<j_N\leq M}\Delta_{j_1,\cdots,j_N}(A)\frac{E_{j_1,\cdots,j_N}(x)}{(1-\frac{\kappa_{j_1}}{\lambda})\cdots(1-\frac{\kappa_{j_N}}{\lambda})}}{\tau(x)}\\ \equiv& e^{-(\lambda x_1+\lambda^2 x_2)}\xi(x,\lambda)\end{aligned} \tag{4.10}$$

[6, (2.12)], [12, Theorem 6.3.8., (6.3.13)], [17, Proposition 2.2, (2.21)]. Here, χ and ξ are referred to as the normalized Sato eigenfunction and the normalized Sato adjoint eigenfunction, respectively. They satisfy the Lax equation and the adjoint Lax equation:

$$\begin{aligned}L\chi(x,\lambda) &\equiv \left(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u_s(x)\right)\chi(x,\lambda)=0,\\ L^\dagger\xi(x,\lambda) &\equiv \left(\partial_{x_2}+\partial_{x_1}^2-2\lambda\partial_{x_1}+u_s(x)\right)\xi(x,\lambda)=0.\end{aligned} \tag{4.11}$$

Proofs for (3.1), (3.2) and (4.9)–(4.11) will be provided in Section 4.2.1 for convenience.

Theorem 4.3. (Direct Scattering Theory [31, 32]) *Given a perturbed* $\mathrm{Gr}(N, M)_{>0}$ *KP soliton* $u_0(x_1, x_2)$ *satisfying*

$$\begin{aligned}&u_0(x_1,x_2)=u_s(x_1,x_2,0)+v_0(x_1,x_2),\\ &u_s(x)\ a\ \mathrm{Gr}(N,M)_{>0}\ \textit{KP soliton defined by}\ \kappa_1,\cdots,\kappa_M\ \textit{and}\ A\in\mathrm{Gr}(N,M)_{>0},\\ &\textstyle\sum_{|l|\leq d+8}|(1+|x_1|+|x_2|)\partial_x^l v_0|_{L^1\cap L^\infty}\ll 1,\ d\geq 0,\\ &z_1=0,\ \{z_n,\kappa_j\}_{1\leq n\leq N,1\leq j\leq M}\ \textit{distinct reals},\ \det\left(\tfrac{1}{\kappa_k-z_h}\right)_{1\leq k,h\leq N}\neq 0,\end{aligned} \tag{4.12}$$

we have

(1) the unique solvability of

$$(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u_0(x_1,x_2))m_0(x_1,x_2,\lambda)=0, \tag{4.13}$$

$$\lim_{|x|\to\infty} m_0(x_1,x_2,\lambda)=\widetilde{\chi}(x_1,x_2,0,\lambda)=\frac{(\lambda-z_1)^{N-1}}{\Pi_{2\leq n\leq N}(\lambda-z_n)}\chi(x_1,x_2,0,\lambda) \tag{4.14}$$

for $\forall\lambda\in\mathbb{C}\backslash\{z_n,\kappa_j\}$.

(2) The forward scattering transform is defined as

$$\mathcal{S}(u_0,\{z_n\})=(\{z_n\},\{\kappa_j\},\mathcal{D},s_c(\lambda)) \tag{4.15}$$

satisfying

$$m_0(x_1,x_2,\lambda)=1+\sum_{n=1}^N\frac{m_{0;z_n,\mathrm{res}}(x_1,x_2)}{\lambda-z_n}+\mathcal{C}T_0m_0\ \in W_0=W_{(x_1,x_2,0)}, \tag{4.16}$$

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$$(e^{\kappa_1 x_1+\kappa_1^2 x_2} m_0(x_1,x_2,\kappa_1^+),\cdots,e^{\kappa_M x_1+\kappa_M^2 x_2} m_0(x_1,x_2,\kappa_M^+))\mathcal{D}=0, \tag{4.17}$$

where $\{z_n\}$ *and* $\{\kappa_j\}$ *are blow-up and multi-valued points of* m_0*, respectively;* $\mathcal{D}$ *are norming constants between values of* m_0 *at* $\lambda=\kappa_j^+=\kappa_j+0^+$ *and can be computed by*

$$\begin{aligned}
\mathcal{D}&=\widetilde{\mathcal{D}}\times\begin{pmatrix}\widetilde{\mathcal{D}}_{11} & \cdots & \widetilde{\mathcal{D}}_{1N}\\ \vdots & \ldots & \vdots\\ \widetilde{\mathcal{D}}_{N1} & \cdots & \widetilde{\mathcal{D}}_{NN}\end{pmatrix}^{-1} diag\,(\kappa_1^N,\cdots,\kappa_N^N), \\
\widetilde{\mathcal{D}}&=diag\,(\frac{\Pi_{2\le n\le N}(\kappa_1-z_n)}{(\kappa_1-z_1)^{N-1}},\cdots,\frac{\Pi_{2\le n\le N}(\kappa_M-z_n)}{(\kappa_M-z_1)^{N-1}})\times\mathcal{D}^\sharp,\\
\mathcal{D}^\sharp&=\left(\mathcal{D}^\sharp_{ji}\right)=\left(\mathcal{D}^\flat_{ji}+\sum_{l=j}^{M}\frac{c_{jl}\mathcal{D}^\flat_{li}}{1-c_{jj}}\right),\\
\mathcal{D}^\flat&=diag\,(\kappa_1^N,\cdots,\kappa_M^N)\,A^T,
\end{aligned}\tag{4.18}$$

with $c_{jl}=-\int\Psi_j(x_1,x_2,0)v_0(x_1,x_2)\varphi_l(x_1,x_2,0)dx_1dx_2$*,* $\Psi_j(x)$*,* $\varphi_l(x)$ *are residues of the adjoint eigenfunction at* κ_{j} *[32, (3.17))] and values of the Sato eigenfunction at* κ_1 *(4.55). Moreover,* $T_0=T|_{x_3=0}$ *and* T *is the continuous scattering operator defined by (4.7),* $s_c(\lambda)$ *is the continuous scattering data, arising from the* $\overline{\partial}$*-characterization*

$$\begin{aligned}
\partial_{\overline{\lambda}}m_0(x_1,x_2,\lambda)&=s_c(\lambda)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}m_0(x_1,x_2,\overline{\lambda}),\ \lambda\notin\mathbb{R},\\
s_c(\lambda)&=\frac{\Pi_{2\le n\le N}(\overline{\lambda}-z_n)}{(\overline{\lambda}-z_1)^{N-1}}\frac{\mathrm{sgn}(\lambda_I)}{2\pi i}\iint e^{-[(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2]}\\
&\quad\times\xi(x_1,x_2,0,\overline{\lambda})v_0(x_1,x_2)m_0(x_1,x_2,\lambda)dx_1dx_2.
\end{aligned}\tag{4.19}$$

The scattering data is a d*-admissible scattering data. Namely, it satisfies the algebraic and analytic constraints:*

$$s_c(\lambda)=\begin{cases}\dfrac{\frac{i}{2}\,\mathrm{sgn}(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|}+\mathrm{sgn}(\lambda_I)h_j(\lambda), & \lambda\in D^\times_{\kappa_j},\\ \mathrm{sgn}(\lambda_I)\hbar_n(\lambda), & \lambda\in D^\times_{z_n},\end{cases}, \tag{4.20}$$

$$\mathcal{D}=\begin{pmatrix}\kappa_1^N & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \kappa_N^N\\ \mathcal{D}_{N+11} & \cdots & \mathcal{D}_{N+1N}\\ \vdots & \ddots & \vdots\\ \mathcal{D}_{M1} & \cdots & \mathcal{D}_{MN}\end{pmatrix}, \tag{4.21}$$

and

$$
\begin{aligned}
&\left|(1-\sum_{j=1}^{M}\mathcal{E}_{\kappa_j})\sum_{|l|\le d+8}\left(|\overline{\lambda}-\lambda|^{l_1}+|\overline{\lambda}^2-\lambda^2|^{l_2}\right)s_c(\lambda)\right|_{L^\infty}\\
&\quad+\sum_{j=1}^{M}\left(|\gamma_j|+|h_j|_{L^\infty(D_{\kappa_j})}\right)+\sum_{n=1}^{N}|\hbar_n|_{C^1(D_{z_n})}\\
&\quad+\left|\mathrm{diag}(q_1,\cdots,q_M)^{-1}\times\mathcal{D}\times\mathrm{diag}(q_1,\cdots,q_N)A_N^T-\mathcal{D}^\flat\right|_{L^\infty}\\
&\le C\sum_{|l|\le d+8}\left|(1+|x_1|+|x_2|)\partial_x^l v_0\right|_{L^1\cap L^\infty},
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
&s_c(\lambda)=\overline{s_c(\overline{\lambda})},h_j(\lambda)=-\overline{h_j(\overline{\lambda})},\hbar_n(\lambda)=-\overline{\hbar_n(\overline{\lambda})},\\
&A_N=(a_{kl})_{1\le k,l\le N},\quad q_j=\frac{\Pi_{2\le n\le N}(\kappa_j-z_n)}{(\kappa_j-z_1)^{N-1}}\text{for } 1\le j\le M.
\end{aligned}
\tag{4.23}
$$

Theorem 4.4. (Linearization Theory) [32, Theorem 5] *If* $\Phi=e^{\lambda x_1+\lambda^2x_2}m(x,\lambda)$ *satisfies the Lax pair (1.2) and*

$$
\partial_{\overline{\lambda}}m(x,\lambda)=s_c(\lambda,x_3)e^{(\overline{\lambda}-\lambda)x_1+(\overline{\lambda}^2-\lambda^2)x_2}m(x,\overline{\lambda}),
\tag{4.24}
$$

$$
(e^{\kappa_1x_1+\kappa_1^2x_2}m(x,\kappa_1^+),\cdots,e^{\kappa_Mx_1+\kappa_M^2x_2}m(x,\kappa_M^+))\mathcal{D}(x_3)=0,
\tag{4.25}
$$

with $\mathcal{D}(x_3)$ *being in the form of (4.2), then*

$$
s_c(\lambda,x_3)=e^{(\overline{\lambda}^3-\lambda^3)x_3}s_c(\lambda),\quad \mathcal{D}_{mn}(x_3)=e^{(\kappa_m^3-\kappa_n^3)x_3}\mathcal{D}_{mn}.
\tag{4.26}
$$

Theorem 4.5. (Inverse Scattering Theory) [33] *Given a* d*-admissible scattering data* $\mathcal{S}=(\{z_n\},\{\kappa_j\},\mathcal{D},s_c(\lambda))$, $\epsilon_0\ll 1$,

(1) there exists uniquely an eigenfunction $m\in W$ *for the system of the CIE and the* $\mathcal{D}$*-symmetry,*

$$
m(x,\lambda)=1+\sum_{n=1}^{N}\frac{m_{z_n,\mathrm{res}}(x)}{\lambda-z_n}+\mathcal{C}Tm,\ \lambda\ne z_n,
\tag{4.27}
$$

$$
(e^{\kappa_1x_1+\kappa_1^2x_2+\kappa_1^3x_3}m(x,\kappa_1^+),\cdots,e^{\kappa_Mx_1+\kappa_M^2x_2+\kappa_M^3x_3}m(x,\kappa_M^+))\mathcal{D}=0,
\tag{4.28}
$$

satisfying

$$
\sum_{0\le l_1+2l_2+3l_3\le d+5}|\partial_x^l\left[m(x,\lambda)-\widetilde{\chi}(x,\lambda)\right]|_W\le C\epsilon_0.
\tag{4.29}
$$

(2) Moreover,

$$
\left(-\partial_{x_2}+\partial_{x_1}^2+2\lambda\partial_{x_1}+u(x)\right)m(x,\lambda)=0,
\tag{4.30}
$$

$$
u(x)\equiv-2\partial_{x_1}\sum_{n=1}^{N}m_{z_n,\mathrm{res}}(x)-\frac{1}{\pi i}\partial_{x_1}\iint Tm\,d\overline{\zeta}\wedge d\zeta,
\tag{4.31}
$$

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$$\sum_{0\le l_1+2l_2+3l_3\le d+4} |\partial_x^l \left[u(x)-u_s(x)\right]|_{L^\infty} \le C\epsilon_0, \tag{4.32}$$

The inverse scattering transform is defined by

$$\mathcal{S}^{-1}(\{z_n,\kappa_j,\mathcal{D},s_c(\lambda)\}) = -\frac{1}{\pi i}\partial_{x_1}\iint Tm\, d\overline{\zeta}\wedge d\zeta - 2\partial_{x_1}\sum_{n=1}^{N} m_{z_n,\mathrm{res}}(x); \tag{4.33}$$

(3) $u:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ *solves the KPII equation.*

It can be seen that when discrete scattering data or continuous scattering data vanish, the forward and inverse scattering transform constructed for perturbed line solitons degenerate into those transforms for rapidly decaying potentials or for N-line solitons.

Using the direct and inverse scattering theories (Theorems 4.3 and 4.5), we solve the Cauchy problem for the KPII:

Corollary 4.6. (The Cauchy Problem) *Given the initial data:*

$$u_0(x_1,x_2) = u_s(x_1,x_2,0) + v_0(x_1,x_2), \tag{4.34}$$

where $u_s(x)$ *is a* $\mathrm{Gr}(N,M)_{>0}$ *KP soliton, and*

$$\textstyle\sum_{|l|\le d+8} |(1+|x_1|+|x_2|)\partial_x^l v_0|_{L^1\cap L^\infty} \ll 1,\ d\ge 0, \tag{4.35}$$

the following results hold:

(1) ***Forward Scattering Transform***
We can construct the forward scattering transform as:

$$\mathcal{S}(u_0,\{z_n\}) = (\{z_n\},\{\kappa_j\},\mathcal{D},s_c(\lambda)) \tag{4.36}$$

where $z_n\in\mathbb{R}$, $\mathcal{D}$ *is an* $\mathrm{M}\times\mathrm{N}$ *matrix, and* $\mathrm{s_c}$ *is a function. The scattering data* S *is* d*-admissible corresponding to* $A\in\mathrm{Gr}(N,M)_{>0}$*. Specifically:*

$$s_c(\lambda) = \begin{cases} \frac{\frac{i}{2}sgn(\lambda_I)}{\overline{\lambda}-\kappa_j}\frac{\gamma_j}{1-\gamma_j|\alpha|} + sgn(\lambda_I)h_j(\lambda), & \lambda\in D^\times_{\kappa_j},\\ sgn(\lambda_I)\hbar_n(\lambda), & \lambda\in D^\times_{z_n}, \end{cases}$$

and the following conditions hold:

$$\begin{aligned} \epsilon_0 \equiv\ & |(1-\textstyle\sum_{j=1}^M \mathcal{E}_{\kappa_j})\sum_{|l|\le d+8}|\left(|\overline{\lambda}-\lambda|^{l_1}+|\overline{\lambda}^2-\lambda^2|^{l_2}\right)s_c(\lambda)|_{L^\infty}\\ & +\textstyle\sum_{j=1}^M(|\gamma_j|+|h_j|_{L^\infty(D_{\kappa_j})})+\sum_{n=1}^N|\hbar_n|_{C^1(D_{z_n})}\\ & +|diag\,(q_1,\ldots,q_M)^{-1}\times\mathcal{D}\times diag\,(q_1,\ldots,q_N)A_N^T-\mathcal{D}^\flat|_{L^\infty}\\ \le\ & C\textstyle\sum_{|l|\le d+8}|(1+|x_1|+|x_2|)\partial_x^l v_0|_{L^1\cap L^\infty}, \end{aligned} \tag{4.37}$$

with $\mathcal{D}^\flat = diag\,(\kappa_1^N,\ldots,\kappa_M^N)\,A^T$, $A_N=(a_{kl})_{1\le k,l\le N}$, $q_j=\frac{\Pi_{2\le n\le N}(\kappa_j-z_n)}{(\kappa_j-z_1)^{N-1}}$.

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(2) ***Solution of the Cauchy Problem for the KPII Equation***
The solution to the Cauchy problem for the KPII equation is given by:

$$u(x) = -2\partial_{x_1} \sum_{n=1}^{N} m_{z_n,res}(x) - \frac{1}{\pi i}\partial_{x_1} \iint Tm\, d\bar{\zeta} \wedge d\zeta, \tag{4.38}$$

and

$$\sum_{0\le l_1+2l_2+3l_3\le d+4} |\partial_x^l [u(x) - u_s(x)] |_{L^\infty} \le C\epsilon_0. \tag{4.39}$$

Here, $m(x,\lambda)$ satisfies the system of the Cauchy integral equation and the $\mathcal{D}$-symmetry:

$$\begin{gathered} m(x,\lambda) = 1 + \sum_{n=1}^{N} \frac{m_{z_n,res}(x)}{\lambda - z_n} + \mathcal{C}Tm, \\ (e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3} m(x,\kappa_1^+), \cdots, e^{\kappa_M x_1+\kappa_M^2 x_2+\kappa_M^3 x_3} m(x,\kappa_M^+))\mathcal{D} = 0 \end{gathered} \tag{4.40}$$

where $\kappa_j^+ = \kappa_j + 0^+$, $\mathcal{C}$ is the Cauchy integral operator, and T *is the continuous scattering operator defined by (4.7).*

4.2. Comments on distinct features

4.2.1. The Lax–Sato formulation of the KP equation

Our approach to establish an IST of perturbed $\mathrm{Gr}(N,M)_{>0}$ KP solitons is based on (3.1), (3.2), (4.9)–(4.11). To verify these formulas, we summarize the Lax–Sato formulation of the KP equation [17, § 2.1–2.4] in this subsection.

- ►**(The KP hierarchy, the KP equation, and the Lax pair):** Suppose the operator $\mathfrak{L}$ can be gauge transformed into the trivial operator $\partial = \partial_{x_1}$, i.e.

$$\partial = \mathcal{W}^{-1}\mathfrak{L}\mathcal{W}, \tag{4.41}$$

where

$$\begin{aligned} \mathcal{W} =& 1 - w_1\partial^{-1} - w_2\partial^{-2} - w_3\partial^{-3} + \cdots, \\ w_j =& w_j(x_1,x_2,x_3,\ldots), \quad x = (x_1,x_2,x_3,\ldots). \end{aligned} \tag{4.42}$$

If the Sato equation

$$\partial_{x_n}\mathcal{W} = B_n\mathcal{W} - \mathcal{W}\partial^n \quad \textit{for}\ n = 1,2,\ldots, \tag{4.43}$$

holds where $B_n = \left(\mathcal{W}\partial^n\mathcal{W}^{-1}\right)_{\ge 0}$, the polynomial part of $\mathfrak{L}^n$ in ∂, then

$$\partial_{x_n}\mathfrak{L} = [B_n, \mathfrak{L}], \tag{4.44}$$

$$\partial_{x_m}B_n - \partial_{x_n}B_m + [B_n, B_m] = 0, \tag{4.45}$$

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and the gauge transform (4.41) transforms the linear system of the vacuum wave function

$$\begin{cases} \partial\phi_0 = \lambda\phi_0, \\ \partial_{x_n}\phi_0 = \partial^n\phi_0 = k^n\phi_0, \quad n = 1, 2, \cdots, \end{cases} \qquad \phi_0(x,\lambda) = \exp(\sum_{n=1}^{\infty}\lambda^n x_n), \tag{4.46}$$

to the KP linear system

$$\begin{cases} \mathfrak{L}\phi = \lambda\phi, \\ \partial_{x_n}\phi = B_n\phi, \quad n = 1, 2, \cdots, \end{cases} \qquad \phi(x,\lambda) = \mathcal{W}\phi_0. \tag{4.47}$$

Note that given a pair (n,m) with $n > m$, from (4.45) (called the Zakharov–Shabat equations), we obtain a system of $n-1$ equations for $u_2, u_3, \ldots, u_n$,

$$\begin{aligned} \mathfrak{L} = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots, \\ u_2 = w_{1,x_1},\ u_3 = w_{2,x_1} + w_1 w_{1,x_1}, \ldots, \end{aligned} \tag{4.48}$$

in the variables x_1, x_m, x_n. For $(n,m) = (3,2)$, the Zakharov–Shabat equations yield the Kadomtsev–Petviashvili equation (1.1) for $u = 2u_2 = 2w_{1,x_1}$, and, from the second equation of (4.47), we derive the Lax pair (1.2).

- ►**(The tau function and wave eigenfunctions):** To derive the τ-function rep of solutions and wave eigenfunctions, let

$$\begin{aligned} \mathcal{W} =& 1 - w_1\partial^{-1} - w_2\partial^{-2} - \cdots - w_N\partial^{-N}, \\ \mathcal{W}_N \equiv& \mathcal{W}\partial^N = \partial^N - w_1\partial^{N-1} - w_2\partial^{N-2} - \cdots - w_N. \end{aligned}$$

Hence the Sato equation (4.43) turns into

$$\partial_{x_n}\mathcal{W}_N = B_n\mathcal{W}_N - \mathcal{W}_N\partial^n \quad \mathit{for}\, n = 1, 2, \ldots \tag{4.49}$$

which yield

$$\partial_{x_n}(\mathcal{W}_N f) = B_n(\mathcal{W}_N f) + \mathcal{W}_N(\partial_{x_n} f - \partial_{x_1}^n f).$$

We conclude that if the Sato equation holds for $\mathcal{W}_N$, then any solution of the N-th-order ODE $\mathcal{W}_N f = 0$ also satisfies the linear heat hierarchy, i.e., $\partial_{x_n} f = \partial_{x_1}^n f$ for $n = 1, 2, \ldots$.

Conversely, if f_j for $j = 1$ up to N satisfy the N-th order ODE and the heat hierarchy, then the Sato equation (4.49) holds. Hence, an explicit KP solution can be found via the τ-function

$$u(x) = 2w_{1,x_1} = 2\partial_{x_1}^2 \ln\tau(x) \equiv 2\partial_{x_1}^2 \ln \mathrm{Wr}(f_1, \cdots, f_N) \tag{4.50}$$

by writing $\mathcal{W}_N f_j = 0$ as

$$\begin{bmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N-1)} \end{bmatrix} \begin{bmatrix} w_N \\ \vdots \\ w_1 \end{bmatrix} = \begin{bmatrix} f_1^{(N)} \\ \vdots \\ f_N^{(N)} \end{bmatrix}.$$

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For the wave function $\phi = \mathcal{W}_N\phi_0$ of the KP linear system (4.47), we begin by expressing it in the determinant form [17, Proposition 2.2]:

$$\phi = W_N\phi_0 = (1 - \frac{w_1}{\lambda} - \frac{w_2}{\lambda^2} - \cdots - \frac{w_N}{\lambda^N})\phi_0 = \frac{1}{\tau}\begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N)} \\ \lambda^{-N} & \lambda^{-N+1} & \cdots & 1 \end{vmatrix}\phi_0.$$

Using elementary column operations, the determinant of the above expression can be rewritten as

$$\frac{(-1)^N}{\lambda^N}\left|\left(f_i^{(j)} - \lambda f_i^{(j-1)}\right)_{1\le i,j\le N}\right|. \tag{4.51}$$

Besides, express $f_i(x)$ in the integral form $f_i(x) = \int_C e^{\sum \zeta^n x_n}\rho_i(\zeta)d\zeta$, which is satisfied by the $\mathrm{Gr}(N,M)_{>0}$ KP solitons. Consequently, the numerator takes on this specific form:

$$\begin{aligned} & f_i^{(j)}(x) - \lambda f_i^{(j-1)}(x) \\ =& -\lambda\int_C \zeta^{j-1}(1 - \frac{\zeta}{\lambda})e^{\sum \zeta^n x_n}\rho_i(\zeta)d\zeta \\ =& -\lambda\int_C \zeta^{j-1}e^{-\sum\frac{\zeta^n}{n\lambda^n}}e^{\sum \zeta^n x_n}\rho_i(\zeta)d\zeta \\ =& -\lambda f_i^{(j-1)}(x_1 - \frac{1}{\lambda}, x_2 - \frac{1}{2\lambda^2}, x_3 - \frac{1}{3\lambda^3}, \ldots). \end{aligned}$$

As a result,

$$\phi(x,\lambda) = \frac{\tau(x_1 - \frac{1}{\lambda}, x_2 - \frac{1}{2\lambda^2}, x_3 - \frac{1}{3\lambda^3}, \ldots)}{\tau(x)}\phi_0(x). \tag{4.52}$$

Moreover, a corresponding formula for the adjoint wave function $\phi^\dagger$ can also be derived [17, § 2.4]:

$$\phi^\dagger(x,\lambda) = \frac{\tau(x_1 + \frac{1}{\lambda}, x_2 + \frac{1}{2\lambda^2}, x_3 + \frac{1}{3\lambda^3}, \ldots)}{\tau(x)}\phi_0^{-1}(x). \tag{4.53}$$

- ► **(The multi-line solitons and Sato eigenfunctions):** The multi-line solitons (3.1) is defined by setting $1 \le n \le 3$, letting

$$\begin{aligned} \begin{pmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_M \end{pmatrix}, \\ A = (a_{ij}) &\in \mathrm{Gr}(N,M)_{\ge 0}, \quad \kappa_1 < \cdots < \kappa_M, \\ E_j(x) &= \exp\theta_j(x) = \exp(\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3) \\ &= \int_C e^{\zeta x_1 + \zeta^2 x_2 + \zeta^3 x_3}\rho_j(\zeta)d\zeta, \quad \rho_j(\zeta) is the point measure at \kappa_j, \end{aligned} \tag{4.54}$$

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in (4.50); and formula (4.9) of $\varphi(x,\lambda)$ and (4.10) of $\psi(x,\lambda)$) are derived by (4.50), (4.52)–(4.54), and

$$e^{\theta_i(x_1-\frac{1}{\lambda},x_2-\frac{1}{2\lambda^2},x_3-\frac{1}{3\lambda^3},\cdots)}=e^{\left(\sum_{n=1}^{\infty}\kappa_i^n x_n\right)-\ln(1-\frac{\kappa_i}{\lambda})}=\frac{\lambda-\kappa_i}{\lambda}e^{\theta_i(x)}.$$

4.2.2. The Lax equation

The Lax equation can be proved by replacing the Sato eigenfunction and Sato adjoint eigenfunction by (4.9) and adapting the procedure (3.28)–(3.42) for the proof of perturbed 1-solitons. Major difficulties and differences occur in proving the orthogonality relation (for the construction of Green's function G) and boundedness of G_d. More precisely,

- ► **(The orthogonality relation) :** [5, 9, 10] Let

$$\begin{aligned}\varphi_j(x)=\varphi(x,\kappa_j),\quad \psi_j(x)=res_{\lambda=\kappa_j}\psi(x,\lambda),\\ \varphi(x,\kappa)=(\varphi_1(x),\ldots,\varphi_M(x)),\quad \psi(x,\kappa)=(\psi_1(x),\ldots,\psi_M(x)).\end{aligned}\tag{4.55}$$

  Define

$$\begin{aligned}\mathcal{D}^\flat&=\operatorname{diag}(\kappa_1^N,\ldots,\kappa_M^N)\,A^T,\\ \mathcal{D}^{\flat,\dagger}&=\begin{pmatrix}-d^T, & I_{M-N}\end{pmatrix}\pi\operatorname{diag}(\kappa_1^{-N},\ldots,\kappa_M^{-N}),\end{aligned}\tag{4.56}$$

  where π is an $M\times M$ permutation matrix and d is an $N\times(M-N)$ matrix satisfying

$$A=\begin{pmatrix}I_N, & d\end{pmatrix}\pi.\tag{4.57}$$

  We can implement various Plücker relations of $\varphi(x,\kappa)$ and $\psi(x,\kappa)$ to prove

$$\mathcal{D}^{\flat,\dagger}\mathcal{D}^\flat=0,\ \ \varphi(x,\kappa)\mathcal{D}^\flat=0,\ \ \mathcal{D}^{\flat,\dagger}\psi(x,\kappa)^T=0$$

  (see [32, Lemma 2.1, 2.2] for detailed proofs). Together with setting

$$P=\mathcal{D}(\mathcal{D}^T\mathcal{D})^{-1}\mathcal{D}^T,\ P'=(\mathcal{D}')^T(\mathcal{D}'\mathcal{D}'^T)^{-1}\mathcal{D}',\ P\oplus P'=I_{M\times M},$$

  we justify the orthogonality relation

$$\sum_{j=1}^{M}\varphi_j(x)\psi_j(x')=0.\tag{4.58}$$

- ► **(Boundedness of G_d) :**

  - following argument to permute and exchange cells, one obtains the decomposition [5, (3.16),(3.17)], [4],

$$G_d(x,x',\lambda)=G_d^1(x,x',\lambda)+G_d^2(x,x',\lambda),\tag{4.59}$$

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with

$$\begin{aligned}
&G_d^1(x,x',\lambda) \\
&= -\frac{\theta(x_2-x_2')}{[(N-1)!]^2(N+1)} \sum_{\{m_i\},\{n_i\}} \operatorname{sgn}(z_{m_N n_N} - z'_{m_N n_N}) \\
&\times e^{-k_{m_N n_N}(x_2-x_2')} V(\{m_i\},n_N) V(n_1,\dots,n_{N-1}) \\
&\times \theta((\lambda_R-\kappa_{m_N})(z_{m_N n_N}-z'_{m_N n_N})) e^{-(\lambda_R-\kappa_{m_N})(z_{m_N n_N}-z'_{m_N n_N})} \\
&\times \frac{\mathcal{D}^\flat(\{m_i\}) \exp(\sum_{l=1}^{N-1} E_{m_l}(x) + E_{n_N}(x))}{\tau(x)} \\
&\times \frac{\mathcal{D}^\flat(\{n_i\}) \exp(\sum_{l=1}^{N-1} E_{n_l}(x') + E_{m_N}(x'))}{\tau(x')},
\end{aligned} \tag{4.60}$$

and

$$\begin{aligned}
&G_d^2(x,x',\lambda) \\
&= +\frac{\theta(x_2-x_2')}{[(N-1)!]^2(N+1)} \sum_{\{m_i\},\{n_i\}} \operatorname{sgn}(z_{m_N n_N} - z'_{m_N n_N}) \\
&\times e^{-k_{m_N n_N}(x_2-x_2')} V(\{m_i\},n_N) V(n_1,\dots,n_{N-1}) \\
&\times \theta((\lambda_R-\kappa_{n_N})(z_{m_N n_N}-z'_{m_N n_N})) e^{-(\lambda_R-\kappa_{n_N})(z_{m_N n_N}-z'_{m_N n_N})} \\
&\times \frac{\mathcal{D}^\flat(\{m_i\}) \exp(\sum_{l=1}^{N} E_{m_l}(x))}{\tau(x)} \frac{\mathcal{D}^\flat(\{n_i\}) \exp(\sum_{l=1}^{N} E_{n_l}(x'))}{\tau(x')},
\end{aligned} \tag{4.61}$$

where $z_{mn} = x_1 + (\kappa_m + \kappa_n)x_2$, $z'_{mn} = x'_1 + (\kappa_m + \kappa_n)x'_2$, $k_{mn} = \lambda_I^2 - (\lambda_R - \kappa_m)(\lambda_R - \kappa_n)$, and

$$V(\{n_i\}) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \kappa_{n_1} & \kappa_{n_2} & \cdots & \kappa_{n_N} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{n_1}^{N-1} & \kappa_{n_2}^{N-1} & \cdots & \kappa_{n_N}^{N-1} \end{pmatrix},$$

$$\mathcal{D}^\flat(\{n_i\}) = \det\begin{pmatrix} \mathcal{D}^\flat_{n_1,1} & \cdots & \mathcal{D}^\flat_{n_1,N} \\ \vdots & \ddots & \vdots \\ \mathcal{D}^\flat_{n_N,1} & \cdots & \mathcal{D}^\flat_{n_N,N} \end{pmatrix},$$

for $m, n \in \{1,\dots,M\}$ and $\{m_i\} = \{m_1,\dots,m_N\}, \{n_1,\dots,n_N\}$ denote unordered set of N indices from $\{1,\dots,M\}$.

Therefore, G_d^2 is uniformly bounded because, after permutation, the cells E_J that appear in the numerators are also present in the denominator of the tau function. Conversely, G_d^1 is uniformly bounded in x due to the Vandermonde matrix $V(\{m_i\}, n_N)$ and the TP condition, though it is not necessarily uniformly bounded in x'. Since m_N may not be distinct from $n_1,\dots,n_{N-1}$. Consequently,

$$|G_d(x,x',\lambda)| < C(C_{x'} + 1), \tag{4.62}$$

and, combining estimates for the continuous Green function G_c,

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$$|G(x,x',\lambda)| < C(C_{x'} + \frac{1}{\sqrt{|x_2 - x_2'|}}). \tag{4.63}$$

To enhance the above estimate, we will utilize the duality between $\mathrm{Gr}(N,M)_{>0}$ and $\mathrm{Gr}(M-N,M)_{>0}$ [18, Section 4.4]:

$$\begin{aligned}&\textit{If } u_s(x) \textit{ is a } \mathrm{Gr}(N,M)_{>0} \textit{ KP soliton,}\\ &\textit{then } u_s(-x) \textit{ is a } \mathrm{Gr}(M-N,M)_{>0} \textit{ KP soliton.}\end{aligned} \tag{4.64}$$

Consider the Lax operators

$$\mathcal{L}_\pm = -\,\partial_{x_2} + \partial_{x_1}^2 + u_s(\pm x_1, \pm x_2, 0), \tag{4.65}$$

$$\mathcal{L}_1 = +\,\partial_{x_2} + \partial_{x_1}^2 + u_s(+x_1, +x_2, 0), \tag{4.66}$$

and the associated Green functions by $\mathcal{G}_\pm(x,x',\lambda)$, $\mathcal{G}_1(x,x',\lambda)$ respectively, namely, $\mathcal{L}_\pm\mathcal{G}_\pm(x,x',\lambda) = \mathcal{L}_1\mathcal{G}_1(x,x',\lambda) = \delta(x-x')$, and define

$$\mathcal{G}_\pm(x,x',\lambda) = e^{\lambda(x_1-x_1')+\lambda^2(x_2-x_2')}G_\pm(x,x',\lambda), \tag{4.67}$$

$$\mathcal{G}_1(x,x',\lambda) = e^{\lambda(x_1'-x_1)+\lambda^2(x_2'-x_2)}G_1(x,x',\lambda). \tag{4.68}$$

Applying the duality theorem (4.64), (4.64), and (4.63), one has

$$|G_\pm(x,x',\lambda)| < C(C_{x'} + \frac{1}{\sqrt{|x_2 - x_2'|}}). \tag{4.69}$$

Thanks to $\mathcal{L}_1 = \mathcal{L}_+^d$,

$$G_1(x,x',\lambda) = G_+(x',x,\lambda) \tag{4.70}$$

[4, (1.6)]. From $\mathcal{L}_1(x_1,x_2) = \mathcal{L}_-(-x_1,-x_2)$,

$$G_1(x,x',\lambda) = G_-(-x,-x',\lambda). \tag{4.71}$$

Combining (4.69)–(4.71), we conclude

$$|G_\pm(x,x',\lambda)| < C(1 + \frac{1}{\sqrt{|x_2 - x_2'|}}). \tag{4.72}$$

4.2.3. The inverse problem

To address the inverse problem, we will examine the limit of the iteration sequence within the eigenfunction space W:

$$\phi^{(k)}(x,\lambda) = 1 + \sum_{n=1}^{N} \frac{\phi^{(k)}_{z_n,\mathrm{res}}(x)}{\lambda - z_n} + \mathcal{C}T\phi^{(k-1)}(x,\lambda), \quad k > 0, \tag{4.73}$$

$$(e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3}\phi^{(k)}(x,\kappa_1^+), \dots, e^{\kappa_M x_1+\kappa_M^2 x_2+\kappa_M^3 x_3}\phi^{(k)}(x,\kappa_M^+))\mathcal{D} = 0, \tag{4.74}$$

$$\phi^{(0)}(x,\lambda) = \widetilde{\chi}(x,\lambda). \tag{4.75}$$

The previous arguments can be adapted with some modifications. However, we must clarify how to implement the $\mathcal{D}$-symmetry to reduce estimates of the residues to those of the Cauchy integral operator at κ_j^+ throughout the iteration process. The lemma is stated as follows.

Proposition 4.7. *Suppose $\mathcal{S} = (\{z_n\}, \{\kappa_j\}, \mathcal{D}, s_c)$ is* d*-admissible and $\phi^{(k)}$, $\phi^{(k)}_{z_n,\mathrm{res}}$ satisfy (4.73), (4.74). Then for* k > 0*,*

$$\begin{pmatrix} \phi^{(k)}_{z_1,\mathrm{res}} \\ \vdots \\ \phi^{(k)}_{z_N,\mathrm{res}} \end{pmatrix} = -B^{-1}\widetilde{A}\begin{pmatrix} 1+\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1+\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)} \end{pmatrix}, \tag{4.76}$$

where

$$\widetilde{A} = \begin{pmatrix} \kappa_1^N e^{\theta_1} & \cdots & 0 & \mathcal{D}_{N+1,1}e^{\theta_{N+1}} & \cdots & \mathcal{D}_{M,1}e^{\theta_M} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_N^N e^{\theta_N} & \mathcal{D}_{N+1,N}e^{\theta_{N+1}} & \cdots & \mathcal{D}_{M,N}e^{\theta_M} \end{pmatrix}, \quad B = \widetilde{A}\begin{pmatrix} \frac{1}{\kappa_1-z_1} & \cdots & \frac{1}{\kappa_1-z_N} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \frac{1}{\kappa_M-z_1} & \cdots & \frac{1}{\kappa_M-z_N} \end{pmatrix}, \tag{4.77}$$

and $e^{\theta_j} = e^{\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3}$. Moreover, for k > 0*,*

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} \left|\partial_x^l \phi^{(k)}_{z_n,\mathrm{res}}\right|_{L^\infty} \le C(1+\epsilon_0 \sum_{0\le l_1+2l_2+3l_3\le d+5} \left|\partial_x^l \phi^{(k-1)}\right|_W), \tag{4.78}$$

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} \left|\partial_x^l \left[\phi^{(k)}_{z_n,\mathrm{res}} - \phi^{(k-1)}_{z_n,\mathrm{res}}\right]\right|_{L^\infty} \le (C\epsilon_0)^k, \tag{4.79}$$

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} \left|\partial_x^l \left[\phi^{(k)}_{z_n,\mathrm{res}} - \widetilde{\chi}_{z_n,\mathrm{res}}\right]\right|_{L^\infty} \le C\epsilon_0. \tag{4.80}$$

Proof. Write the $\mathcal{D}$-symmetry and the evaluation at κ_j^+ of $\phi^{(k)}$ as a linear system for $M+N$ variables $\{\phi^{(k)}(x,\kappa_j^+), \phi^{(k)}_{z_n,\mathrm{res}}(x)\}$,

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$$\begin{pmatrix} \kappa_1^N e^{\theta_1} & \cdots & 0 & \mathcal{D}_{N+1,1}e^{\theta_{N+1}} & \cdots & \mathcal{D}_{M,1}e^{\theta_M} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & \kappa_N^N e^{\theta_N} & \mathcal{D}_{N+1,N}e^{\theta_{N+1}} & \cdots & \mathcal{D}_{M,N}e^{\theta_M} & 0 & \cdots & 0 \\ -1 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{\kappa_1-z_1} & \cdots & \frac{1}{\kappa_1-z_N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -1 & \frac{1}{\kappa_M-z_1} & \cdots & \frac{1}{\kappa_M-z_N} \end{pmatrix} \begin{pmatrix} \phi^{(k)}(x,\kappa_1^+) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \phi^{(k)}(x,\kappa_M^+) \\ \phi^{(k)}_{z_1,\mathrm{res}}(x) \\ \vdots \\ \phi^{(k)}_{z_N,\mathrm{res}}(x) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1-\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ -1-\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)} \end{pmatrix}. \tag{4.81}$$

Solving $\phi^{(k)}(x,\kappa_j^+)$ in terms of $\phi^{(k)}_{z_n,\mathrm{res}}(x)$ and plugging the outcomes into (4.81) yields

$$B\begin{pmatrix} \phi^{(k)}_{z_1,\mathrm{res}}(x) \\ \vdots \\ \phi^{(k)}_{z_N,\mathrm{res}}(x) \end{pmatrix} = -\widetilde{A}\begin{pmatrix} 1+\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1+\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)} \end{pmatrix}, \tag{4.82}$$

with B and $\widetilde{A}$ defined by (4.77). By the d-admissible condition, the system (4.81) is just determined and is equivalent to (4.76).

Next, the d-admissible condition implies that defining $\mathfrak{A}$ through:

$$\mathfrak{A}^T = diag\,(\kappa_1^N,\ldots,\kappa_M^N)^{-1}\times\mathfrak{D}^\sharp,$$

then $\mathfrak{A}\in \mathrm{Gr}(N,M)_{>0}$. Let $\widetilde{\chi}'(x,\lambda)$ be the normalized Sato eigenfunction with data $(\{z_n\},\{\kappa_j\},\mathfrak{A},0)$, we have:

$$\widetilde{\chi}'(x,\lambda) = 1+\sum_{n=1}^{N}\frac{\widetilde{\chi}'_{z_n,\mathrm{res}}(x)}{\lambda-z_n}, \tag{4.83}$$

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$$(e^{\kappa_1 x_1+\kappa_1^2 x_2+\kappa_1^3 x_3}\widetilde{\chi}'(x,\kappa_1),\dots,e^{\kappa_M x_1+\kappa_M^2 x_2+\kappa_M^3 x_3}\widetilde{\chi}'(x,\kappa_M))\mathcal{D}=0, \tag{4.84}$$

and, from the d-admissible condition, $\forall k$,

$$|\widetilde{\chi}'_{z_n,\mathrm{res}}(x)-\widetilde{\chi}_{z_n,\mathrm{res}}(x)|_{C^k}\le C_k\epsilon_0. \tag{4.85}$$

Moreover, using previous argument,

$$\begin{pmatrix}\widetilde{\chi}'_{z_1,\mathrm{res}}(x)\\ \vdots\\ \widetilde{\chi}'_{z_N,\mathrm{res}}(x)\end{pmatrix}=-B^{-1}\widetilde{A}\begin{pmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ 1\end{pmatrix}, \tag{4.86}$$

with B and $\widetilde{A}$ defined by (4.77). Let $E_j=e^{\theta_j}=e^{\kappa_j x_1+\kappa_j^2 x_2+\kappa_j^3 x_3}$ and write

$$\begin{aligned}\widetilde{A}=&\mathcal{D}^T diag\,(E_1,\dots,E_M),\\ B=&\mathcal{D}^T diag\,(E_1,\dots,E_M)\begin{pmatrix}\frac{1}{\kappa_1-z_1} & \cdots & \frac{1}{\kappa_1-z_N}\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ \frac{1}{\kappa_M-z_1} & \cdots & \frac{1}{\kappa_M-z_N}\end{pmatrix}.\end{aligned} \tag{4.87}$$

From Sato's theory, (4.9), (4.3), (4.86), (4.87), elementary row and column operations and matching the coefficients of $E_1\times\cdots\times E_N$,

$$\begin{aligned}&B^{-1}=\frac{1}{\tau'(x)}\begin{pmatrix}b_{11} & \cdots & b_{1N}\\ \vdots & \ddots & \vdots\\ b_{N1} & \cdots & b_{NN}\end{pmatrix},\\ &b_{kl}=\sum_{J(kl)=(j_{(kl),1},\cdots,j_{(kl),N-1})}\Lambda_{J(kl)}E_{J(kl)}(x),\quad 1\le j_{(kl),1}<\cdots<j_{(kl),N-1}\le M,\\ &\tau'(x)\ \textit{is the tau function with data}\ \kappa_j,\ \mathfrak{A},\\ &|\Lambda_{J(kl)}|=|\Lambda_{J(kl)}(\{z_n\},\{\kappa_j\},\mathfrak{A})|<C.\end{aligned} \tag{4.88}$$

As a consequence,

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$$\tau'(x)\widetilde{\chi}'_{z_h,\mathrm{res}}(x) \tag{4.89}$$
$$=\textit{the } h^{\mathrm{th}}\textit{-row of}\begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{NN} \end{pmatrix}\begin{pmatrix} \kappa_1^N E_1 + \cdots + \mathcal{D}_{N+1,1}E_{N+1} + \cdots + \mathcal{D}_{M,1}E_M \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \kappa_N^N E_1 + \cdots + \mathcal{D}_{N+1,N}E_{N+1} + \cdots + \mathcal{D}_{M,N}E_M \end{pmatrix}$$
$$=(\kappa_1^N E_1 + \cdots + \mathcal{D}_{N+1,1}E_{N+1} + \cdots + \mathcal{D}_{M,1}E_M)\sum_{|J(h1)|=N-1}\Lambda_{J(h1)}E_{J(h1)}(x)$$
$$+\cdots+(\kappa_N^N E_1 + \cdots + \mathcal{D}_{N+1,N}E_{N+1} + \cdots + \mathcal{D}_{M,N}E_M)\sum_{|J(hN)|=N-1}\Lambda_{J(hN)}E_{J(hN)}(x)$$
$$\equiv(\widetilde{a}_{11}E_1 + \cdots + \widetilde{a}_{1M}E_M)\sum_{|J(h1)|=N-1}\Lambda_{J(h1)}E_{J(h1)}(x)$$
$$+\cdots+(\widetilde{a}_{N1}E_1 + \cdots + \widetilde{a}_{NM}E_M)\sum_{|J(hN)|=N-1}\Lambda_{J(hN)}E_{J(hN)}(x).$$

Since $E_J(hl)$ are $N-1$ cells. According to the formula of the Sato eigenfunction,

$$\begin{aligned} 0 =&\widetilde{a}_{1k}E_k\sum_{k\in J(h1),\ |J(h1)|=N-1}\Lambda_{J(h1)}E_{J(h1)}(x)+\cdots \\ &+\widetilde{a}_{Nk}E_k\sum_{k\in J(hN),|J(hN)|=N-1}\Lambda_{J(hN)}E_{J(hN)}(x). \end{aligned}\tag{4.90}$$

Using (4.76), (4.87)–(4.90), multi-linearity, and estimates of the CIO's,

$$\tau'(x)\phi^{(k)}_{z_h,\mathrm{res}}(x) = \tau'(x)\widetilde{\chi}'_{z_h,\mathrm{res}}(x)$$
$$+\textit{the h-row of}\begin{pmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{NN} \end{pmatrix}\begin{pmatrix} \widetilde{a}_{11}E_1\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} + \cdots + \widetilde{a}_{1M}E_M\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \widetilde{a}_{N1}E_1\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} + \cdots + \widetilde{a}_{NM}E_M\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)} \end{pmatrix}$$
$$=\tau'(x)\widetilde{\chi}'_{z_h,\mathrm{res}}(x)$$
$$+(\widetilde{a}_{11}E_1\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} + \cdots + \widetilde{a}_{1M}E_M\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)})\sum_{|J(h1)|=N-1}\Lambda_{J(h1)}E_{J(h1)}(x)$$
$$+\cdots+(\widetilde{a}_{N1}E_1\mathcal{C}_{\kappa_1^+}T\phi^{(k-1)} + \cdots + \widetilde{a}_{NM}E_M\mathcal{C}_{\kappa_M^+}T\phi^{(k-1)})\sum_{|J(hN)|=N-1}\Lambda_{J(hN)}E_{J(hN)}(x)$$
$$=\sum_{|J(h)|=N}\widetilde{\Lambda}_{J(h)}E_{J(h)}(x),$$

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with

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} |\partial_x^l \widetilde{\Lambda}_{J(h)}| < C(1+\sum_{j=1}^{M}\sum_{0\le l_1+2l_2+3l_3\le d+5} |\partial_x^l \mathcal{C}_{\kappa_j^+} T\phi^{(k-1)}|).$$

Along with the TP condition of $\mathfrak{A}$, yield

$$\sum_{0\le l_1+2l_2+3l_3\le d+5} |\partial_x^l \phi^{(k)}_{z_n,\mathrm{res}}(x)| \le C(1+\sum_{j=1}^{M}\sum_{0\le l_1+2l_2+3l_3\le d+5} |\partial_x^l \mathcal{C}_{\kappa_j^+} T\phi^{(k-1)}|).$$

Combining with (4.85), we prove (4.78)–(4.80). □

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