A NOTE ON THE DIAMETER OF A GRAPH

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The distance $d(x, y)$ between vertices x, y of a graph G is the length of the shortest path from x to y in G. The diameter $\delta(G)$ of G is the maximum distance between any pair of vertices in G. i.e. $\delta(G) = \max_{x \in G} \max_{y \in G} d(x, y)$. In this note we obtain an upper bound for $\delta(G)$ in terms of the numbers of vertices and edges in G. Using this bound it is then shown that for any complement-connected graph G with N vertices

$$\delta(G) + \delta(\bar{G}) \leq N + 1$$

where $\bar{G}$ is the complement of G.

THEOREM. *Let δ be the diameter of an undirected connected graph G. If G has N vertices $\{x_i\}_1^N$ and E edges then*

$$2\delta - 3 - \left(\frac{\delta^2 - \delta - 4}{N}\right) \leq \frac{N^2 - 2E}{N} .$$

Proof. Let $x_1, x_2, \ldots, x_{\delta+1}$ be a diametral path. If $m > \delta+1$, x_m can be joined to at most three vertices of this path. For otherwise, suppose x_m is joined to x_i and to x_{i+k} $(k > 2)$. Then $x_1, x_2, \ldots, x_i, x_m, x_{i+k}, \ldots, x_{\delta+1}$ is a path of length $\delta - k + 2 < \delta$, contradicting the supposition that $x_1, x_2, \ldots, x_{\delta+1}$ is a diametral path.

Hence $E \leq \delta$ (the diametral path)

$+ 3(N-\delta-1)$ (the above-mentioned connections)

$+\frac{1}{2}(N-\delta-1)(N-\delta-2)$ (x_m joined to x_n for $m, n > \delta+1$, $m \neq n$)

i.e. $$2\delta - 3 - \left(\frac{\delta^2 - \delta - 4}{N}\right) \leq \frac{N^2 - 2E}{N}$$ as stated.

Note. This upper bound is best possible in the following sense. Given N, E there exists a graph G with N vertices, E edges and diameter δ such that

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(1) $$2\delta - 3 - \left(\frac{\delta^2 - \delta - 4}{N}\right) \leq \frac{N^2 - 2E}{N} < 2(\delta+1) - 3 - \left(\frac{(\delta+1)^2 - (\delta+1) - 4}{N}\right).$$

To construct such a G, let G' be the graph on N vertices $\{x_i\}_1^N$ obtained by taking a path $x_1, x_2, \ldots x_{\delta+2}$ and joining x_m $(\forall m \geq \delta+3)$ to x_n $(\forall n \geq \delta, n \neq m)$. Then G' has diameter $\delta+1$ and E' edges, where $E' = \delta + 1 + 3(N-\delta-2) + \frac{1}{2}(N-\delta-2)(N-\delta-3)$. The inequalities (1) imply that $0 < E - E' \leq N - \delta$. Let $E - E' = k$. Then G is obtained from G' by adding the following k edges: if $k < N - \delta$ join $x_{\delta-1}$ to $x_{\delta+r}$, $r = 2, \ldots, k+1$; if $k = N - \delta$ join $x_{\delta-1}$ to $x_{\delta+r}$, $r = 2, \ldots, N - \delta$, and x_δ to $x_{\delta+2}$.

COROLLARY 1. In an undirected connected graph,

$$\delta < 1 + \frac{N^2 - 2E}{N}$$

Proof. If $\delta = 1$, $2E = N(N-1)$ and therefore $1 + \frac{N^2-2E}{N} = 2 > \delta = 1$. If $\delta = 2$, $2E < N(N-1)$ and hence $1 + \frac{N^2-2E}{N} > 2 = \delta$. When $\delta \geq 3$, $\delta^2 - \delta - 4 > 0$. Therefore, on using the trivial bound $\delta \leq N - 1$, the Theorem gives

$$\frac{N^2 - 2E}{N} > 2\delta - 3 - \left(\frac{\delta^2 - \delta - 4}{\delta + 1}\right)$$

$$= 2\delta - 3 - (\delta - 2) + \frac{2}{\delta+1} > \delta - 1.$$

COROLLARY 2. If both G and its complement $\bar{G}$ are connected graphs,

$$\delta(G) + \delta(\bar{G}) \leq N + 1.$$

Proof. From Corollary 1

$$\delta(G) < 1 + \frac{N^2 - 2E}{N}.$$

Similarly $$\delta(\bar{G}) < 1 + \frac{N^2 - 2\bar{E}}{N} \quad \text{where } E + \bar{E} = \frac{1}{2}N(N-1).$$

Therefore $$\delta(G) + \delta(\bar{G}) < N + 3$$

i.e.

(2) $$\delta(G) + \delta(\bar{G}) \leq N + 2.$$

If $\delta(G) \leq 2$, then, since $\delta(\bar{G}) \leq N - 1$, we have immediately that $\delta(G) + \delta(\bar{G}) \leq N + 1$.

If $\delta(G) = 3$, then $\delta(\bar{G}) \leq N - 2$. For there is only one graph $\bar{G}$ with $\delta(\bar{G}) = N - 1$ (the simple path) and for this graph $\delta(G) = 2$. Hence in this case $\delta(G) + \delta(\bar{G}) \leq N + 1$. By symmetry Corollary 2 also holds if $\delta(\bar{G}) \leq 3$.

We now assume that $\delta(G) > 3$, $\delta(\bar{G}) > 3$.

By the Theorem

$$2\delta - 3 \leq \frac{N^2 - 2E}{N} + \left(\frac{\delta^2 - \delta - 4}{N}\right)$$

$$2\bar{\delta} - 3 \leq \frac{N^2 - 2\bar{E}}{N} + \left(\frac{\bar{\delta}^2 - \bar{\delta} - 4}{N}\right)$$

where $\delta \equiv \delta(G)$, $\bar{\delta} \equiv \delta(\bar{G})$.

Therefore $\quad 2(\delta + \bar{\delta}) - 6 \leq N + 1 + \left(\frac{\delta^2 + \bar{\delta}^2 - \delta - \bar{\delta} - 8}{\delta + \bar{\delta} - 2}\right)$

by addition and (2).

Hence $\quad 2(\delta + \bar{\delta}) - 6 \leq N + 1 + \delta + \bar{\delta} - 5 - \frac{2(\delta - 3)(\bar{\delta} - 3)}{\delta + \bar{\delta} - 2}$.

i.e. $\quad \delta + \bar{\delta} \leq N + 2 - \frac{2(\delta - 3)(\bar{\delta} - 3)}{\delta + \bar{\delta} - 2} < N + 2$.

Therefore $\quad \delta + \bar{\delta} \leq N + 1$.

Note added in proof: The author has recently noticed that these results are essentially contained in Lemma 1.1 and Lemma 3 of [1]. In fact Lemma 3 of [1] implies that, apart from the case $\delta(G) = \delta(\bar{G}) = 3$, $\delta(G) + \delta(\bar{G}) = \max\{\delta(G) + 2, \delta(\bar{G}) + 2\}$.

REFERENCE

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