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UNIFORM APPROXIMATION BY MEROMORPHIC FUNCTIONS ON CLOSED SETS WITH CONTINUOUS EXTENSION INTO THE BOUNDARY

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1. Introduction.

Notation. For S a subset of the extended plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ let S^0 be its interior, $\bar{S}$ its closure in $\mathbf{C}^*$ and $\partial S = \bar{S} \backslash S^0$ its boundary. $H(S)$ and $M(S)$ will denote the sets of all restrictions to S of functions which are, on a neighborhood of S, holomorphic or meromorphic, respectively. $A(S)$ will stand for the set of functions from S to $\mathbf{C}^*$ which are continuous on S and whose restriction to S^0 are holomorphic. Finally, for S compact, $R(S)$ will denote the set of functions allowing a uniform approximation on S by rational functions with poles outside S.

In the sequel we shall always consider the following as given: an open proper subset G of $\mathbf{C}^*$, a (relatively) closed subset F of G and a function f mapping F into $\mathbf{C}$.

The problem. The starting point of this paper is an investigation by A. Stray [**8**] who considers, for a given subset E of $\partial F \cap \partial G$, the functions in $A(F)$ which have a continuous extension into E. Stray discusses the geometrical condition to be satisfied in order that a function of this kind can be approximated uniformly on F by functions from $H(G)$; these approximating functions can be chosen in such a way that they also admit a continuous extension into E. This result suggested the following question: suppose f is a uniform limit of functions belonging to $H(G)$ or $M(G)$. Is it possible to select the approximating function m in such a way that the *difference function $m - f$ can be extended continuously into $\bar{F}$*, including the points of $\partial F \cap \partial G = \bar{F} \backslash F$ for which f itself has no continuous extension?

Definition. The function $f : F \to \mathbf{C}^*$ is said to be *UCE-approximable* on F by functions from $M(G)$ if, given $\epsilon > 0$, there are functions m and e with the following properties:

$$m \in M(G), \quad e \in A(F) \cap C(\bar{F})$$
$$m(z) - f(z) = e(z) \quad (z \in F)$$
$$|e(z)| < \epsilon \quad (z \in \bar{F}).$$

(The letters UCE stand for "uniform" and "continuous extension").

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The main result. We shall prove in this paper that the necessary and sufficient conditions which are known to hold for the uniform approximation of a function f on F by functions from $H(G)$ or $M(G)$ (see [**1**]–[**8**]) are sufficient for the UCE-approximation.

Properties of the UCE-approximation. It is easily seen that convergence of the given function f is preserved by the approximating function m in the following sense: if $\{z_n\}$ is a sequence of points in F with limit point t (t can be in particular a point of $\bar{F}\backslash F$) for which $f(z_n)$ converges, for $n \to \infty$, to a finite or infinite limit a, then $m(z_n)$ also converges, namely to $a + e(t)$. This implies the more general result: if F_1 is a subset of F which admits t as an accumulation point then the *cluster sets* of the restrictions $f|F_1$ and $m|F_1$ relative to t are *congruent*; the first is mapped onto the second by the translation with vector $e(t)$, which is of absolute value less than ϵ. Special points of $\bar{F}\backslash F$ known from the discussion of the boundary behavior, as e.g. Weierstrass–Fatou and Meier-points, are the same for f and m.

In approximating f on F uniformly it is usual (and sufficient) to consider only domains G. If G is an open set with components $G_1, G_2, \ldots$ and if m_n is the meromorphic approximating function on G_n of $f|G_n$ then $m_1, m_2, \ldots$ can be considered as restrictions of a function m, meromorphic on G and approximating the given function f on F. In case of UCE-approximations one obtains a more general statement by taking G as an open set rather than as a domain. One has to consider the possibility of a point of $\bar{F}\backslash F$ lying on the boundary of two different components of G (even being an interior point of $\bar{F}$). The function $m - f$ can then be extended continuously.

2. Two lemmas. If g is holomorphic on the open domain $O \subset \mathbf{C}^*$ and if z_1, z_2 are two points of O, let $d(g; z_1, z_2)$ be defined as follows:

$$d(g; z_1,z_2) = \begin{cases} \dfrac{g(z_1) - g(z_2)}{z_1 - z_2} & \text{for } z_1 \neq z_2 \\ g'(z_1) \quad \text{for } z_1 = z_2 \neq \infty \\ \lim\limits_{z\to\infty} g'(z) = 0 & \text{for } z_1 = \infty \quad \text{or} \quad z_2 = \infty \end{cases}$$

LEMMA A. *If K is a compact proper subset of* $\mathbf{C}^*$ *and if $g \in M(K)$, then, given $\epsilon > 0$, there is a rational function r satisfying the following conditions: for $z, z_1, z_2 \in K$,*

$$|r(z) - g(z)| < \epsilon \quad \textit{and} \quad |d(r - g; z_1,z_2)| < \epsilon.$$

Proof. Since g can have only finitely many poles on K, there is a rational function q such that $g - q \in H(K)$. We may therefore assume $g \in H(K)$. Moreover we may suppose $\infty \notin K$. If $\infty \in K$ we use the transformation $w = 1/(z - a)$ for an $a \in \mathbf{C}\backslash K$ and the fact that if $h = r - g$ and

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$\tilde{h}(\omega) = h(z)$, then

$$|d(\tilde{h}; \omega_1, \omega_2)| = |z_1 - a||z_2 - a||d(h; z_1, z_2)| > \Delta^2|d(r - g; z_1, z_2)|, \quad \Delta > 0.$$

If $\infty \notin K$ we choose an open neighborhood U of K with rectifiable boundary Γ and such that $g \in H(\bar{U})$. Let l denote the length of Γ and δ its distance from K. Given $\epsilon > 0$, a positive number η can be chosen in such a way that $\eta < \epsilon$ and $\eta < 2\pi\epsilon\delta^2 l^{-1}$. By Runge's Theorem, there is a rational function r satisfying

$$|r(z) - f(z)| < \eta \quad \text{for } z \in K.$$

This implies

$$|d(r - f; z_1, z_2)| = 1/(2\pi)\left|\int_\Gamma \frac{r(t) - f(t)}{(t - z_1)(t - z_2)}\, dt\right| < (l\eta)/(2\pi\delta^2) < \epsilon$$

for $z_1, z_2 \in K$.

In [**7**] I stated and proved the "Fusion Lemma" (Lemma 1). Lemma B below is a stronger version; it is fundamental for the present paper.

LEMMA B. *Let K_1, K_2 and K be compact subsets of the extended plane* $\mathbf{C}^*$ *with $K_1 \cap K_2 = \emptyset$. If r_1 and r_2 are any two rational functions satisfying, for some $\epsilon > 0$,*

$$(1) \qquad |r_1(z) - r_2(z)| < \epsilon \quad \text{for } z \in K,$$

then there is a positive number a not depending on r_1 and r_2 and a rational function r such that, for $j = 1, 2$:

$$(2) \qquad |r(z) - r_j(z)| < a\epsilon \quad \text{for } z \in K_j \cup K,$$

$$(2') \qquad |d(r - r_j; z_1, z_2)| < a\epsilon \quad \text{for } z_1, z_2 \in K_j.$$

Proof. We shall modify the proof of Lemma 1 of [**7**] in two points; the resulting arguments will prove Lemma B.

First modification. Choose $a > 2$ such that (4) of [**7**] is satisfied as well as

$$(4') \qquad (1/\pi)\int\int_E \left|\frac{\partial\phi(\zeta)}{\partial\bar{\zeta}}\right| \frac{d\xi d\eta}{|\zeta - z_1||\zeta - z_2|} < a - 1 \quad \text{for } z_1, z_2 \in K_1 \cup K_2.$$

This is always possible because E has positive distance from $K_1 \cup K_2$. (4′) and (6) of [**7**] imply, for functions g defined by (7) of [**7**], that

$$(8') \qquad d(g; z_1, z_2) = (1/\pi)\left|\int\int_E \frac{q_1(\zeta)}{(\zeta - z_1)(\zeta - z_2)} \frac{\partial\phi}{\partial\bar{\zeta}}\, d\xi d\eta\right| < (a - 1)\epsilon \quad \text{for } z_1, z_2 \in K_1 \cup K_2.$$

Second modification. On page 106 of [**7**] a rational function r_3 was introduced

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by means of Runge's Theorem. Using Lemma A above r_3 can be so chosen that

$$(9') \quad |r_3(z) - f(z)| < \epsilon \quad \text{for } z \in K_1 \cup K_2 \cup K \quad \text{and}$$
$$|d(r_3 - f; z_1, z_2)| < \epsilon \quad \text{for } z_1, z_2 \in K_1 \cup K_2.$$

Hence, using (8′) and (9′),

$$(10') \quad |d(r_3 - f + g; z_1, z_2)| < a\epsilon \quad \text{for } z_1, z_2 \in K_1 \cup K_2.$$

Since on K_1 we have $r - r_1 = r_3 - q = r_3 - f + g$ and on K_2 we have $r - r_2 = r_3 = r_3 - f + g$, considering the estimations for $r - r_1$ and $r - r_2$ in [**7**], Lemma B is proved.

3. UCE-approximations by meromorphic functions.

THEOREM I. The following statements are equivalent:

(a) *f can be uniformly approximated on F by functions from $M(G)$ having no poles on F.*

(b) *If K is a compact subset of F then $f|K \in R(K)$.*

(c) *f is UCE-approximable on F by functions from $M(G)$ having no poles on F.*

Proof. (a) $\Rightarrow$ (b) is rather obvious. If m is a function without poles on F and meromorphic on G, then $m|K$ is holomorphic. According to Runge's Theorem $m|K$ and hence $f|K$ can be uniformly approximated by rational functions without poles on F.

(c) $\Rightarrow$ (a) is trivial. It remains to consider (b) $\Rightarrow$ (c). In [**7**, Theorem I] we showed that (a) is a consequence of (b) (called (*) in [**7**]). We shall modify the proof given there in such a way as to derive a proof of (b) $\Rightarrow$ (c); we list only the modifications.

We may suppose F bounded. The general case can be reduced to this one by means of the transformation $w = 1/(z - z_0)$ with $z_0 \in G\backslash F$; the facts expressed by the statements (b) and (c) are invariant under this transformation. The sequence of numbers $a_n > 0$ occurring on page 106 of [**7**] is chosen now in such a way that a_n satisfies Lemma B for the sets $G_n, \mathbf{C}^*\backslash G_{n+1}$ and F_n. By considering (12) of [**7**] one can now choose r_n such that, together with (13) and (14) of [**7**]:

$$(14') \quad |d((r_n - q_{n+1}); z_1, z_2)| < \epsilon_n \quad \text{for } z_1, z_2 \in \mathbf{C}^*\backslash G_{n+1}.$$

(14) of [**7**] implies the uniform convergence, on $\mathbf{C}^*\backslash G$ and for $n \to \infty$, of the rational functions

$$h_n(z) = \sum_{1}^{n-1} (r_\nu(z) - q_{\nu+1}(z)).$$

In particular, h_n converges on the boundary Γ of G to a function continuous on Γ:

$$\varphi(t) = \lim_{n\to\infty} h_n(t), \quad t \in \Gamma.$$

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In the following estimations we assume $z \in \bar{F}_n \backslash F_{n-1}$ ($\subseteq \mathbf{C}^* \backslash G_n$) and $t \in \Gamma$. (14′) and (10) of [7] imply

$$|d(h_n; t, z)| < \sum_1^{n-1} \epsilon_\nu < \epsilon,$$

which is equivalent to

$$|h_n(t) - h_n(z)| < |t - z|\epsilon.$$

On the other hand, by (14) of [7]:

$$|h_n(t) - \varphi(t)| = \left| \sum_n^\infty (r_\nu(t) - q_{\nu+1}(t) \right| < \sum_n^\infty \epsilon_\nu,$$

hence

$$|h_n(z) - \varphi(t)| < |t - z|\epsilon + \sum_n^\infty \epsilon_\nu.$$

(11) and (13) of [7], together with

$$m(z) = R_n(z) + q_n(z) + \sum_n^\infty (r_\nu(z) - q_\nu(z))$$

imply

$$|m(z) - f(z) - \varphi(t)| \leqq |h_n(z) - \varphi(t)| + |q_n(z) - f(z)| + \sum_n^\infty |r_\nu(z) - q_\nu(z)| < |t - z|\epsilon + \epsilon_n + 2 \sum_n^\infty \epsilon_\nu.$$

Note that m is independent of n.

Since F has been assumed bounded and since $\lim_{n\to\infty} \sum_n^\infty \epsilon_\nu = 0$ this has the following consequence: the difference of $m(z) - f(z)$ and $\varphi(t)$ converges to 0 if z converges on F to a point t of $\bar{F} \backslash F$. Setting $e(z) = m(z) - f(z)$ for $z \in F$ and $e(z) = \varphi(z)$ for $z \in \bar{F} \backslash F$ we have $e(z) \in A(F) \cap C(\bar{F})$. As in the proof of Theorem I of [7], $|e(z)| < \epsilon$ for $z \in F$. But $e(z) \in C(\bar{F})$, so we even have $|e(z)| \leqq \epsilon$ for $z \in \bar{F}$. This proves Theorem I.

Remarks. Since h_n converges uniformly on $\mathbf{C}^* \backslash G$, for $n \to \infty$, it is obvious that $e(z)$ is even in $A((\mathbf{C}^* \backslash G) \cup F)$ and $|e(z)| < \epsilon$ if $z \in ((\mathbf{C}^* \backslash G) \cup \bar{F})$.

An immediate consequence of Theorem I is

THEOREM II. *If*

(**) $\quad R(F \cap \bar{G}_1) = A(F \cap \bar{G}_1)$

holds for every subdomain G_1 of G with $\bar{G}_1 \subset G$, then every function from $A(F)$ can be UCE-approximated by a function from $M(G)$.

Remark. Condition (**) has been shown to be necessary for the weaker statement of Theorem II of [7] (see also [4]). Hence it is also necessary for Theorem II above to hold.

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There is a slightly stronger version of Theorem I as well as of Theorem II. In (a) omit the condition "having no poles on F". Lemma B holds even in the case of r_1 and r_2 admitting poles on $K_1 \cup K_2 \cup K$; consequently the rational functions used to construct m in the proof of Theorem I may have poles on F. Condition (b) can then be weakened to: if K is a compact subset of F, $f|K$ is approximable by rational functions with or without poles on K.

4. UCE-approximation by holomorphic functions. Let G^* be the 1-point-compactification of G. Arakeljan [**1**] proved: a function $f \in A(F)$ is uniformly approximable on F by functions from $H(G)$ if and only if

(α) $G^* \backslash F$ is connected, and

(β) $G^* \backslash F$ is locally connected.

THEOREM III. *If conditions* (α) *and* (β) *are satisfied then every function from* $A(F)$ *can be UCE-approximated on* F *by functions from* $H(G)$.

Remark. Condition (α) is sufficient for (**) of Theorem II to hold. In fact, (α) implies that no set $F \cap \bar{G}_1$ considered in Theorem II will dissect the plane which in turn implies, by results of Mergelyan, the possibility of uniformly approximating $f|F \cap \bar{G}_1$ by polynomials. Hence (α) is a sufficient condition for f to be UCE-approximable by functions from $M(G)$. On the other hand, (α) is obviously not necessary for this. If e.g. F is a circle, (**) holds, but (α) does not

To prove Theorem III, the proof of Theorem I could be modified in the following way: choose the functions r_n successively in such a way that $r_n - q_n$ have no poles on G_n. We shall, however, use another approach by first proving Lemma C below. This lemma might be useful for other purposes as well. Theorem III is an immediate consequence of Theorem II and Lemma C.

LEMMA C. (Pole pushing lemma). *If conditions* (α) *and* (β) *are satisfied and if* m *is a function in* $M(G)$ *without poles on* F, *the restriction* $m|F$ *is UCE-approximable on* F *by functions in* $H(G)$. *In fact we can choose the approximating function so that* $e \in A(\bar{F})$.

Remark. Lemma C will be proved by a simple modification of the proof used in [**5**] (for $G = \mathbf{C}$).

Proof. By (α) and (β), G can be exhausted by domains $G_1, G_2, \ldots$ with the following properties: no component of $G \backslash (F \cup \bar{G}_n)$ is compact, i.e. every component extends to the boundary of G. If z_1, z_2 belong to the same component of $G \backslash (F \cup \bar{G}_n)$ and if $p_1(t)$ is a polynomial without constant term then, according to Runge, there is another polynomial $p_2(t)$ without constant term such that

$$|p_1(1/(z - z_1)) - p_2(1/(z - z_2))| < \epsilon \quad \text{for } z \in F \cup \bar{G}_n.$$

If therefore $\mu_1 \in M(G)$ has a pole in z_1 with $p_1(1/(z - z_1))$ as its principal part and if one defines

$$\mu_2(z) = \mu_1(z) - p_1(1/(z - z_1)) + p_2(1/(z - z_2)),$$

then $\mu_2 \in M(G)$ and μ_2 has a pole in z_2 instead of z_1. The other poles of μ_2, different from z_1, are just the remaining poles of μ_1. Furthermore, $\mu_1 - \mu_2$ is rational with exactly two poles, viz. z_1 and z_2.

Now select a sequence of positive numbers ϵ_n with $\sum_1^\infty \epsilon_n < \epsilon$. Start with $m_0 = m$, $G_0 = \emptyset$. Determine successively functions $m_1, m_2, \ldots$ from $M(G)$ with the following properties: the poles of m_n are outside of $F \cup \bar{G}_n$; $m_n - m_{n-1}$ is rational;

(1) $\quad |m_n(z) - m_{n-1}(z)| < \epsilon_n \quad$ for $z \in \bar{F} \cup \bar{G}_n$.

This determination is possible as follows from the preceding discussion. The sequence m_n converges uniformly on each domain $\bar{G}_n$. The limit function

$$g = m_n + \sum_{n+1}^{\infty} (m_\nu - m_{\nu-1})$$

is holomorphic on G_n since this is true for $m_n, m_{n+1}, \ldots$ Hence g is holomorphic on $G = \bigcup_1^\infty G_n$. The difference

$$e = g - m = \sum_{1}^{\infty} (m_\nu - m_{\nu-1})$$

is a function meromorphic on G with no poles on F and is continuous on $\bar{F}$. (Each $m_\nu - m_{\nu-1}$ is rational, has no poles on F and the sum $\sum_1^\infty (m_\nu - m_{\nu-1})$ converges on $\bar{F}$ because of (1)). Finally,

$$|e(z)| < \sum_{1}^{\infty} \epsilon_\nu < \epsilon \quad \text{for } z \in \bar{F}.$$

Remark. Function e of Lemma C can even be constructed in such a way as to be continuous on $F \cup (\mathbf{C}^*\backslash G)$ and holomorphic in each interior point of this set as well as in each point of F. The meromorphic functions needed for the proof can indeed be determined so as to satisfy (1) on the compact set $\bar{G}_n \cup F \cup (\mathbf{C}^*\backslash G)$.

5. Simultaneous UCE-approximation of a function $f \in M(F)$ and a finite number of its derivatives by functions from $M(G)$ and their derivatives.

THEOREM IV. *Given a function $f \in M(F)$, a natural number q and $\epsilon > 0$ there is a function $m \in M(G)$ and a function $e \in A(\bar{F})$ whose derivatives e', e'', ..., $e^{(q)}$ exist on $\bar{F}\backslash\{\infty\}$ with the following properties:*

$$e(z) = m(z) - f(z) \quad \text{for } z \in F$$
$$|e(z)| < \epsilon \quad \text{for } z \in \bar{F}$$
$$|e^{(k)}(z)| < \epsilon \quad k = 1, 2, \ldots, q, \text{ for } z \in \bar{F}, z \neq \infty.$$

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If conditions (α) *and* (β) *of Theorem III are satisfied and if* $f \in H(F)$ *then* m *can be taken from* $H(G)$.

Remarks. (a) $e|F \in H(F)$.

(b) If $\infty \in \bar{F}$ then $e(\infty) = 0$.

(c) An essential part of the method of proof presented below has already been given in [**5**].

Proof. Since each point of F has a neighborhood on which f is meromorphic there are countably many subdomains $D_1, D_2, \ldots$ of G with the following properties:

(a) $F \subseteq \bigcup D_n$.

(b) $\bar{D}_i \cap \bar{D}_k = \emptyset \quad i, k \in \mathbf{N}, i \neq k$.

(c) The domains D_i accumulate at most on the boundary of G, i.e. any compact subset of G intersects only finitely many of them.

(d) The set b of boundary points of $\bigcup D_n$ contained in G is composed of countably many segments which can accumulate at most on the boundary of G.

(e) f is meromorphic in each point of $\bigcup D_n$ and holomorphic in each point of b. (Note that the poles of f do not have an accumulation point in F).

Furthermore, we select a sequence $G_1, G_2, \ldots$ of subdomains of G which exhaust G:

$$\bar{G}_n \subseteq G_{n+1}, \quad \bigcup G_n = G.$$

Let $b_n = b \cap \bar{G}_n$ and let l_n be the sum of the lengths of the finitely many segments which make up b_n. Finally, suppose δ_n has been chosen such that

$$(1) \qquad 0 < \delta_n < 1, \quad |t - z| > \delta_n \quad \text{for } t \in b_n \text{ and } z \in \bar{F}$$

and let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of positive numbers with

$$(2) \qquad k! \sum_1^\infty (\epsilon_n l_n)/(\delta_n^{k+1}) < 2\pi\epsilon \quad k = 1, 2, \ldots, q.$$

$$(3) \qquad \sum_1^\infty \epsilon_n l_n < 1.$$

Since b has area 0, by a special case of Theorem I of [**7**] (which may be proved for $G = \mathbf{C}$ and $G =$ disk by elementary means) there is a function $m_1 \in M(G)$ for which

$$(4) \qquad |m_1(z) - f(z)| < \epsilon_n \quad n = 1, 2, \ldots \text{ for } z \in b_n.$$

m_1 approximates f on b but not necessarily on $\bigcup D_n$. By means of a certain integral we will succeed in constructing, starting with m_1, a function m approximating f on F.

In the integrals below we suppose any part of the boundary of a domain D_n to be oriented with D_n on its left.

If O is an open subset of $\mathbf{C}^*$ which has positive distance Δ from b it is well known that

$$J_n(z) = (1/2\pi i) \int_{b_n} (m_1(t) - f(t))/(t - z)\, dt$$

is a function holomorphic on 0. Because the length of b_n is l_n and because of (4),

$$|J_n(z)| < (\epsilon_n l_n)/(2\pi\Delta) \quad \text{for } z \in O.$$

This, together with (3), implies the uniform convergence of $\sum_1^\infty J_n(z)$ on O. Extending the integral over the whole of b, viz.

$$J(z) = \sum_1^\infty J_n(z),$$

it is therefore holomorphic on every open set O, $\bar{O} \subseteq C^*\backslash\bar{b}$, hence also on $C^*\backslash\bar{b}$. In particular, $J(z)$ is a holomorphic function on $G\backslash(\bigcup_1^\infty D_n \cup b)$. The restriction of $J(z)$ on the latter set can be analytically extended to a function m_2 meromorphic on the whole of G. In fact,

$$m_2(z) = J(z) - m_1(z) + f(z) \quad \text{for } z \in \bigcup_1^\infty D_n,$$

while as before

$$m_2(z) = J(z) \quad \text{for } z \in G\backslash\left(\bigcup_1^\infty D_n \cup b\right).$$

The proof makes use of Cauchy's Integral Theorem and his Integral Formula applied to a small circular neighborhood of any point of b. (See [**5**, pp. 108/9]).

We shall now show that $m = m_1 + m_2$ is the approximating function we wanted to construct.

Since $F \subseteq \bigcup_1^\infty D_n$ we have

(5) $$m(z) - f(z) = J(z) \quad \text{for } z \in F.$$

The following arguments makes use of (1), (2) and (4). The sequence

$$S_\nu(z) = \sum_{n=1}^{\nu} J_n(z) = (1/2\pi i) \sum_{n=1}^{\nu} \int_{b_n} (m_1(t) - f(t))/(t - z)\, dt, \qquad \nu = 1, 2, \ldots$$

of functions holomorphic on $\bar{F}$ converges uniformly on $\bar{F}$, hence $e = \lim_{\nu\to\infty} S_\nu$ is a function continuous on $\bar{F}$ with

$$e|F = \sum_1^\infty J_n = J.$$

Furthermore, by (2):

$$|e(z)| < \sum_{n=1}^{\infty} (\epsilon_n l_n)/(2\pi\delta_n) < \epsilon \quad \text{for } z \in \bar{F}.$$

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For a fixed number $k \leqq q$ the sequence of functions

$$S_\nu^{(k)}(z) = \sum_{n=1}^{\infty} J_n^{(k)}(z) = (k!/2\pi i) \sum_{n=1}^{\nu} \int_{b_n} (m_1(t) - f(t)/(t-z)^{k+1} dt$$
$$\nu = 1, 2, \ldots$$

holomorphic on $\bar{F}$ is also uniformly convergent on $\bar{F}$. Therefore

$$e^{(k)} = \lim_{\nu \to \infty} S_\nu^{(k)}$$

is continuous on $\bar{F}$ with $e^{(k)}|F = J^{(k)}$ and

$$|e^{(k)}(z)| < (k!/2\pi) \sum_{1}^{\infty} \epsilon_n l_n / \delta_n^{k+1} < \epsilon \quad \text{for } z \in \bar{F}.$$

By taking $m(z) = f(z) + e(z)$ for $z \in F$ we have, by (5), that m is the approximating function we were looking for.

Note that, for $\infty \in \bar{F}$, we have $S_\nu(\infty) = e(\infty) = 0$ and $S_\nu^{(k)}(\infty) = \lim_{z \to \infty} S_\nu^{(k)}(z) = 0$. One usually does not, however, write $e^{(k)}(\infty) = \lim_{z \to \infty} e^{(k)}(z)$.

6. Combination of UCE-approximations on two different closed sets. Several authors used auxiliary functions to obtain asymptotic or tangential approximations. Lemma D below introduces such a function in case of UCE-approximations.

LEMMA D. *Suppose F satisfies condition (**) of Theorem II. Let $f \in A(F)$ and suppose F is the disjoint union of F_1 and F_2, while $h \in A(F)$ is a function with the following properties: h can be extended on F_i to $\bar{F}_i \backslash F_i$ $(i = 1, 2)$; $0 < |h(z)| < 1$ for $z \in F$. Then there is $m \in M(G)$ and $e \in A(F)$ such that e can be extended on F_i to $\bar{F}_i$ $(i = 1, 2)$ and such that $e(z) = m(z) - f(z)$, $|e(z)| \leqq |h(z)|$ for $z \in F$.*

Proof. $2h^{-1}$ either belongs to $A(F)$ or to the slightly more general class of functions mentioned at the end of Section 3, in case h has zeros on F. Hence Theorem II is applicable and guarantees the existence of functions $m_1 \in M(G)$ and $e_1 \in A(F) \cap C(\bar{F})$ with

$$e_1(z) = m_1(z) - 2h^{-1}(z) \quad \text{for } z \in F$$
$$|e_1(z)| < 1 \quad \text{for } z \in \bar{F}.$$

Therefore

$$|m_1(z)| > 2|h^{-1}(z)| - 1 > |h^{-1}(z)|.$$

Another application of Theorem II shows the existence of functions $m_2 \in M(G)$ and $e_2 \in A(F) \cap C(\bar{F})$ with

$$e_2(z) = m_2(z) - m_1(z) f(z) \quad \text{for } z \in F$$
$$|e_2(z)| < 1 \quad \text{for } z \in \bar{F}.$$

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Then $m = m_2/m_1 \in M(G)$ and for $e(z) = m(z) - f(z)$ we have

$$e(z) = (m_2(z) - m_1(z)f(z))/m_1(z) = e_2(z)/(e_1(z) + 2h^{-1}(z)) \quad \text{for } z \in F$$
$$|e(z)| \leqq |e_2(z)h(z)/(2 + e_1(z)h(z))|$$
$$\leqq |h(z)|/(2 - |e_1(z)h(z)|) \leqq |h(z)| \quad \text{for } z \in F.$$

Since $e_1, e_2 \in A(F) \cap C(\bar{F})$ the function $e|F$ belongs to $A(F)$ and can be extended on a subset of F to $\partial F \cap \partial G$ if and only if h can be so extended. Our arguments imply that this holds for $e|F_1$ and $e|F_2$.

Note that if F satisfies (α) and (β) of Theorem III, m can be chosen in $H(G)$.

Theorems V and VI below may prove useful in case one wants to construct meromorphic functions with prescribed boundary behavior.

THEOREM V. (Generalization of a Theorem of [**2**] and a Theorem of [**7**]). *Let F_1 and F_2 be two disjoint subsets of G with the following properties: F_2 is nowhere dense; $F_2 \cap \partial G \neq \emptyset$; $F = F_1 \cup F_2$ satisfies (**) of Theorem II. Let moreover f be a function from $A(F)$, η a positive real number and $\epsilon(z)$ a function positive and continuous on F_2 converging to 0 in case z converges to ∂G on F_2.*

Then there is a function m from $M(G)$ and a function e with continuous extensions on F_i to $\bar{F}_i$ $(i = 1, 2)$ with the following properties:

$$m(z) - f(z) = e(z) \quad \textit{for } z \in F$$
$$|e(z)| < \eta \quad \textit{for } z \in F_1$$
$$|e(z)| < \epsilon(z) \quad \textit{for } z \in F_2.$$

If $G^*\backslash F$ is connected and locally connected, m can be chosen in $H(G)$.

Proof. We may suppose $\eta < 1$ and $\epsilon(z) < 1$. We choose the auxiliary function h by setting $h = \eta$ on F_1 and $h = \epsilon$ on F_2. Theorem V is now proved by Lemma D.

Remark. $m - f$ is continuously extendable on F_i to $\bar{F}_i$ $(i = 1, 2)$, but the values in a point of $\bar{F}_1 \cap \bar{F}_2$ may be different. We obtain a UCE-approximation on F_1 and a tangential (Carleman) approximation on F_2. If, however, $\bar{F}_1 \cap \bar{F}_2 = \emptyset$ we have a UCE-approximation on the whole of $F_1 \cup F_2$.

THEOREM VI. *Let F_1, F_2 be two disjoint subsets of G, closed in G. $\bar{F}_1 \cap \bar{F}_2 \neq \emptyset$ may hold. Suppose F_1 satisfies (**) of Theorem II. Then to every pair of functions $f_1 \in A(F_1)$, $f_2 \in M(F_2)$ there is a function m UCE-approximating f_1 on F_1 as well as f_2 on F_2, the latter in the special manner described in Theorem IV (simultaneous approximations of derivatives).*

Proof. According to Theorem II there is $m_1 \in M(G)$ UCE-approximating f_1 on F_1. On the other hand there is $m_2 \in M(G)$ UCE-approximating f_2 on F_2 as described in Theorem IV. Since $F_1 \cap F_2 = \emptyset$ m_1 and m_2 can be taken as

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restrictions of a function m_3 meromorphic on $F_1 \cup F_2$. Finally, Theorem IV guarantees the existence of m approximating m_3 as described in Theorem IV.

Remark. Similarly to Theorem V we have a UCE-approximation on the whole of F if $\bar{F}_1 \cap \bar{F}_2 = \emptyset$. Otherwise this need not hold in general.

7. Corollaries equivalent to Theorems I, II, and IV.

We obtain the corollaries by taking the open set G as "large" as possible.

COROLLARY 1. *Let F be a proper subset of $\mathbf{C}^*$ that differs from $\bar{F}$ by a set E compact in $\mathbf{C}^*$. We assume as given a function f such that $f|K$ is uniformly approximable on K by rational functions, where K may be any compact subset of F.*

Then to $\epsilon > 0$ there is $m \in M(\mathbf{C}^*\backslash E)$ with the properties: $m - f$ can be extended continuously to $E = \bar{F}\backslash F$; $|m(z) - f(z)| < \epsilon$ for $z \in \bar{F}$.

Corollaries 2 and 3 are derived from Corollary 1 by substituting new conditions for f and F.

COROLLARY 2. *If $R(F \cap \bar{G}_1) = A(F \cap \bar{G}_1)$ holds for subdomains G_1 of $C^*\backslash E$ with $\bar{G}_1 \subset C^*\backslash E$, then every function from $A(F)$ can be UCE approximated by a function from $M(\mathbf{C}^*\backslash E)$.*

COROLLARY 3. *Theorem IV holds for F and $\mathbf{C}^*\backslash E$.*

Proofs for the three corollaries are immediate using Theorems I, II and IV by choosing $G = \mathbf{C}^*\backslash E$. It is easily seen that, conversely, the three theorems are consequences of the corresponding corollaries.

Evidently, Theorem III also allows the use of a set F as described above and hence the derivation of a corollary. This one will, however, not be equivalent to Theorem III in general because if conditions (α) and (β) are satisfied for $G = \mathbf{C}^*\backslash E$ they need not hold for smaller sets G in which F is closed.

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Editor's note: The Editor has learned with regret that Dr. Roth died on 22 July 1977, shortly after this paper was received. Any communication regarding the paper should be directed to Prof. Dr. P. Wilker, Universität Bern, 3012 Bern Sidlerstr. 5, Switzerland.

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