

A GENERALIZATION OF QUASI-MONOTONE SEQUENCES

by M. M. ROBERTSON

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1. Introduction

The object of this note is to generalize the notion of quasi-monotony for sequences of real numbers and to prove corresponding generalizations of certain known theorems. First, we recall the definition of quasi-monotony.

Definition 1. A sequence $\{a_n\}$ is said to be quasi-monotone if and only if $n^{-\beta}a_n \downarrow 0$ for some $\beta \geq 0$ or equivalently if and only if $a_n \geq 0$ and $\Delta a_n \geq -\alpha n^{-1}a_n$ for some $\alpha \geq 0$. (Here $\Delta a_n = a_n - a_{n+1}$).

The generalized notion referred to in the title is to be called (ϕ, δ) -monotony and this gives rise to

Definition 2. A sequence $\{a_n\}$ is said to be (ϕ, δ) -monotone if and only if $a_n \rightarrow 0$, $a_n \geq 0$ ultimately and $\Delta a_n \geq -\delta_{n+1}$, where the δ_n form a sequence of non-negative numbers, $\{\phi_n\}$ is a positive monotone increasing sequence and $\Sigma \phi_n \delta_n < \infty$.

It is easily seen that every monotone decreasing null sequence is (ϕ, δ) -monotone for all sequences $\{\phi_n\}, \{\delta_n\}$ such that $\Sigma \phi_n \delta_n < \infty$. Also, we observe that a quasi-monotone sequence $\{a_n\}$ in which $a_n \rightarrow 0$ is (ϕ, δ) -monotone with $\delta_{n+1} = \alpha n^{-1}a_n$ for all sequences $\{\phi_n\}$ such that $\Sigma \phi_{n+1} n^{-1}a_n < \infty$. In a recent note, Boas (1) defined δ -quasi-monotone sequences and obtained a number of theorems involving them. These results of Boas are concerned with the particular monotone increasing sequences $\{n^\gamma\} (\gamma \geq 0)$ and $\{\log n\}$ and, in our terminology, these δ -quasi-monotone sequences are (n^γ, δ) -monotone and $(\log n, \delta)$ -monotone respectively. The following results on (ϕ, δ) -monotone sequences are first established in this paper.

Theorem 1. If the sequence $\{a_n\}$ is (ϕ, δ) -monotone and $\Sigma a_n \Delta \phi_n$ converges, then $a_n \phi_n \rightarrow 0$.

Theorem 2. If the sequence $\{a_n\}$ is (ϕ, δ) -monotone and $\Sigma a_n \Delta \phi_n$ converges, then the series $\Sigma \phi_{n+1} \Delta a_n$ is absolutely convergent.

Theorem 3. If $\{\lambda_n\}$ is a strictly increasing sequence of positive integers such that $\Delta \lambda_n = O(\Delta \lambda_{n-1})$ as $n \rightarrow \infty$ and if the sequence $\{a_n\}$ is (n, δ) -monotone, then the two series Σa_n and $\Sigma a_{\lambda_n} \Delta \lambda_n$ are either both convergent or both divergent.

The conclusion of Theorem 1 clearly holds for every convergent sequence $\{\phi_n\}$ and, as Theorem 2 is deduced from Theorem 1, this theorem is also true

for convergent $\{\phi_n\}$. It is easily seen that Theorem 1 extends Olivier's theorem which shows that, for monotone decreasing null sequences $\{a_n\}$, $\Sigma a_n < \infty$ implies that $na_n \rightarrow 0$. Theorem 3 provides an extension of the well known Cauchy condensation test for convergence. The extension of Cauchy's test for quasi-monotone sequences was proved by Shah (2) and Szász (4).

In order to state the next result, we require another definition.

Definition 3. A sequence $\{a_n\}$ is said to be (ϕ, δ) -positive if and only if it is the sequence of differences of a (ϕ, δ) -monotone sequence.

Theorem 4. If the sequence $\{a_n\}$ is (ϕ, δ) -positive and $\Sigma \phi_{n+1} a_n$ converges, then the series $\Sigma \phi_{n+1} a_n$ is absolutely convergent.

In the note of Boas mentioned above, several results were established concerning the integrability of trigonometric series. Here, we generalize two of these results to the case where the trigonometric sine or cosine coefficients form a (ϕ, δ) -monotone sequence. Previous results of this nature for quasi-monotone sequences were proved by Shah in (3).

Theorem 5. Suppose that $\eta(x)$ is a non-negative function such that $\eta(x) \in L(0, \pi)$ and

$$\phi_n = n \int_0^{1/n} \eta(x) dx + \int_{1/n}^\pi x^{-1} \eta(x) dx$$

is a monotone increasing sequence. If $\{\lambda_n\}$ is a (ϕ, δ) -monotone sequence and $\Sigma \lambda_n \Delta \phi_n$ converges, then $\Sigma \lambda_n \cos nx$ is convergent for all values of x (except perhaps at integral multiples of 2π) and $\eta(x) \Sigma \lambda_n \cos nx \in L(0, \pi)$.

Theorem 6. Suppose that $\eta(x)$ is a non-negative function such that

$$x\eta(x) \in L(0, \pi)$$

and

$$\phi_n = n^2 \int_0^{1/n} x\eta(x) dx + \int_{1/n}^\pi x^{-1} \eta(x) dx$$

is a monotone increasing sequence. If $\{\lambda_n\}$ is a (ϕ, δ) -monotone sequence and $\Sigma \lambda_n \Delta \phi_n$ converges, then $\Sigma \lambda_n \sin nx$ is convergent for all values of x and

$$\eta(x) \Sigma \lambda_n \sin nx \in L(0, \pi).$$

2. Proofs of the first four theorems

As has been already remarked, the conclusion of Theorem 1 is immediate when $\{\phi_n\}$ is convergent and so we assume that $\phi_n \uparrow \infty$. We note that

$$\Delta(\phi_r a_r) = \phi_r a_r - \phi_{r+1} a_{r+1} = a_r \Delta \phi_r + \phi_{r+1} \Delta a_r,$$

and so, for $m < n$,

$$\begin{aligned} \phi_m a_m - \phi_n a_n &= \sum_{r=m}^{n-1} a_r \Delta \phi_r + \sum_{r=m}^{n-1} \phi_{r+1} \Delta a_r \\ &\geq \sum_{r=m}^{n-1} a_r \Delta \phi_r - \sum_{r=m}^{n-1} \phi_{r+1} \delta_{r+1}. \end{aligned} \tag{2.1}$$

Since $\sum a_n \Delta \phi_n$ and $\sum \phi_n \delta_n$ converge, to every $\varepsilon > 0$ there corresponds an integer N such that

$$\sum_{r=m}^{n-1} a_r \Delta \phi_r - \sum_{r=m}^{n-1} \phi_{r+1} \delta_{r+1} \geq -\varepsilon$$

and therefore,

$$\phi_m a_m - \phi_n a_n \geq -\varepsilon$$

for all m, n such that $N \leq m < n$. Hence, since ε is arbitrary, the sequence $\{\phi_n a_n\}$ can have at most one limit point. This limit point must be zero, as otherwise we shall show that a contradiction must occur.

Let us suppose that $\phi_n a_n \rightarrow l > 0$. Then, for all sufficiently large n , $a_n > \frac{1}{2} l \phi_n^{-1}$ and so, since $\sum a_n \Delta \phi_n$ converges, $\sum \phi_n^{-1} \Delta \phi_n$ is convergent. Since the sequence $\{\phi_n\}$ is monotone increasing, we write $-c_n = \phi_n^{-1} \Delta \phi_n$. Then we have $c_n \geq 0$, $\sum c_n$ convergent and $\phi_{n+1} / \phi_n = 1 + c_n$. Therefore,

$$\lim_{n \rightarrow \infty} \phi_n = \phi_1 \prod_1^{\infty} (1 + c_n),$$

which is the above mentioned contradiction since $\prod(1 + c_n)$ converges because of the convergence of $\sum c_n$ and $\phi_n \uparrow \infty$. This completes the proof.

To prove Theorem 2, we write $(\Delta a_n)^- = \max(0, -\Delta a_n)$ and obtain

$$\phi_{n+1} |\Delta a_n| = \phi_{n+1} \Delta a_n + 2\phi_{n+1} (\Delta a_n)^-$$

Since $0 \leq (\Delta a_n)^- \leq \delta_{n+1}$ and $\sum \phi_n \delta_n$ converges, we see that $\sum \phi_{n+1} (\Delta a_n)^-$ is convergent. The proof is completed by showing that $\sum \phi_{n+1} \Delta a_n$ is convergent, and this follows by the Cauchy principle of convergence from (2.1) because both $\{\phi_n a_n\}$ and $\sum a_n \Delta \phi_n$ converge.

We now prove Theorem 3. First, we assume that $\sum a_{\lambda_n} \Delta \lambda_n$ is convergent. We have $\Delta a_r \geq -\delta_{r+1}$ and so $a_{r+1} \leq a_r + \delta_{r+1}$. It follows that

$$a_r \leq a_{\lambda_n} + \sum_{\lambda_n+1}^r \delta_k$$

for $\lambda_n < r < \lambda_{n+1}$. This shows that

$$\sum_{\lambda_n}^{\lambda_{n+1}-1} a_r \leq (\lambda_{n+1} - \lambda_n) a_{\lambda_n} + \sum_{\lambda_n+1}^{\lambda_{n+1}-1} \delta_k (\lambda_{n+1} - k)$$

and so, since

$$\lambda_{n+1} - k < \lambda_{n+1} - \lambda_n < C(\lambda_n - \lambda_{n-1}) < Ck$$

for some fixed positive number C , we have

$$\sum_1^{\infty} a_n \leq - \sum_1^{\infty} a_{\lambda_n} \Delta \lambda_n + \sum_1^{\infty} n \delta_n.$$

Thus $\sum a_n$ converges since both $\sum a_{\lambda_n} \Delta \lambda_n$ and $\sum n \delta_n$ are convergent. The converse result is proved in a similar fashion.

Next we prove Theorem 4. Since the sequence $\{a_n\}$ is (ϕ, δ) -positive, there

is a (ϕ, δ) -monotone sequence $\{A_n\}$ such that $a_n = \Delta A_n = A_n - A_{n+1}$. Also, $A_n \rightarrow 0$ and so Σa_n is convergent and $A_n = \sum_n^\infty a_k$. Then the proof follows from Theorem 2 if we show that $\Sigma A_n \Delta \phi_n$ is convergent.

Now, there is an integer N such that $A_n \geq 0$ for all $n > N$ so that, in order to show that $\Sigma A_n \Delta \phi_n$ converges, we have only to show that $-\sum_{N+1}^m A_n \Delta \phi_n$ is bounded above for all $m > N$. We have

$$-\sum_{N+1}^m A_n \Delta \phi_n = -\sum_{n=N+1}^m \Delta \phi_n \sum_{k=n}^\infty a_k \leq -\sum_{n=N+1}^m \Delta \phi_n \sum_{k=n}^\infty (a_k + \delta_{k+1})$$

for all $m > N$. Since $a_k + \delta_{k+1} \geq 0$ for all k , we can reverse the order of summation and obtain

$$\begin{aligned} -\sum_{n=N+1}^m \Delta \phi_n \sum_{k=n}^\infty (a_k + \delta_{k+1}) &= -\sum_{k=N+1}^m \sum_{n=N+1}^k \Delta \phi_n (a_k + \delta_{k+1}) - \sum_{k=m+1}^\infty \sum_{n=N+1}^m \Delta \phi_n (a_k + \delta_{k+1}) \\ &= \sum_{k=N+1}^m (\phi_{k+1} - \phi_{N+1})(a_k + \delta_{k+1}) + (\phi_{m+1} - \phi_{N+1}) \sum_{k=m+1}^\infty (a_k + \delta_{k+1}) \\ &\leq \sum_{k=N+1}^\infty \phi_{k+1} (a_k + \delta_{k+1}), \end{aligned}$$

and so $\Sigma A_n \Delta \phi_n$ converges because both $\Sigma \phi_{n+1} a_n$ and $\Sigma \phi_n \delta_n$ converge.

3. Proofs of Theorems 5 and 6

Since the proofs of these theorems are very similar, we prove only Theorem 6. We write $c_k(x) = 1 - \cos(k + \frac{1}{2})x$. Then we obtain

$$\begin{aligned} -2 \sin \frac{1}{2}x \sum_1^n \lambda_k \sin kx &= \sum_1^n \lambda_k \{ \cos(k + \frac{1}{2})x - \cos(k - \frac{1}{2})x \} \\ &= \sum_1^n \lambda_k \{ c_{k-1}(x) - c_k(x) \} \\ &= \lambda_1 c_0(x) - \lambda_n c_n(x) + \sum_1^{n-1} (\lambda_{k+1} - \lambda_k) c_k(x). \end{aligned}$$

Since $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\sum_1^\infty \lambda_n \sin nx = \frac{1}{2} \operatorname{cosec} \frac{1}{2}x \{ -\lambda_1 c_0(x) + \sum_1^\infty (\Delta \lambda_n) c_n(x) \} \tag{3.1}$$

whenever the right-hand side is convergent.

Now, since $\Sigma \phi_n \delta_n$ and $\Sigma \lambda_n \Delta \phi_n$ converge, it follows by Theorem 2 that $\Sigma \phi_{n+1} \Delta \lambda_n$ is absolutely convergent. Thus $\Sigma \Delta \lambda_n$ is absolutely convergent and, as $|c_k(x)| \leq 2$, (3.1) shows that the series $\Sigma \lambda_n \sin nx$ is uniformly convergent

in any closed interval which excludes all integral multiples of 2π . Also, clearly the series converges to zero at integral multiples of 2π .

To show that $\eta(x)\sum\lambda_n \sin nx \in L(0, \pi)$, we see from (3.1) that

$$\begin{aligned} \int_0^\pi \eta(x) \left| \sum_1^\infty \lambda_n \sin nx \right| dx &\leq \frac{1}{2} |\lambda_1| \int_0^\pi \eta(x)c_0(x) \operatorname{cosec} \frac{1}{2}x dx \\ &\quad + \int_0^\pi \eta(x) \sum_1^\infty |\Delta\lambda_n| c_n(x) \operatorname{cosec} \frac{1}{2}x dx \\ &= \frac{1}{2} |\lambda_1| \int_0^\pi \eta(x) \tan \frac{1}{4}x dx + \sum_1^\infty |\Delta\lambda_n| \int_0^\pi \eta(x)c_n(x) \operatorname{cosec} \frac{1}{2}x dx \\ &\leq \frac{1}{4} |\lambda_1| \int_0^\pi x\eta(x) dx + 8 \sum_1^\infty |\Delta\lambda_n| \left\{ n^2 \int_0^{1/n} x\eta(x) dx + \int_{1/n}^\pi x^{-1}\eta(x) dx \right\}. \end{aligned}$$

This latter quantity is finite since $x\eta(x) \in L(0, \pi)$ and $\sum\phi_n |\Delta\lambda_n|$ is convergent.

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DEPARTMENT OF MATHEMATICS,
 UNIVERSITY OF SURREY,
 LONDON, S.W.11