ANALYTIC FUNCTIONS ON SOME RIEMANN SURFACES, II

KIKUJI MATSUMOTO

To Professor Kinjiro Kunugi on the occasion of his 60th birthday

1. In their paper [12], Toda and the author have concerned themselves in the following

THEOREM OF KURAMOCHI. *Let R be a hyperbolic Riemann surface of the class $O_{HB}(O_{HD}$, resp.). Then, for any compact subset K of R such that $R-K$ is connected, $R-K$ as an open Riemann surface belongs to the class $O_{AB}(O_{AD}$, resp.)* (Kuramochi [4]).

They have raised there the question as to whether there exists a hyperbolic Riemann surface, which has no Martin or Royden boundary point with positive harmonic measure and has yet the same property as stated in Theorem of Kuramochi, and given a positive answer to the Martin part of this question.

The main purpose of this paper is to show that the Royden part is also answered in the positive. In the sequel, we shall investigate covering properties of analytic functions on Riemann surfaces of the class O°_{AD}, which was introduced by Kuroda in his paper [6], give an extension of the D part of Theorem of Kuramochi and, using this extension, give an example of a Riemann surface which answers the Royden part of the question in the positive.

2. Let R be a Riemann surface and let G be a domain on R with smooth relative boundary ∂G clustering nowhere in R. For simplicity, we shall call such a domain G a subregion of R. If G admits no non-constant single-valued analytic function with a finite Dirichlet integral and with real part vanishing continuously on its relative boundary ∂G, we say that G belongs to the class SO_{AD}.

Let $w=f(p)$ be a non-constant single-valued analytic function in a subregion G with relative boundary ∂G of a Riemann surface R. We suppose that

Received May 10, 1963.

https://doi.org/10.1017/S0027763000011235 Published online by Cambridge University Press

this function is continuous on $G \cup \partial G$ and that, for a certain point $w = w^*$ in the w-plane and for a certain positive number ρ, the value of this function $f(p)$ at every point of G lies in the open disc (c_ρ): $|w - w^*| < \rho$[1] and the values of $f(p)$ on ∂G fall on the circumference c_ρ of the disc (c_ρ). Further we suppose that the spherical area of the Riemannian image Φ of G by $w = f(p)$ is finite[2].

We denote by E the set of points w in (c_ρ) which are not covered by Φ. Then, this set E is closed with respect to (c_ρ) and if G and each subregion $G'(\subset G)$ of R belong to the class SO_{AD}, E is totally disconnected. This can be proved as follows. First we prove that E has no interior point. In fact, we may suppose without any loss of generality that $w^* \neq \infty$, and contrary suppose that E contains an interior point w_1. Then for a sufficiently small $\rho_1 > 0$, the disc (c_1): $|w - w_1| < \rho_1$ is contained completely in (c_ρ) and is not covered by Φ at all. Obviously there exists a non-constant single-valued analytic function $\varphi(w)$ in $(c_\rho) - (\overline{c_1})$, where $(\overline{c_1})$ is the closure of (c_1), such that its real part vanishes continuously on c_ρ and $|\varphi'(w)|$ is bounded: $|\varphi'(w)| \leqq M < +\infty$. The function $\varphi(f(p))$ is a non-constant single-valued analytic function in G with real part vanishing continuously on ∂G and has a Dirichlet integral

$$\iint_G |f'(p)|^2 |\varphi'(f(p))|^2 dxdy \leqq M^2 \iint |f'(p)|^2 dxdy < +\infty,$$

where we denote by $x + iy$ the local parameter at p. This contradicts that G belongs to the class SO_{AD}. Next suppose that E contains a non-degenerate continuum γ. Then we can find a disc (c_2) $(\subset (c_\rho))$ such that the open set $(c_2) - \gamma$ has at least two connected components. Consequently each connected piece Φ' of Φ lying over (c_2) does not cover a set with interior points. The subregion G' in G corresponding to Φ' belongs to the class SO_{AD} by our assumption, so that the same argument as above leads us to a contradiction.

3. Now we shall be concerned with the covering properties of analytic functions on a Riemann surface of the class O_{AD}°. The class O_{AD}° is the class of open Riemann surfaces, any subregion of which belongs to the class SO_{AD}. The following inclusions hold.

---

1) When $w^* = \infty$, (c_ρ) is the disc $|w| > 1/\rho$.

2) In the sequel we shall say simply that $f(p)$ has a finite spherical area, when the spherical area of Φ is finite.

$$\begin{array}{ccccc} O_{HB} & \subsetneqq & O^{\circ}_{AB} & \subsetneqq & O_{AB} \\ \cap\!\!\!\!\!\! \shortparallel & & \cap\!\!\!\!\!\! \shortparallel & & \cap\!\!\!\!\!\! \shortparallel \\ O_{HD} & \subsetneqq & O^{\circ}_{AD} & \subsetneqq & O_{AD} \end{array}$$

On the other hand, no inclusion holds between O_{HD} and O°_{AB} or O_{AB} and between O°_{AD} and O_{AB}.

Let R be a Riemann surface and let $w=f(p)$ be a non-constant single-valued analytic function defined on R. We consider the Riemannian image Φ of R by $w=f(p)$ over the w-plane, that is, the covering surface over the w-plane formed by elements $[p, f(p)]$. Then the following extension of Kuroda's theorem [6, Theorem 3] is an immediate consequence of the fact just stated in §2.

THEOREM 1. *Suppose that R belongs to the class $O^{\circ}{}_{AD}$ and the spherical area of Φ is finite. Then Φ has the Iversen property.*

4. Let R be an arbitrary open Riemann surface and let $\{G_n\}_{n=1,2,\dots}$ denote a sequence of non-compact subregions of R with compact relative boundary ∂G_n such that $G_n \supset G_{n+1} \cup \partial G_{n+1}$ for each n and $\bigcap_{n=1}^{\infty} G_n = \emptyset$. We classify such sequences with the following equivalence relation: Two sequences $\{G^1_n\}_{n=1,2,\dots}$ and $\{G^2_n\}_{n=1,2,\dots}$ are equivalent if and only if, for any m, there is an n such that $G^1_n \supset G^2_m$, and vice versa. Each of these equivalence classes is a boundary component of R in Kerékjártó-Stoïlow's sense, and we consider it as an ideal boundary point of R. A neighborhood of this ideal boundary point P means the union of P and an open set of R containing $G_m \in \{G_n\}_{n=1,2,\dots}$ for some m, where $\{G_n\}_{n=1,2,\dots}$ is a representative member of P. Then, for a subdomain D of R such that P can be approached by a sequence of points in D and for a single-valued analytic function $w=f(p)$ of D, we can consider the cluster set $C_D(f, P)$ of $f(p)$ at P.

Let R be an open Riemann surface of the class O°_{AD} and let K be a compact subset such that $R-K$ is connected. Let $w=f(p)$ be a non-constant single-valued analytic function on $R-K$ with a finite spherical area. We consider a disc (c_ρ): $|w-w^*|<\rho$[3] such that, for some relatively compact subregion $R_0 \supset K$ of R with smooth boundary ∂R_0, $f(p)$ takes no value in the closure of

[3] In case $w^*=\infty$, we consider as (c_ρ) a domain $|w|>\rho$.

(c_ρ) on ∂R_0. Then each connected component Δ of the inverse image $f^{-1}((c_\rho))$ on $R-\overline{R}_0$, if exists, belongs to the class SO_{AD} and it follows from the fact stated in §2 that the set $(c_\rho)-f(\Delta)$ is totally disconnected. Therefore by a localization of so-called Stoïlow's principle on Iversen's property[4] we have

THEOREM 2. *Let R be an open Riemann surface belonging to the class O°_{AD}. Then, for any single-valued analytic function $w=f(p)$ with a finite spherical area defined on $R-K$, where K is a compact subset such that $R-K$ is connected, the cluster set $C_{R-K}(f, P)$ is total or reduces to one point at each ideal boundary point P of R.*

As a corollary, we have the following theorem due to Kuramochi [5].

THEOREM 3. *Let R be an open Riemann surface belonging to the class O°_{AD}, let K be a compact subset such that $R-K$ is connected and let $w=f(p)$ be a single-valued analytic function on $R-K$ with a finite Dirichlet integral. Then $f(p)$ has a limit at each ideal boundary point of R.*

Proof. Let R_0 be a relatively compact subregion of R such that $R_0 \supset K$ and $R-\overline{R}_0$ is connected. Then we have

$$|\mathfrak{Re}\, f(p)| < \sup_{p\in\partial R_0} |\mathfrak{Re}\, f(p)| < +\infty \quad \text{and} \quad |\mathfrak{Im}\, f(p)| < \sup_{p\in\partial R_0} |\mathfrak{Im}\, f(p)| < +\infty$$

on $R-\overline{R}_0$, and hence $f(p)$ is bounded on $R-\overline{R}_0$. Here we shall prove the first inequality. Suppose that $\mathfrak{Re}\, f(p_0) > c > \sup_{p\in\partial R_0} \mathfrak{Re}\, f(p)$ for a point $p_0 \in R-\overline{R}_0$. Then the restriction $f_\Delta(p)$ of $f(p)$ to the connected component Δ of the open set $\{p;\ \mathfrak{Re}\, f(p) > c\}$ containing p_0 is non-constant, single-valued and analytic and its real part is constant on the relative boundary $\partial\Delta$ of Δ, that is, $\mathfrak{Re}\, f_\Delta(p)=c$ on $\partial\Delta$. Obviously the Dirichlet integral of $f_\Delta(p)$ over Δ is finite. On the other hand, Δ belongs to the class SO_{AD}, because R belongs to the class O°_{AD}; this is a contradiction. By the same reasoning we have $\mathfrak{Re}\, f(p) \geqq \inf_{p\in\partial R_0} \mathfrak{Re}\, f(p)$, so that $|\mathfrak{Re}\, f(p)| \leqq \sup_{p\in\partial R_0} |\mathfrak{Re}\, f(p)| < +\infty$.

Therefore at any ideal boundary point P of R, the cluster set $C_{R-K}(f, P)$ can not be total and hence, by Theorem 2, it reduces to one point, since the finiteness of the spherical area is derived from the finiteness of the Dirichlet integral. Our proof is now complete.

---

[4] Cf. K. Noshiro [11], Chapt. IV, § 2.

5. Let R be an arbitrary open Riemann surface and let $\{R_n\}_{n=0,1,2,\dots}$ be a normal exhaustion of R such that $R-\overline{R}_0$ is connected. For an ideal boundary point P of R this exhaustion determines a representative member $\{G_n\}$ of P such that every G_n is a connected component of $R-\overline{R}_n$. We consider the harmonic function $\omega_{n,m}(p)$ $(0<\omega_{n,m}(p)<1)$ in $R_m-\overline{G}_n-\overline{R}_0(m\geqq n)$ with boundary values such that

$$\omega_{n,m}(p)=\begin{cases}0 & \text{on } (\partial R_m-G_n)\cup\partial R_0\\ 1 & \text{on } \partial G_n.\end{cases}$$

$\omega_{n,m}(p)$ increases as $m\to\infty$, so that $\omega_{n,m}(p)$ tends to a harmonic function $\omega_n(p)$. Now let n tend to infinity. Then $\omega_n(p)$ decreases and tends to a non-negative harmonic function $\omega_P(p)$ defined on $R-\overline{R}_0$. $\omega_P(p)\equiv 0$ or $\omega_P(p)>0$ in $R-\overline{R}_0$, and we say that the harmonic measure of an ideal boundary point P of R is zero or positive according as the first or the second case occurs, respectively. Of course this property of P does not depend on the choice of the exhaustion $\{R_n\}$ of R.

Now we have an extension of Theorem of Kuramochi stated in §1.

THEOREM 4. *Let R be a Riemann surface of the class O°_{AD} which has at least one ideal boundary point with positive harmonic measure. Then, for any compact subset K of R such that $R-K$ is connected, $R-K$ belongs to the class O_{AD}.*

Using Theorem 3, we can give a proof of this theorem quite similar to that in the case of the paper [12].

6. Constantinescu and Cornea have clarified in their paper [1] that in the D part of Kuramochi's theorem, $\underline{HD}$-minimal functions play an essential role. On the other hand, Kusunoki and Mori [7] and Nakai [9] have proved the equivalence of the existence of an $\underline{HD}$-minimal function and the existence of a point with positive harmonic measure in Royden's harmonic boundary. But for the proof of Theorem 4, we do not need the existence of $\underline{HD}$-minimal functions. In fact, we can give an example of a Riemann surface which does not belong to the class U_{HD}, that is, has no Royden boundary point with positive harmonic measure, and yet satisfies the conditions of Theorem 4. Here U_{HD}, which was introduced by Constantinescu and Cornea [1], is the class of hyperbolic Riemann surfaces admitting at least one $\underline{HD}$-minimal function.

7. Before constructing the example, we shall study the sufficient condition due to Nakai [10] for a Riemann surface not to belong to the class U_{HD}.

First we shall state Nakai's result. Let R be a Riemann surface. We denote by $[C_1, C_2]$ a pair of mutually disjoint simple closed curves C_1 and C_2 on R satisfying the following two conditions:

(1) C_1 and C_2 are dividing cycles of R, i.e., the open set $R-C_i$ $(i=1, 2)$ consists of two components,

(2) the union of C_1 and C_2 is the boundary of a relatively compact domain (C_1, C_2) of R such that (C_1, C_2) is of genus one.

We say that two such pairs $[C_1, C_2]$ and $[C_1', C_2']$ are equivalent if there exists such a third pair $[C_1'', C_2'']$ that

$$(C_1, C_2) \cap (C_1', C_2') \supset (C_1'', C_2''),$$

or if there exists a chain of pairs

$$[C_1, C_2], [C_1^{(1)}, C_2^{(1)}], \ldots, [C_1^{(n)}, C_2^{(n)}], [C_1', C_2']$$

such that each pair of this chain is equivalent to its next one in the above sense. Then this relation is actually an equivalence relation, so that we divide the totality of these pairs $[C_1, C_2]$ into equivalence classes. Calling each equivalence class H a handle of R, we observe that R has at most a countable number of handles.

An annulus A in R is said to be associated with a handle H of R, $A \in H$ in notation, if there exists a representative $[C_1, C_2]$ of H with $\bar{A} \subset (C_1, C_2)$ such that each boundary component of the relative boundary of A rounds the hole of (C_1, C_2) that is,

(3) each boundary component of the relative boundary of A is not adividing cycle of the domain (C_1, C_2),

(4) each boundary component of the relative boundary of A is not homotopic to any component of an arbitrary level curve of the harmonic function in (C_1, C_2) with boundary value 0 on C_1 and 1 on C_2.

According to Nakai, we say that a Riemann surface R is an almost finite Riemann surface or that R is of almost finite genus, if there exists a sequence $\{A_n\}$ of annuli in R satisfying the following condition:

(5) $A_n \in H_n$, where $\{H_n\}$ is the totality of handles in R,

(6) $A_n \cap A_m = \emptyset$ if $n \neq m$,

(7) $\sum_n 1/\text{mod } A_n < +\infty$,

where mod A_n is the modulus of the annulus A_n.

Nakai's result can be stated as follows.

THEOREM OF NAKAI. *Any almost finite Riemann surface does not belong to the class* U_{HD} (Nakai [10]).

8. Here we shall give an alternative proof which does not use the theory of Royden's compactification.

Let G be a subregion on a Riemann surface R, let u be a positive harmonic function on R and let U be a positive harmonic function on G vanishing continuously on ∂G such that there exists at least one positive superharmonic function on R dominating U on G (we shall call such a function U admissible). We denote by $I_G(u)$ and $E_G(U)$ the upper envelope of the non-negative subharmonic function on G dominated by u and vanishing continuously on ∂G and the lower envelope of the positive superharmonic functions on R dominating U on G, respectively. (These operations was introduced by Kuramochi [3] and Heins [2]). Further we denote by γ_n the closed Jordan curve in A_n which divides A_n into two annuli $A_{n,1}$ and $A_{n,2}$ such that

$$\text{mod } A_{n,1} = \text{mod } A_{n,2} = \text{mod } A_n/2.$$

Nakai's theorem can be derived from the following four propositions.

PROPOSITION 1. *Any Riemann surface of planar character does not belong to the class* U_{HD}.

PROPOSITION 2. $G = R - \bigcup_n \gamma_n$ *is a subregion on* R *of planar character.*

PROPOSITION 3. *If* $R \notin O_G$ *and*

$$\sum_n 1/\text{mod } A_n < +\infty,$$

then

$$E_G I_G 1 = 1 \text{ and } I_G 1 \in HD.$$

PROPOSITION 4. *Let u be an $\underline{HD}$-minimal function on R. If there exists an admissible function U on G having a finite Dirichlet integral such that $E_G(U)$ dominates u on R, then $I_G(u)$ is also $\underline{HD}$-minimal on G.*

Now contrary suppose that an almost finite Riemann surface R admits an

https://doi.org/10.1017/S0027763000011235 Published online by Cambridge University Press

HD-minimal function u. It is known that u is bounded,[5] so that we assume that $0<u<1$. Put $U=I_G1$. Then, by Proposition 3, U has a finite Dirichlet integral and $E_GU=E_GI_G1=1\geqq u$. Therefore U satisfies all conditions of Proposition 4 and so we can conclude that I_Gu is HD-minimal on G. On the other hand, by Proposition 2, G is of planar character; this contradicts Proposition 1. Thus R can not belong to the class U_{HD}.

9. Proposition 1 is an immediate consequence of Theorem 11′ in Constantinescu and Cornea [1], Proposition 2 is obvious and Proposition 4 is Lemma 6 in Matsumoto [8], and so it remains for us to prove Proposition 3.

Proof of Proposition 3. Let $w_n(p)$ be the continuous function on R such that

$$w_n(p)=\begin{cases}\text{harmonic} & \text{on } A_n-\gamma_n\\ 1 & \text{on } \gamma_n\\ 0 & \text{on } R-A_n,\end{cases}$$

and let $g(p)$ denote the least harmonic majorant of $\sum_n w_n(p)$ on $G=R-\bigcup_n \gamma_n$. Then

$$g(p)+I_G1(p)\equiv 1 \text{ on } G$$

and

$$D(g)\leqq \sum_n D(w_n)=8\pi\sum_n 1/\text{mod } A_n<+\infty,$$

where we denote by $D(u)$ the Dirichlet integral of u taken over R. Hence, I_G1 is the constant zero or

$$I_G1\in HD.$$

Next suppose that $E_GI_G1<1$. Then, putting $v=1-E_GI_G1>0$, we have

$$I_Gv=I_G(1-E_GI_G1)=I_G1-I_GE_GI_G1=0,$$

because $I_GE_G(U)=U$ for each admissible U. Now we define the continuous function $v_n(p)$ by

$$v_n(p)=\begin{cases}\text{harmonic} & \text{on } A_n-\gamma_n\\ v(p) & \text{on } \gamma_n\\ 0 & \text{on } R-A_n,\end{cases}$$

---

[5] See Constantinescu and Cornea [1].

https://doi.org/10.1017/S0027763000011235 Published online by Cambridge University Press

and denote by $\hat{v}_n(p)$ the least harmonic majorant of $v_n(p)$ on G. Then we have for each m

$$D(\sum_{n\geqq m}\hat{v}_n)\leqq D(\sum_{n\geqq m}v_n)=\sum_{n\geqq m}D(v_n)<\sum_{n\geqq m}D(w_n)=8\pi\sum_{n\geqq m}1/\text{mod }A_n,$$

and, since $I_G v=0$, have also

$$v(p)=\sum_n \hat{v}_n(p) \qquad \text{on } R.$$

Let $\{R_k\}_{k=0,1,2,\ldots}$ be a normal exhaustion of R such that $R_0\supset\gamma_1$. Defining the continuous function u_k by

$$u_k(p)=\begin{cases}\text{harmonic on } R_k-\gamma_1\\ 0 \qquad\qquad \text{on } \gamma_1\\ v \qquad\qquad \text{on } R-R_k,\end{cases}$$

we see that u_k tends to $v-\hat{v}_1$ uniformly on any compact subset of $R-\gamma_1$ as $k\to\infty$. Further we see that

$$D(u_k-u_j)\to 0 \qquad \text{as } k,\ j\to\infty,$$

and hence

$$D(v-\hat{v}_1)=\lim_{k\to\infty}D(u_k)\geqq D(v).$$

In fact,

$$D(u_k-u_j)=D(u_k)-D(u_j) \qquad \text{if } k>j,$$

and

$$D(u_k)\geqq D(u_{k+1})\geqq D(v) \qquad \text{for each } k,$$

so that

$$D(u_k-u_j)=D(u_k)-D(u_j)\to 0 \qquad \text{as } k,\ j\to\infty.$$

Similarly we have

$$D(v-\sum_{n=1}^{m+1}v_n)\geqq D(v-\sum_{n=1}^{m}\hat{v}_n)\geqq D(v-\hat{v}_1) \qquad \text{for each } m.$$

On the other hand, $D(v-\hat{v}_1)>0$, because, if $I_G 1>0$, v is not constant, so that $D(v-\hat{v}_1)\geqq D(v)>0$ and, if $I_G 1=0$, then $v=1$ and $v-\hat{v}$ is the harmonic measure of the ideal boundary of the hyperbolic Riemann surface R with respect to $R-\gamma_1$, so that $D(v-\hat{v}_1)>0$. This contradicts that

$$D(v-\sum_{n=1}^{m}v_n)=D(\sum_{n\geqq m}v_n)<8\pi\sum_{n\geqq m}1/\text{mod }A_n\to 0 \qquad \text{as } m\to\infty.$$

Thus we have

$$E_G I_G 1 = 1,$$

and hence we see that $I_G 1$ can not be the constant zero and $I_G 1 \in HD$.

10. From § 8, we have

THEOREM 5. *Let R be an open Riemann surface. If there exists a subregion G on R which is of planar character and satisfies that*

$$E_G I_G 1 = 1 \quad \text{and} \quad I_G 1 \in HD,$$

then R does not belong to the class U_{HD}.

Remark 1. Let G be an open subset of R, each connected component G_i of which is a subregion on R. We define $I_G u$ by

$$I_G u = I_{G_i} u \qquad \text{on each } G_i$$

and, for an admissible U, $E_G U$ by

$$E_G U = \sum_i E_{G_i} U_i,$$

where we say that U is admissible for G if the restriction U_i of U to each G_i is admissible in the usual sense. Then we see that, under these difinitions, the connectivity of G in Theorem 5 is not necessary and the second condition $I_G 1 \in HD$ can be relaxed as follows:

$$I_{G_i} 1 \in HD \qquad \text{for each } G_i.$$

Remark 2. Using Lemma 5 in [8], we can prove the following analogue of Theorem 5.

THEOREM 6. *Let R be an open Riemann surface. If there exists a subregion G on R which is of planar character and satisfies that*

$$E_G I_G 1 = 1,$$

then R admits no bounded minimal harmonic function.

11. We shall construct an example of a Riemann surface not belonging to the class U_{HD} and yet satisfying the conditions of Theorem 4.

Construction of the example. Our example can be obtained if we take the slits sufficiently small in the example given in [12].

Let E be a Cantor set on the closed interval I_0: $[-1/2, 1/2]$ with constant successive ratios ξ_n, $0<\xi_n=2\ell<1$. Then E is of logarithmic positive capacity. In the quite similar manner in [12], we construct a Riemann surface $\tilde{F}$ with only one ideal boundary component as a covering surface of the complementary domain F of E with respect to the extended w-plane. But, this time, we take as $A_{n,k}(n=1, 2, \ldots ; k=1, 2, \ldots, 2^n)$ the following ring domains on F.

$$A_{n,k}=\{\ell^n(1-\ell)<|w-w_{n,k}|<\ell^{n-1}(1-\ell)/2\}$$

and take the slits $S_{n,k}(n=1, 2, \ldots ; k=1, 2, \ldots, 2^n)$ so small that there exists an annulus $D_{n,k}$ which separates $S_{n,k}$ from E and satisfies that

$$D_{n,k}\cap D_{n',k'}=\emptyset \qquad \text{if } (n, k)\neq(n', k')$$

and

$$\sum_{k,n} 1/\text{mod } D_{n,k}<+\infty.$$

Then $\tilde{F}$ is a desired Riemann surface.

To show that $\tilde{F}$ is a desired Riemann surface. By the same reasoning in [12], $\tilde{F}$ has only one ideal boundary component with positive harmonic measure. Therefore it is enough to prove that

(i) $\tilde{F}\in O^{\circ}_{AD}$,
(ii) $\tilde{F}\notin U_{HD}$.

Proof of (i). Kuroda [6] proved that if a Riemann surface R admits a sequence of ring domains $B_{n,k}(n=1,2, \ldots ; k=1,2, \ldots, \nu(n))$ such that, for each n, all of $B_{n+1,k}$ $(k=1, 2, \ldots, \nu(n+1))$ together separate the ideal boundary of R from all of $B_{n,k}(k=1, 2, \ldots, \nu(n))$ and

$$\sum_n \log \mu_n=+\infty,$$

then R belongs to the class O°_{AD}. Here μ_n denotes the minimum harmonic modulus of ring domains $B_{n,k}$ $(k=1,2, \ldots, \nu(n))$.

For the sequence of ring domains $\{A_{n,k}\}$ considered above, $\mu_n=(\ell^{n-1}(1-\ell)/2)/\ell^n(1-\ell)=1/2\ell>1$. Hence $\tilde{F}\in O^{\circ}_{AD}$.

Proof of (ii). Let $c_{n,k}$ denote the inner boundary of the annulus $D_{n,k}$ and let $(c_{n,k})$ denote the interior of $c_{n,k}$. Then for each $(c_{n,k})$ $(n=1,2, \ldots ;$ $k=1,2, \ldots, 2^n)$, there exists only one connected piece $(\tilde{c}_{n,k})$ of $\tilde{F}$ over $(c_{n,k})$

which is two-sheeted. We delete from $\widetilde{F}$ all of the closures $(\overline{\widetilde{c}_{n,k}})$ of these two-sheeted connected pieces $(\widetilde{c}_{n,k})$ $(n = 1, 2, \ldots ; k = 1, 2, \ldots, 2^n)$ and obtain an open subset G of $\widetilde{F}$, each connected component of which is a subregion of $\widetilde{F}$. By our assumption

$$\sum_{n,k} 1/\text{mod } D_{n,k} < +\infty,$$

the same argument as in §9 leads us to the conclusion that

$$E_G I_G 1 = 1 \quad \text{and} \quad I_G 1 \in HD.$$

Therefore we see by Remark 1 of Theorem 5 that $\widetilde{F}$ does not belong to the class U_{HD}.

References

[1] C. Constantinescu and A. Cornea: Über den idealen Rand und einige seiner Anwendungen bei Klassifikation der Riemannschen Flächen, Nagoya Math. J. **13** (1958), 166–233.

[2] M. Heins: On the Lindelöfian principle, Ann. of Math. **61** (1955), 440–473.

[3] Z. Kuramochi: Relations between harmonic dimensions, Proc. Japan Acad. **30** (1954), 576–580.

[4] Z. Kuramochi: On the behaviour of analytic functions on abstract Riemann surfaces, Osaka Math. J. **7** (1955), 109–127.

[5] Z. Kuramochi: Representation of Riemann surfaces, Osaka Math. J. **11** (1959), 71–82.

[6] T. Kuroda: On some theorems of Sario, Bull. Math. Soc. Sci. Math. Phys. R.P.R. **2** (50) (1958), 411–417.

[7] Y. Kusunoki and S. Mori: On the harmonic boundary of an open Riemann surface, II, Mem. Coll. Sci. Univ. Kyoto **33** (1960), 209–223.

[8] K. Matsumoto: On subsurfaces of some Riemann surfaces, Nagoya Math. J. **15** (1959), 261–274.

[9] M. Nakai: A measure on the harmonic boundary of a Riemann surface, Nagoya Math. J. **17** (1960), 181–218.

[10] M. Nakai: Genus and classification of Riemann surfaces, Osaka Math. J. **14** (1962), 153–180.

[11] K. Noshiro: Cluster Sets, Berlin (1960).

[12] N. Toda and K. Matsumoto: Analytic functions on some Riemann surfaces, Nagoya Math. J. **22** (1963), 211–217.

Mathematical Institute
Nagoya University

https://doi.org/10.1017/S0027763000011235 Published online by Cambridge University Press